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Cofinitely δ_{ss}-Supplemented Modules

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ABSTRACT: This paper explores the class of cofinitely δ_{ss}-supplemented modules introduced as a natural extension of δ_{ss}-supplemented modules. The primary aim is to investigate structural and closure properties of this broader class. It has been verified that the collection of cofinitely δ_{ss}-supplemented modules retains the same property under both arbitrary sums and the construction of factor modules. Furthermore, a module P is characterized as amply cofinitely δ_{ss}-supplemented precisely when each maximal submodule A of P such that P/A is singular possesses ample δ_{ss}-supplements within P. Left δ_{ss}-perfect rings have been characterized via cofinitely δ_{ss}-supplemented modules and this characterization has been presented as equivalent conditions.

Key Words: (Ample) δ_{ss}-supplements, strongly δ-local modules, cofinite submodules, left δ_{ss}-perfect rings.

Contents

1. Introduction

In this text, we represent an associative ring with identity element by S and all unitary left S-modules by P. The expression $A \leq P$ indicates that A is a submodule of P. $A \leq P$ is named *cofinite*, if the factor module P/A is a finitely generated module (refer to [1]). $A \leq P$ is termed *small* in P, written as $A \ll P$, provided for each proper submodule B of P, the submodule $A + B$ does not equal to P. Dually, $A \leq P$ is named *essential* in P, written $A \trianglelefteq P$, provided $A \cap K \neq 0$ for each nonzero $K \leq P$. P is said to be *singular* in case $P \cong B/A$ for some module B and for its essential submodule A. $Soc(P)$ and $Rad(P)$ are the socle and the radical of a module P, respectively (see [9]). Zhou introduced the submodule $\delta(P) = \cap\{A \leq P \mid P/A$ is singular and simple$\}$ in [10]. Within the same study $A \leq P$ is named *δ-small* in P, indicated by $A \ll_\delta P$, provided for each proper submodule B of P satisfying that P/B is singular, the submodule $A + B$ does not equal to P. Each small submodule and non-singular semisimple submodule of P satisfies the δ-small condition. [10, Lemma 1.5] provides that $\delta(P)$ equals the sum of all δ-small submodules of P. In Zhou and Zhang's paper the submodule $Soc_s(P) = \sum\{A \ll P \mid A$ is simple$\}$ is defined (refer to [11]). Accordingly, Nişancı Türkmen and Türkmen proposed the notion of $Soc_\delta(P) = \sum\{A \ll_\delta P \mid A$ is simple$\}$ in [6].

A module P which does not equal to zero is termed *local* in case there exists a proper submodule of P which contains whole proper submodules of P. This notion naturally extends to rings: a ring S is called *local* whenever S is a local module as a module ${}_SS$ (see [9, 41.3]). The condition for a module to be local can also be characterized by the behavior of its radical, specifically, for a module P to be local is equivalent to its radical $Rad(P)$ being both maximal and a small submodule in P (see [9, 41.4]). An enhanced form of local module was introduced by Kaynar et al., who defined a module as *strongly local* when it satisfies the local condition and, in addition, it has a semisimple radical. If the module ${}_SS$ adheres to this stronger version, then the ring S is described *strongly local* (see [3]).

Büyükaşık and Lomp proposed a different generalization known as δ-locality. In this context, a module P is classified as *δ-local* when the submodule $\delta(P)$ is both δ-small and is maximal submodule in P (refer to [2]). Whereby, this notion was refined further: a module P is named *strongly δ-local* if it satisfies δ-locality and its submodule $\delta(P)$ is included in the socle of P, denoted by $\delta(P) \leq Soc(P)$ (refer to [6]). As demonstrated in [6, Lemma 2.2] this is equivalent to $\delta(P)$ being semisimple and being maximal submodule in P.

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Let P denote a module and suppose A is a submodule of P. A submodule B is named a *supplement* of A in P provided it is a minimal element (with respect to inclusion) among all submodules $K \leq P$ for which the equality $P = A + K$ holds. This is equivalent to the requirement that $P = A + B$ and the intersection $A \cap B$ is small in B, denoted by $A \cap B \ll B$. A module P is classified as *supplemented* provided each submodule has at least one supplement in P. Moreover, P is described as *amply supplemented* provided that for all submodules A and B of P with $P = A + B$, there exists a supplement of A included in B (refer to [9, Section 41] for additional information).

