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Rough ideals in ADL's

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ABSTRACT: In this paper, we initiate the study of rough sets within the context of ADL's. Here, we introduce the notions of rough ideals and rough prime ideals in an ADL and study the special properties of images and preimages of such ideals under the homomorphism of ADL's.

Key Words: Rough set, ADL, rough ideals, rough prime ideals, homomorphism.

Contents

1. Introduction

The notation of rough sets was first introduced by Pawlak [9] as an extension of general set theory. Rough set theory is still a new mathematical approach to deal with uncertainity or vagueness and imprecise data. In this approach, Frege's idea of vagueness or imprecision is expressed by a boundary region of a set, and not by a partial membership, like in fuzzy set theory which was proposed by Zadeh[17]. After the notion of rough sets, the algebric approach of these was studied by several authors. Biswas and Nanda [1] introduced the notions of rough groups and rough subgroups. Kuroki [6] investigated the rough ideals in semigroups. Morderson [7] applied rough set theory to fuzzy ideal theory in rings. Davvaz [2, 3] introduced the notion of rough subring with respect to an ideal (fuzzy ideal) of a ring. Degang, Wenxiu, Yeung and Tsang [4] studied rough approximations on a complete distributive lattices. Kazanci and Davvaz [5] introduced the concepts of rough prime (primary) ideals and rough fuzzy prime (primary) ideals in commutative rings. Xiao, Zhou [14] have introduced the notion of rough ideals in lattices. Further, Xiao and Guo [15] investigated rough sets induced by ideals in lattices. Yang, Zhu and Xin [16] studied rough sets based on fuzzy ideals in distributive lattices.

Swamy and Rao [13] have introduced the concept of an Almost Distributive Lattice (abbreviated as ADL) as a common abstraction of most of the existing ring theoretic and lattice theoretic generalization of Boolean rings and Boolean algebras. An ADL A is an algebra $(A, \wedge, \vee, 0)$ satisfying the following axioms: for all a, b and $c \in A$,

(1) $0 \wedge a = 0$
(2) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
(3) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(4) $(a \vee b) \wedge b = b$
(5) $(a \vee b) \wedge a = a$
(6) $a \vee (a \wedge b) = a$

For example, let R be a commutative regular ring with identity (that is, R is a commutative ring with identity in which, for each $a \in R$ there exists an uniquc idempotent $a_0 \in R$ such that $aR = a_0R$). If we

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define, for any $a, b \in R$,

$$a \wedge b = a_0 b \text{ and } a \vee b = a + b + a_0\ b$$

then $(R, \wedge, \vee, 0)$ is an ADL. An ADL satisfies all the axioms of a distributive lattice, except the axioms :

$$a \wedge b = b \wedge a,\ a \vee b = b \vee a \text{ and } (a \wedge b) \vee c = (a \vee c) \wedge (b \vee c).$$

It is proved that these three conditions are equivalent in any ADL. If any one of these axioms hold, then the ADL becomes a distributive lattice. A nonempty set I of an ADL $(A, \wedge, \vee, 0)$ is said to be an ideal if $a \vee b \in I$ and $a \wedge x \in I$ for all $a, b \in I$ and $x \in A$. The partial order $\leqslant$ on an ADL A is defined by $a \leqslant b$ if and only if $a \wedge b = a$ (equivalently, $a \vee b = b$). The set of all ideals of an ADL A forms a distributive lattice under the operation $\wedge$ and $\vee$ defined by

$$I \wedge J = I \cap J \text{ and } I \vee J = \{a \vee b : a \in I, b \in J\}$$

for all ideals I and J of A. Actually, the set of all ideals forms a complete lattice under the set inclusion.

In recent trends, several researchers have done much work on ADL's, for example, Santhi Sundar Raj, U.M.Swamy, Natnael, Srikanya, Prakasam Babu and Ramanuja Rao [12,10,8,11] have introduced certain fuzzy concepts in ADL's, like fuzzy ideals and filters, fuzzy prime ideals, fuzzy prime spectrums, fuzzy initial and final segments. In this paper, we shall introduce the notion of rough sets in ADL,s by a pair of lower and upper approximations of a subset with respect to a congruence relation. Mainly, we introduce, the notions of rough ideals and rough prime ideals and investigate certain properties of these. Through out this paper, A stands for a non-trivial ADL $(A, \wedge, \vee, 0)$.

2. Rough Subsets in an ADL

Let θ be a congrurence on A, that is, θ is an equivalence relation on A such that $(a, b) \in \theta$ and $(c, d) \in \theta$ implies

$$(a \wedge c, b \wedge d) \in \theta \text{ and } (a \vee c, b \vee d) \in \theta$$

(equivalently, $(a, b) \in \theta$ and $c \in A$ implies

$$(a \wedge c, b \wedge d), (c \wedge a, d \wedge b), (a \vee c, b \vee c) \text{ and } (c \vee a, c \vee b) \in \theta\big).$$

The θ-congruence class containing an element x of A denoted by $[x]_\theta$. Therefore,

$$[x]_\theta = \{a \in A : (a, x) \in \theta\}.$$

Let X be a nonempty subset of A. Then the sets

$$\theta_-(X) := \{x \in A : [x]_\theta \subseteq X\}$$

and

$$\theta^-(X) := \{x \in A : [x]_\theta \cap X \neq \emptyset\}$$

are called the θ-lower and θ-upper approximations of A.

Let $\mathcal{P}(A)$ denote the set of all subsets of A. For any nonempty subset X of A,

$$\theta(X) = \big(\theta_-(X),\ \theta^-(X)\big)$$

is called a rough set with respect to the congruence relation θ on A or a θ-rough subset of $\mathcal{P}(A) \times \mathcal{P}(A)$ or simply a rough set of A if $\theta_-(X) \neq \theta^-(X)$.

Let us recall (from [9]) some basic properties of approximations in any universe, which will be needed in the main text of the paper.

Theorem 2.1 *Let θ and ϕ be equvalence relations on any unuverse U. If X and Y are nonempty subsets of U, then the following holds:*

(1) $\theta_-(X) \subseteq X \subseteq \theta^-(X)$;
(2) $\theta^-(X \cup Y) = \theta^-(X) \cup \theta^-(Y)$;
(3) $\theta_-(X \cup Y) = \theta_-(X) \cap \theta_-(Y)$;
(4) $X \subseteq Y$ *implies* $\theta_-(X) \subseteq \theta_-(Y)$;
(5) $X \subseteq Y$ *implies* $\theta^-(X) \subseteq \theta^-(Y)$;
(6) $\theta_-(X \cup Y) \supseteq \theta_-(X) \cup \theta_-(Y)$;
(7) $\theta^-(X \cap Y) \subseteq \theta^-(X) \cap \theta^-(Y)$;
(8) $\theta \subseteq \phi$ *implies* $\theta_-(X) \supseteq \phi_-(X)$;
(9) $\theta \subseteq \phi$ *implies* $\theta^-(X) \subseteq \phi^-(X)$;
(10) $(\theta \cap \phi)^-(X) \subseteq \theta^-(X) \cap \phi^-(X)$;
(11) $(\theta \cap \phi)_-(X) \supseteq \theta_-(X) \cap \phi_-(X)$.

(12) $\theta^-\big(\theta^-(X)\big) = \theta^-(X)$

If we define the join and meet of any two subsets X and Y of an ADL A as follows:

$$X \vee Y = \{x \vee y : x \in X, y \in Y\};$$
$$X \wedge Y = \{x \wedge y : x \in X, y \in Y\}.$$

Then one can easily verified that, $[\ a\ ]_\theta \vee [\ b\ ]_\theta \subseteq [\ a \vee b]_\theta$ and $[\ a\ ]_\theta \wedge [\ b\ ]_\theta \subseteq [\ a \wedge b]_\theta$ for any congruence relation θ on A and for all $a, b \in A$. A congruence relation θ on an ADL A is called join complete if $[\ a \vee b]_\theta \subseteq [\ a\ ]_\theta \vee [\ b\ ]_\theta$. Similarly, θ is said to be meet-complete if $[\ a \wedge b]_\theta \subseteq [\ a\ ]_\theta \wedge [\ b\ ]_\theta$ for all $a, b \in A$. If θ is a congrurence on A and X and Y are non-empty subsets of A, then it can be easily seen that

$$\theta^-(X) \vee \theta^-(Y) \subseteq \theta^-(X \vee Y) \text{ and } \theta^-(X) \wedge \theta^-(Y) \subseteq \theta^-(X \wedge Y).$$

The reverse inequalities may not be hold good even in the case of lattices (see example 1 in [14]). However, we prove the following.