According to the definition provided in [3], a submodule B qualifies as an *ss-supplement* of A in a module P provided the equality $P = A + B$ holds and the inclusion $A \cap B \leq Soc_s(B)$ is satisfied. As established by [3, Lemma 3] this is equivalent to the condition that $P = A + B$ and $A \cap B$ is both semisimple and is a small submodule in B, that is $A \cap B \ll B$. An alternative formulation of the same condition is stated that $P = A + B$ and $A \cap B \leq Rad(B)$ and $A \cap B$ is semisimple. A module P is named *ss-supplemented* when each submodule within it possesses an *ss*-supplement. Furthermore, provided for all submodules A and B of P with $P = A + B$, an *ss*-supplement of A exists within B, then P is named *amply ss-supplemented*. In [5] the concept of *cofinitely ss-supplemented* modules is introduced as follows: a module P is a this type of module if each of its cofinite submodules possesses an *ss*-supplement. Moreover, P is named *amply cofinitely ss-supplemented* provided that, for any cofinite submodule A of P satisfying $P = A + B$ for some $B \leq P$, A possesses an *ss*-supplement included in B.

The notion of δ_{ss}-supplemented module, as detailed in [6], refers to a module P for which each submodule A possesses a δ_{ss}-supplement B in P, i.e. $P = A + B$ and $A \cap B \leq Soc_\delta(B)$. In this context, P is further called *amply δ_{ss}-supplemented* when for any equality $P = A + B$ with $B \leq P$, there exists a δ_{ss}-supplement of A that is entirely included in B. According to [6, Lemma 3.3], the condition that B is a δ_{ss}-supplement of A in P is an equivalent condition where $P = A + B$, $A \cap B$ is semisimple, and $A \cap B \ll_\delta B$. This is also equivalent to the alternative characterization in which $P = A + B$, $A \cap B \leq \delta(B)$ and $A \cap B$ is a semisimple submodule. Additionally, [6, Theorem 5.3] introduces the notion of *left δ_{ss}-perfect rings* as those rings S for which each left S-module satisfies (amply) δ_{ss}-supplemented property. δ_{ss}-supplemented R-modules are classified in the specialized form with the help of an ideal of the ring S in [7].

In this study, we broaden the concept of δ_{ss}-supplemented modules by introducing and examining the more inclusive class of cofinitely δ_{ss}-supplemented modules. We show that this property is preserved under taking both factor modules, and arbitrary sums: that is if a module is cofinitely δ_{ss}-supplemented, then so are its factor modules and arbitrary sums of cofinitely δ_{ss}-supplemented submodules of a module is again cofinitely δ_{ss}-supplemented. Since in finitely generated modules each submodule is cofinite, it immediately follows that for such a module P with a cofinite submodule $\delta(P)$, the factor module $P/\delta(P)$ is semisimple. Furthermore, we establish a characterization involving singular factors: a cofinite submodule A of P where P/A is singular, possesses a δ_{ss}-supplement in P if and only if $P/Loc_\delta(P)$ has not a maximal submodule $C/Loc_\delta(P)$ with P/C singular. Here $Loc_\delta(P)$ denotes the sum of all submodules of P that are strongly δ-local. Additionally, we prove that a ring S satisfies the condition that each left S-module is cofinitely δ_{ss}-supplemented if and only if each such module is expressible as a sum of all strongly δ-local submodules or all projective semisimple ones. Furthermore, we give a main characterization for left δ_{ss}-perfect rings in terms of cofinitely δ_{ss}-supplemented modules.

2. Cofinitely δ_{ss}-Supplemented Modules

Definition 2.1 *We call a module P cofinitely δ_{ss}-supplemented module in case for each cofinite submodule A of P, there exists a submodule B of P such that $P = A + B$, $A \cap B \ll_\delta B$ and $A \cap B$ is semisimple.*

We also call a module P amply cofinitely δ_{ss}-supplemented module in case each cofinite submodule A of P with $P = A + B$ for some $B \leq P$, A has a δ_{ss}-supplement in P contained in B.