Lemma 2.1 *Let θ be a congruence on A and X and Y be nonempty subsets of A. If θ join and meet-complete, then:*

(1) $\theta^-(X) \vee \theta^-(Y) = \theta^-(X \vee Y)$ *and* $\theta_-(X) \vee \theta_-(Y) \subseteq \theta_-(X \vee Y)$*;*
(2) $\theta^-(X) \wedge \theta^-(Y) = \theta^-(X \wedge Y)$ *and* $\theta_-(X) \wedge \theta_-(Y) \subseteq \theta_-(X \wedge Y)$*.*

Proof: (1) Let $x \in \theta^-(X \vee Y)$, it means that there exists $a \in X \vee Y$ such that $(a,\ x) \in \theta$. Suppose $a = p \vee q$ where $p \in X$, $q \in Y$, then $x \in [\ p \vee q\ ]_\theta$ so that $x = e \vee f$ where $e \in [\ p\ ]_\theta$ and $f \in [\ q\ ]_\theta$. So, $p \in [\ e\ ]_\theta \cap X$ and $q \in [\ f\ ]_\theta \cap Y$ which implies $x \in \theta^-(X) \vee \theta^-(Y)$. Thus $\theta^-(X \vee Y) \subseteq \theta^-(X) \vee \theta^-(Y)$. Similarly, $\theta_-(X) \vee \theta_-(Y) \subseteq \theta_-(X \vee Y)$

(2) Its proof is similar to that of (1). □

The composition $\theta \circ \phi$ of any two congruences $\theta,\ \phi$ on A, defined by

$$\theta \circ \phi = \ \big\{(a,b) \in A \times A : (a,c) \in \phi \ and\ (c,d) \in \theta \ \ for some\ c \in A\big\}.$$

It can be easily verified that $\theta \circ \phi$ is a congruence on A if and only if $\theta \circ \phi = \phi \circ \theta$.

Theorem 2.2 *Let θ and ϕ be congruences on A with $\theta \circ \phi = \phi \circ \theta$. If X is a nonempty subset of A, then:*

(1) $\theta^-(X) \vee \phi^-(X) \subseteq (\theta \circ \phi)^-(X)$*;*
(2) $\theta^-(X) \wedge \phi^-(X) \subseteq (\theta \circ \phi)^-(X)$*.*

Proof: (1) Let $a \in \theta^-(X) \vee \phi^-(X)$. Then $a = b \vee c$ for some $b \in \theta^-(X)$ and $c \in \phi^-(X)$. It follows that there exists $x \in X$ and $y \in Y$ such that $(b,x) \in \theta$ and $(c,y) \in \phi$ so that $(x \vee y, b \vee y) \in \theta$ and $(b \vee y, b \vee c) \in \phi$ and hence $(x \vee y, b \vee c) \in \theta \circ \phi$ which implies $x \vee y \in [b \vee c]_{\theta\circ\phi} \cap X$ and hence $a \in (\theta \circ \phi)^-(X)$.

(2) Proof is similar to (1). □

Note that the converse of above do not hold even in the case of general lattices (see example (1) of [14])

Theorem 2.3 *Let θ and ϕ be congruences on A with $\theta \circ \phi = \phi \circ \theta$. If X is a nonempty subset of A, then $\theta_-\big(\phi_-(X)\big) = \phi_-\big(\theta^-(X)\big)$.*

Proof: Let $x \in \theta_-\big(\phi_-(X)\big)$ and $x \notin \phi_-\big(\theta_-(X)\big)$.Then $(x,y) \in \phi$ for some $y \in A$ and $y \notin \theta_-(X)$. It follows that $(y,z) \in \theta$ and $z \notin X$. Now $(x,z) \in \theta \circ \phi$ so $(x,a) \in \theta$ and $(a,z) \in \phi$ for some $a \in A$. As $a \in \phi_-(X)$, we have $z \in X$; which is absurd. So $x \in \phi_-\big(\theta_-(X)\big)$. Therefore, $\theta_-\big(\phi_-(X)\big) \subseteq \phi_-\big(\theta_-(X)\big)$. Other inclusion is similar. □

Theorem 2.4 *Let θ and ϕ be congruences on A. If X is a nonempty subset of A, then the following are equivalent :*

(1) $\theta \circ \phi = \phi \circ \theta$*;*
(2) $\theta^-\big(\phi^-(X)\big) = \phi^-\big(\theta^-(X)\big)$*.*

Proof: (1) $\Rightarrow$ (2). Let $x \in \theta^{-}\big(\phi^{-}(X)\big)$. Then there exists $y, z \in A$ such that $(x, y) \in \theta$ and $(y, z) \in \phi$ so $(x, z) \in \phi \circ \theta = \theta \circ \phi$. It follows that $(x, a) \in \phi$ and $(a, z) \in \theta$ for some $a \in A$. This shows $x \in \phi^{-}\big(\theta^{-}(X)\big)$. Therefore $\theta^{-}\big(\phi^{-}(X)\big) \subseteq \phi^{-}\big(\theta^{-}(X)\big)$. Reverse inclusion is similar.

(2) $\Rightarrow$ (1). Let$(x, a) \in \phi$ and $(a, y) \in \theta$. Then $y \in \theta^{-}\big(\phi^{-}(\{x\})\big) = \phi^{-}\big(\theta^{-}(\{x\})\big)$ which implies $[\ y\ ]_{\phi} \cap \theta^{-}\big(\{x\}\big)$ is nonempty. It follows that, there exists $d \in A$ such that $(x, d) \in \theta$ and $(d, y) \in \phi$ and hence $(x, y) \in \phi \circ \theta$. Therefore $\theta \circ \phi \subseteq \phi \circ \theta$. Similarly, $\phi \circ \theta \subseteq \theta \circ \phi$. □

3. Rough Ideals in an ADL

Let θ be a congruence relation on an ADL A. Then a nonempty subset X of A is called a lower (upper) rough ideal of A if $\theta_{-}(X)$ $\big(\theta^{-}(X)\big)$ is an ideal of A. X is said to be a rough ideal of A if it is both upper and lower rough ideal of A.

Lemma 3.1 *Let θ be a congrurence on A and X a nonempty subset of A.*

(1) *If $\theta_{-}(X)$ is an ideal of A, then $[\ 0\ ]_{\theta} \subseteq X$;*
(2) *If $\theta^{-}(X)$ is an ideal of A, then $[\ 0\ ]_{\theta} \cap X \neq \emptyset$*

Proof: (1) and (2) followes by the fact that every ideal of A contains 0 □

Lemma 3.2 *Let θ be a congrurence on A and I be an ideal of A.*

(1) *If $[\ 0\ ]_{\theta} \cap I = [\ 0\ ]_{\theta}$, then $\theta_{-}(I) = \emptyset$;*
(2) *If $[\ 0\ ]_{\theta} \subseteq I$, then $\theta_{-}(I) = I$.*

Proof: (1) Suppose that $[\ x\ ]_{\theta} \subseteq I$ and $y \in [\ 0\ ]_{\theta}$, then $(x, x \vee y) \in \theta$ so that $x \vee y \in I$ and hence $y \in I$. This shows that $[\ 0\ ]_{\theta} \cap I = [\ 0\ ]_{\theta}$; a contradiction. Thus $\theta_{-}(I) = \emptyset$.

(2) By theorem 2.1(1), $\theta_{-}(I) \subseteq I$. Let $x \in I$ and $y \in [\ x\ ]_{\theta}$. Then $(x, y) \in \theta$; that is $(x \vee 0, y \vee y) \in \theta$, so that $(0, y) \in \theta$ and hence $y \in I$. This shows that $I \subseteq \theta_{-}(I)$. □

Theorem 3.1 *Let θ be a congrurence on A.*

(1) *If I is an ideal of A, then I is an upper rough ideal.*
(2) *If θ is a join-complete and I is an ideal of A, then I is a lower rough ideal provided $\theta_{-}(I)$ is nonempty.*

Proof: (1) Clearly $0 \in \theta^{-}(I)$. Let $a, b \in \theta^{-}(I)$ then $x \in [\ a\ ]_{\theta}$ and $y \in [\ b\ ]_{\theta}$ for some $x, y \in I$. Now $x \vee y \in I$ and $x \vee y \in [\ a \vee b\ ]_{\theta}$. This follows $[\ a \vee b\ ]_{\theta} \cap I \neq \emptyset$ and hence $a \vee b \in \theta^{-}(I)$. Also, we have $a \wedge x \in I$, it follows that $[\ a \wedge b\ ]_{\theta} \cap I \neq \emptyset$ so that $a \wedge x \in \theta^{-}(I)$. Thus $\theta^{-}(I)$ is an ideal of A.