Proposition 2.1 *Let P be a (an amply) cofinitely δ_{ss}-supplemented module. Then each homomorphic image of P is a (an amply) cofinitely δ_{ss}-supplemented module.*

Proof: Suppose that $h : P \to T$ is a homomorphism and A is a cofinite submodule of $h(P)$. Then $P/h^{-1}(A) \cong (P/Ker(h))/(h^{-1}(A)/Ker(h))$ such that $P/Ker(h) \cong h(P)$ and $h^{-1}(A)/Ker(h) \cong A$.

Therefore, $P/h^{-1}(A)$ is finitely generated. By the assumption, there exists a submodule B such that $P = h^{-1}(A) + B$, $h^{-1}(A) \cap B \ll_\delta B$ and $h^{-1}(A) \cap B$ is semisimple. Thus, $h(P) = h(h^{-1}(A)) + h(B)$ and $h(h^{-1}(A)) = A$ as A is a submodule of $h(P)$. It follows that $h(P) = A + h(B)$. Also, we infer from $h^{-1}(A) \cap B \ll_\delta B$ that $h(h^{-1}(A)) \cap h(B) \ll_\delta h(B)$ by [10, Lemma 1.3(2)]. This means that $A \cap h(B) \ll_\delta h(B)$. Moreover, as $h^{-1}(A) \cap B$ is semisimple, then $A \cap h(B)$ is semisimple by [4, Corollary 8.1.5]. Hence $h(P)$ is a cofinitely δ_{ss}-supplemented module.

By modifying this method, we can prove that if P is an amply cofinitely δ_{ss}-supplemented module, then $h(P)$ is an amply cofinitely δ_{ss}-supplemented module. □

Corollary 2.1 *If P is a (an amply) cofinitely δ_{ss}-supplemented module, then this property is inherited by all factor modules of P.*

Lemma 2.1 *Let P be a finitely generated module. Then P is a cofinitely δ_{ss}-supplemented module if and only if P is a δ_{ss}-supplemented module.*

Proof: The sufficiency is clear. To prove the necessity, suppose that A is any submodule of P. Since P is a finitely generated module, then each submodule of P is cofinite. Thus A has a δ_{ss}-supplement in P, by assumption. Consequently P is a δ_{ss}-supplemented module. □

Proposition 2.2 *Suppose that P is a cofinitely δ_{ss}-supplemented module such that $\delta(P)$ is a cofinite submodule of P. Then the factor module $P/\delta(P)$ is a semisimple module.*

Proof: By Corollary 2.1, $P/\delta(P)$ is a cofinitely δ_{ss}-supplemented module. By assumption and Lemma 2.1, $P/\delta(P)$ is a δ_{ss}-supplemented module. Thus $P/\delta(P)$ is a δ-supplemented module. Since $\delta(P/\delta(P)) = 0$, then $P/\delta(P)$ has no nonzero δ-small submodules. Hence $P/\delta(P)$ is a semisimple module. □

Lemma 2.2 *Assume A and B are submodules of a module P with A being cofinitely δ_{ss}-supplemented and B cofinite submodule of P. If $A + B$ possesses a δ_{ss}-supplement in P, then B also has a δ_{ss}-supplement in P.*

Proof: Let us consider W as a δ_{ss}-supplement of $A + B$ in P. Then we obtain

$$A/A \cap (W + B) \cong (A + W + B)/(W + B) = P/(W + B)$$

is a finitely generated module as P/B is finitely generated. By assumption, A has a submodule Y which is a δ_{ss}-supplement of the submodule $A \cap (W + B)$. Thus $P = A + B + W = (A \cap (W + B) + Y) + B + W = B + W + Y$ and $Y \cap (A \cap (W + B)) = Y \cap (W + B)$. Therefore we infer that $B \cap (W + Y) \leq W \cap (B + Y) + Y \cap (B + W) \leq W \cap (A + B) + Y \cap (B + W)$. Here we conclude that $B \cap (W + Y) \ll_\delta W + Y$ by [10, Lemma 1.3] and $B \cap (W + Y)$ is semisimple by [4, Corollary 8.1.5]. Hence $W + Y$ is a δ_{ss}-supplement of B in P. □