(2) Let $a, b \in \theta^{-}(I)$, then $a \vee b \in \theta^{-}(I)$. For any $x \in A$, $a = a \vee (a \wedge x)$ it follows that $[\ a\ ]_{\theta} \vee [\ a \wedge x\ ]_{\theta} \subseteq I$. If $y \in [\ a \wedge x\ ]_{\theta}$, then $a \vee y \in I$ and hence $y \in I$. So, $[\ a \wedge x\ ]_{\theta} \subseteq I$ and hence $a \wedge x \in \theta_{-}(I)$.Thus $\theta_{-}(I)$ is an ideal of A. □

Theorem 3.2 *Let θ and ϕ be congruences on A. If I is an ideal of A, then*

$$(\theta \cap \phi)^{-}(I) = \theta^{-}(I) \cap \phi^{-}(I).$$

Proof: Let $x \in \theta^{-}(I) \cap \phi^{-}(I)$ then there exist $a, b \in I$ such that $(a, x) \in \theta$ and $(b, x) \in \phi$. Now $a \vee b \in I$ and $(a \vee b) \wedge x \in I$. As $(x \vee b) \wedge x = (b \vee x) \wedge x = x$ and $(a \vee x) \wedge x = x$, we have $\big((a \vee b) \wedge x, x\big) \in \theta \cap \emptyset$. and hence $[\ x\ ]_{\theta\ \cap\ \phi} \cap I \neq \emptyset$ and so $x \in (\theta \cap \phi)^{-}(I)$. Thus $\theta^{-}(I) \cap \phi^{-}(I) \subseteq (\theta \cap \phi)^{-}(I)$. Reverse inclusion holds in theorem 2.1 (10). □

The proof of the following is straight forward, so we omit the details.

Theorem 3.3 *Let θ be congruence on A. If I and J are ideal of A, then:*

(1) $\theta^-(I \cap J) = \theta^-(I) \cap \theta^-(J)$;
(2) $\theta^-(I \wedge J) = \theta^-(I) \wedge \theta^-(J)$.

Theorem 3.4 *Let θ be a congruence on A and $\theta\_\mathcal{I}(A)$ denotes the set of all lower rough ideals of A. Then:*

(1) *$\theta\_\mathcal{I}(A)$ is closed under arbitrary intersection;*
(2) *$\big(\theta\_\mathcal{I}(A), \subseteq\big)$ is a complete lattice in which for any $\{X_\alpha : \alpha \in \triangle\} \subseteq \theta\_\mathcal{I}(A)$,*

$$glb\ \{X_\alpha : \alpha \in \triangle\} = \bigcap_{\alpha\in\triangle} X_\alpha;$$

$$lub\ \{X_\alpha : \alpha \in \triangle\} = \bigcap\{X \in \theta\_\mathcal{I}(A) : \bigcup_{\alpha\in\triangle} X_\alpha \subseteq X\}.$$

Proof: It follows by the facts that $\theta_-\big(\{0\}\big) = \{0\}$, $\theta_-(A) = A$ and every lower rough ideal contains 0. □

Recall (from [13]) that an ideal P of an ADL $(A, \wedge, \vee, 0)$ is called a prime ideal of A if $P \neq A$ and for any $a, b \in A$, $a \wedge b \in P$ implies $a \in P$ or $b \in P$. Now we shall introduce the notation of rough prime ideals in ADL's.

Definition 3.1 *Let θ be a congruence relation on an ADL A. Then a subset X of A is said to be a lower (upper) ruogh prime ideal of A if $\theta_-(X)$ $\big(\theta^-(X)\big)$ is a prime ideal of A.*

Proposition 3.1 *Let θ be a meet-complete congruence on A and P be a prime ideal of A such that $\theta^-(P) \neq A$. Then P is a upper rough prime ideal of A.*

Proof: Suppose $a \notin \theta^-(P)$ and $b \notin \theta^-(P)$. Then, it follows, $a \notin P$ and $b \notin P$ and hence $a \wedge b \notin P$. If $x \in [\ a \wedge b\ ]_\theta \cap P$, then $x = t \wedge s$ for some $t \in [\ a\ ]_\theta$ and $s \in [\ b\ ]_\theta$. Also, $t \in P$ or $s \in P$. It follows that $[\ a\ ]_\theta \cap P \neq \emptyset$ or $[\ b\ ]_\theta \cap P \neq \emptyset$; a contradiction to our supposition. Thus $a \wedge b \notin \theta^-(P)$. □

Proposition 3.2 *Let θ be a congruence on A and P be a prime ideal of A such that $\theta_-(P) \neq \emptyset$. Then P is a lower rough prime ideal of A .*

Proof: Suppose $a \notin \theta_-(P)$ and $b \notin \theta_-(P)$, it means that $[a]_\theta \nsubseteq P$ and $[b]_\theta \nsubseteq P$. So there exist $x \in [\ a\ ]_\theta$ and $y \in [\ b\ ]_\theta$ such that $x \notin P$ and $y \notin P$. Now $x \wedge y \notin P$ and $x \wedge y \in [\ a \wedge b\ ]_\theta$. This implies $a \wedge b \notin \theta_-(P)$. □

4. Problem of homomorphism

By a homomorphism of an ADL $(A, \wedge, \vee, 0)$ into an ADL $(B, \wedge, \vee, 0)$, we mean a mapping $f : A \longrightarrow B$ satisfying: $f(a \vee b) = f(a) \vee f(b)$, $f(a \wedge b) = f(a) \wedge f(b)$ and $f(0) = 0$, for all $a, b \in A$. If $f : A \longrightarrow B$ is a homomorphism of ADL's, then it can be easily verified that

$$kerf = \{(a, b) \in A \times A : f(a) = f(b)\}$$

is a congruence notation on A. Now, the following is a straight forward verification, so we omit the details.

Theorem 4.1 *Let f be a homomorphism of an ADL A onto an ADL B, and let $\theta = kerf$ and $X \subseteq A$. Then :*

(1) $f\big(\theta^-(X)\big) = f(X)$;
(2) *If f is one-one, then $f\big(\theta_-(X)\big) = f(X)$.*

In the following we disscuss certain properties of images and inverse images of lower and upper approximations of a set under a homomorphism.

Theorem 4.2 *Let f be a homomorphism of an ADL A onto an ADL B. If θ_A be a congruence relation on A and $\theta_B = \{(a,b) \in B \times B : \big(f^{-1}(a), f^{-1}(b)\big) \in \theta_A\}$. Then for any subset X of B,*

(1) *θ_B is a congruence relation on B*
(2) *$\theta_A^-\big(f^{-1}(X)\big) = f^{-1}\big(\theta_B^-(X)\big)$;*
(3) *$f^{-1}\big(\theta_{B-}(X)\big) \subseteq \theta_{A-}\big(f^{-1}(X)\big)$; furthermore, if f is one-one, then*
(4) *$\theta_{A-}\big(f^{-1}(X)\big) = f^{-1}\big(\theta_{B-}(X)\big)$.*

Proof: (1) It is clear.

(2)
$$\begin{aligned} x \in \theta_A^-\big(f^{-1}(X)\big) &\iff [\,x\,]_{\theta_A} \cap f^{-1}(X) \neq \emptyset \\ &\iff f\big([\,x\,]_{\theta_A}\big) \cap X \neq \emptyset \\ &\iff \big[f(x)\big]_{\theta_B} \cap X \neq \emptyset \\ &\iff x \in f^{-1}\big(\theta_B^-(X)\big) \end{aligned}$$

(3)
$$\begin{aligned} x \in f^{-1}\big(\theta_{B-}(X)\big) &\Rightarrow f(x) \in \theta_{B-}(X) \\ &\Rightarrow \big[f(x)\big]_{\theta_B} \subseteq X \\ &\Rightarrow f\big([\,x\,]_{\theta_A}\big) \subseteq X \\ &\Rightarrow [\,x\,]_{\theta_A} \subseteq f^{-1}(X) \\ &\Rightarrow x \in \theta_{A-}\big(f^{-1}(X)\big). \end{aligned}$$