Proposition 2.3 *An arbitrary sum of cofinitely δ_{ss}-supplemented submodules of a module P so is.*

Proof: Assume that P_i is a family of cofinitely δ_{ss}-supplemented submodules of P for each $i \in I$ where I is any index set such that $T = \sum_{i \in I} P_i$. Let A be a cofinite submodule of T. Since T/A is finitely generated, any element $t + A$ of T/A has the form $t + A = s_1 t_1 + ... + s_n t_n + A$ where $\{t_1 + A, t_2 + A, ..., t_n + A\}$ is a generating set of P/A. Moreover, t is an element of P such that $t = k_{i_1} + k_{i_2} + ... + k_{i_{h(i)}}$, where k_{i_w} is an element of some P_{i_w} for each $i_w \in I$. Therefore, $t = s_1(k_{1_1} + ... + k_{1_{h(1)}}) + ... + s_n(k_{n_1} + ... + k_{n_{h(n)}}) + a$, where $a \in A$. Then $T = \sum_{j \in J} P_j + A$ for a finite set $J = \{1_1, ..., 1_{h_1}, 2_1, ..., n_{h(n)}\}$. Thus $T = \sum_{j \in J} P_j + A = P_{1_1} + \sum_{j \in J - \{1_1\}} P_j + A$. Here P_{1_1} is a cofinitely δ_{ss}-supplemented module and $P_{1_1} + \sum_{j \in J - \{1_1\}} P_j + A$ possesses 0 δ_{ss}-supplement. Since J is finite, this iterative method allows us to conclude from Lemma 2.2 that A has a δ_{ss}-supplement within T. □

A module P is *T-generated* provided that there is an epimorphism $g : T^{(I)} \to P$ for any index set I. Now we conclude the following result of Proposition 2.3 and Corollary 2.1.

Corollary 2.2 *If P is a cofinitely δ_{ss}-supplemented module, then each P-generated module inherits this property.*

Proposition 2.4 *Let P be a cofinitely δ_{ss}-supplemented module. Then each cofinite submodule of $P/\delta(P)$ is a direct summand.*

Proof: $P/\delta(P)$ has cofinite submodules formed by $A/\delta(P)$, where A is a cofinite submodule of P. Thus there exists a submodule B of P such that $P = A + B$, $A \cap B \ll_\delta B$ and $A \cap B$ is semisimple. Note that $A \cap B \leq \delta(P)$. Hence we infer $P/\delta(P) = (A/\delta(P)) \oplus (B + \delta(P)/\delta(P))$ meaning that $A/\delta(P)$ is a direct summand of $P/\delta(P)$. □

From now on, we shall use $Cof_{\delta_{ss}}(P)$ and $Loc_\delta(P)$ to indicate the sum of all cofinitely δ_{ss}-supplemented submodules of P and the sum of all strongly δ-local submodules of P, respectively. Observe from [6, Lemma 4.1] that strongly δ-local modules are cofinitely δ_{ss}-supplemented.

Theorem 2.1 *Let P be a module. Then the conditions stated below are all equivalent:*

1. *Each cofinite submodule A of P with singular P/A has a δ_{ss}-supplement in P.*
2. *Each maximal submodule A of P with singular P/A has a δ_{ss}-supplement in P.*
3. *$P/Loc_\delta(P)$ does not include a maximal submodule $C/Loc_\delta(P)$ with singular P/C.*
4. *$P/Cof_{\delta_{ss}}(P)$ does not include a maximal submodule $C/Cof_{\delta_{ss}}(P)$ with singular P/C.*

Proof: (1) $\Longrightarrow$ (2) Obvious.