On the other hand,
$$\begin{aligned} x \in \theta_{A-}\big(f^{-1}(X)\big) &\Rightarrow [\,x\,]_{\theta_A} \subseteq f^{-1}(X) \\ &\Rightarrow f\big([\,x\,]_{\theta_A}\big) \subseteq X \quad \text{(since f is one-to-one)} \\ &\Rightarrow \big[f(x)\big]_{\theta_B} \subseteq X \\ &\Rightarrow f(x) \in \theta_{B-}(X) \\ &\Rightarrow x \in f^{-1}\big(\theta_{B-}(X)\big) \end{aligned}$$
□

It is easy to prove that, if θ_A is both meet and join complete congruence on an ADL A and f is a homomorphism of A onto an ADL B, then the congruence θ_B defined above is both meet and join complete. Now we have the following.

Theorem 4.3 *Let f be a homomophism of an ADL A onto an ADL B. Let θ_A be meet and join complete congruence on A. Then, for any subset X of A, we have the following:*

(1) *$f\big(\theta_A^-(X)\big) = \theta_B^-\big(f(X)\big)$;*
(2) *$f\big(\theta_{A-}(X)\big) \subseteq \theta_{B-}\big(f(X)\big)$; further more, if f is one-one, then $f\big(\theta_{A-}(X)\big) = \theta_{B-}\big(f(X)\big)$.*

Proof: (1) Let $a \in f\big(\theta_A^-(X)\big)$, then there exist $x \in \theta_A^-(X)$ such that $f(x) = a$, so $[\,x\,]_{\theta_A} \cap X \neq \emptyset$ and hence there exists $y \in [\,x\,]_{\theta_A} \cap X$, $f(y) \in f(X)$ and $\big(f(y), f(x)\big) \in \theta_B$. Hence $f(y) \in \big[f(x)\big]_{\theta_B} \cap f(X)$ which implies $a \in \theta_B^-\big(f(X)\big)$. Thus $f\big(\theta_A^-(X)\big) \subseteq \theta_B^-\big(f(X)\big)$.
On the other hand, let $a \in \theta_B^-\big(f(X)\big)$. Then, there exists $x \in X$ such that $f(x) \in [\,a\,]\theta_B$.
Let $y \in A$ such that $f(y) = a$. Then $\big(f(x), f(y)\big) \in \theta_B$ so that $(x, y) \in \theta_A$. This implies $y \in \theta_A^-\big(X\big)$ and hence $a \in f\big(\theta_A^-(X)\big)$. Thus $\theta_B^-\big(f(X)\big) \subseteq f\big(\theta_A^-(X)\big)$.

(2) Proof is similar to (1) and theorem 4.2 (3). □

Theorem 4.4 *Let f be a homomorphism of an ADL A onto an ADL B. Let θ_A be a congruence on A and $\theta_B = \{(a,b) \in B \times B : \big(f^{-1}(a), f^{-1}(b)\big) \in \theta_A\}$. Then, for any subset X of B,*

(1) *$\theta_B^-(X)$ is an ideal of B if and only if $\theta_A^-\big(f^{-1}(X)\big)$ is an ideal of A;*
(2) *$\theta_B^-(X)$ is a prime ideal of B if and only if $\theta_A^-\big(f^{-1}(X)\big)$ is a prime ideal of A.*

Proof: (1) $x, y \in \theta_A^-\big(f^{-1}(X)\big)$ implies $x, y \in f^{-1}\big(\theta_B^-(X)\big)$ (by theorem 4.2 (2)) so $f(x), f(y) \in \theta_B^-(X)$. Since $\theta_B^-(X)$ is an ideal of B and f is homomorphism,
we get that $x \vee y$ and $x \wedge a \in f^{-1}\big(\theta_B^-(X)\big) = \theta_A^-\big(f^{-1}(X)\big)$ for any $a \in A$. Thus $\theta_A^-\big(f^{-1}(X)\big)$ is an ideal of A. Converse also follows theorem 4.2 (2).

(2) $a \wedge b \in \theta_A^-\big(f^{-1}(X)\big)$ implies $a \wedge b \in f^{-1}\big(\theta_B^-(X)\big)$ (by theorem 4.2 (2)) and hence $f(a) \wedge f(b) \in \theta_B^-(X)$. Since $\theta_B^-(X)$ is a prime ideal of B, $f(a) \in \theta_B^-(X)$ or $f(b) \in \theta_B^-(X)$. It follows that $a \in \theta_A^-\big(f^{-1}(X)\big)$ *or* $b \in \theta_A^-\big(f^{-1}(X)\big)$. Thus $\theta_A^-\big(f^{-1}(X)\big)$ is a prime ideal of A.

Conversely suppose that $a \wedge b \in \theta_B^-(X)$ and $f(x) = a, f(y) = b$. Then $x \wedge y \in f^{-1}\big(\theta_B^-(X)\big) = \theta_A^-\big(f^{-1}(X)\big)$. As $\theta_A^-\big(f^{-1}(X)\big)$ is a prime ideal of A, $a \in \theta_B^-(X)$ or $b \in \theta_B^-(X)$. Thus $\theta_B^-(X)$ is a prime ideal of B. □

The above theorem can be extended to the case of lower approximations, provided f is one-one.

5. Conclusion

This paper is intended to develop the theory of rough sets in Almost Distributive Lattices. Here, we have introduced the notion of rough ideals as a generalized notion of ideals of an ADL. It is known that, interchanging the operations $\wedge$ and $\vee$ in a lattice $\big(L, \wedge, \vee\big)$ yields a lattice again. However, in an ADL $\big(A, \wedge, \vee, 0\big)$, interchanging the operations $\wedge$ and $\vee$ does not yield an ADL again; for, the operation $\vee$ does not distributive over $\wedge$ from right. In this context, we can define the notion of rough filters and develop the theory of rough filters in ADL's separately. In future, we find rough sets induced by both crisp and fuzzy ideals (filters) of ADL's.

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References

1. Biswas R, Nanda S (1994) Rough groups and rough subgroups. Bull Pol Acad Sci Math 42: 125-254.
2. Davvaz B (2004) Roughness in rings. Inf Sci 164: 147-163.
3. Davvaz B (2006) Roughness based on fuzzy ideals. Inf Sci 176: 2417-2437.
4. Degang C, Wenxiu Z, Yeung D, Tsang ECC (2006) Rough approximations on a complete completely distributive lattice with applications to generalized rough sets. Inf Sci 176: 1829-1848.
5. Kazanci O, Davvaz B (2008) On the structure of rough prime (primary) ideals and rough fuzzy prime (primary) ideals in commutative rings. Inf Sci 178: 1343-1354.
6. Kuroki N (1997) Rough ideals in semigroups. Inf Sci 100: 139-63.
7. Mordeson N (2001) Rough set theory applied to (fuzzy) ideal theory. Fuzzy sets Syst 121: 315-324.
8. Natnael T.A, Srikanya G, Santhi Sndar Raj Ch (2021), L-fuzzy prime spectrum of ADL's. Advances in Fuzzy systems, Article ID 5520736.
9. Pawlak Z (1982) Rough sets. Int J Comput Inf Sci 11: 341-356.
10. Santhi Sundar Raj Ch, Natnael T.A., Swamy U. M (2018) Fuzzy Prime ideals of ADLs. International Journal of Computing Science and Applied Mathematics, 4(2): 32-36.
11. Srikanya G, Prakasam Babu G, Ramanuja Rao K, Santhi Sundar Raj Ch (2023), Fuzzy Initial and Final Segments in ADL's. International Journal of Analysis and Applications, 21:67.
12. Swamy U. M, Santhi Sundar Raj Ch, and Natnael Teshale A., Fuzzy Ideals of almost distributive lattices, Annals of Fuzzy Mathematics and Informatics, 14, No.4 (2017) 371-379.
13. Swamy U.M and Rao C G, Almost distributive lattice, J. Australian Math. Soc. (Ser. A) 31 (1981), 77-91.
14. Xiao Q, Li Q, Zhou X (2012) Rough ideals in lattices. Neural comput and Applia 21: S245-S253.
15. Xiao Q, Li Q, Guo L (2014) Rough sets induced by ideals in lattices. Information Science 271: 82-92.
16. Yang Y, Zhu K, Xin X (2020) Rough sets based on fuzzy ideals in distributive lattices. Open Math 18: 122-137
17. L. A. Zadeh (1965), Fuzzy sets. Inf Control 8: 338-353.

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