(2) $\Longrightarrow$ (3) Assume that A is a maximal submodule of P with singular P/A. Then P has a submodule B such that $P = A+B$, $A\cap B \ll_\delta B$ and $A\cap B$ is semisimple. Note that $P/A = (A+B)/A \cong B/(A\cap B)$. So $A\cap B$ is a maximal submodule of B with singular $B/(A\cap B)$. Thus $A\cap B = \delta(B)$ and $\delta(B) \leq Soc(B)$. Therefore, B is a strongly δ-local submodule of P. Hence $B \leq Loc_\delta(P)$, and so $Loc_\delta(P)$ is not a submodule of A. Consequently, $P/Loc_\delta(P)$ does not include a maximal submodule as desired.

(3) $\Longrightarrow$ (4) Assume contrary that $P/Cof_{\delta_{ss}}(P)$ includes a maximal submodule $C/Cof_{\delta_{ss}}(P)$ with singular P/C. Consider the epimorphism $h : P/Loc_\delta(P) \to P/Cof_{\delta_{ss}}(P)$. Following this way, $h^{-1}(C/Cof_{\delta_{ss}}(P))$ is a maximal submodule of $P/Loc_\delta(P)$ with singular $(P/Loc_\delta(P))/(h^{-1}(C/Cof_{\delta_{ss}}(P)))$. This is a contradiction. So the claim holds.

(4) $\Longrightarrow$ (1) Assume that A is a cofinite submodule of P with singular P/A. Thus a finitely generated factor module $P/(A + Cof_{\delta_{ss}}(P)) = (P/A)/(A + Cof_{\delta_{ss}}(P)/A)$ is singular. Then by (4) $P = A + Cof_{\delta_{ss}}(P)$. Here since P/A is finitely generated, then $P = A + P_1 + P_2 + ... + P_k$ where P_i is a cofinitely δ_{ss}-supplemented submodule for each $k \in \mathbb{Z}^+$ $(1 \leq i \leq k)$. Thus A has a δ_{ss}-supplement in P from Lemma 2.2 and Proposition 2.3. □

In what follows we denote by $\mathcal{M}(A)$ the collection of maximal submodules W of a module P which includes the submodule A with singular P/W. For instance, $\mathcal{M}(P) = \emptyset$ and $\mathcal{M}(0)$ means that the collection of all maximal submodules W of P with singular P/W (this set could be also empty). Accordingly, let β denotes a relation defined by $A\beta B$ if and only if $\mathcal{M}(A) = \mathcal{M}(B)$ on the set of submodules of P. β is an equivalence relation on the collection of submodules of P.

Recall from [8] that for a module P the submodule $Soc_p(P) = \sum\{A \leq P \mid A \text{ is simple and projective}\}$ is defined and it is clearly observed that $Soc_p(P)$ is the largest projective semisimple submodule of P.

Theorem 2.2 *Let P be a module. Then the conditions stated below are all equivalent:*

1. *P is an amply cofinitely δ_{ss}-supplemented module.*
2. *Each submodule A of P has ample δ_{ss}-supplements in P with cyclic P/A.*
3. *Each maximal submodule A of P with singular P/A has ample δ_{ss}-supplements in P.*

4. *$A\beta(Loc_\delta(A) \cup Soc_p(A))$ for each submodule A of P.*

5. *$(Sp)\beta(Loc_\delta(Sp) \cup Soc_p(Sp))$ for each $p \in P - \delta(P)$.*

Proof: The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) are clear.

(3) $\Longrightarrow$ (1) Assume that A is a cofinite submodule of P. If $A = P$, then A has ample δ_{ss}-supplements in P. Thus we suppose that $A \neq P$. Let X be an intersection of all essential maximal submodules of P including the submodule A. This means that $X/A = \delta(P/A)$. Since P/A is finitely generated, $X/A \ll_\delta P/A$ by [10, Lemma 1.5]. Let M be any essential maximal submodule of P such that X is included in M. By assumption, there exists a δ_{ss}-supplement K of M in P, i.e. $P = M + K$, $M \cap K \ll_\delta K$ and $M \cap K$ is semisimple. Thus, we have $P/X = (M/X) \oplus ((K + X)/X)$, because $(M/X) \cap ((K + X)/X) \ll_\delta P/X$ and $(M/X) \cap ((K + X)/X) = 0$. Thus P/X is a finitely generated and semisimple module, and so X is a finite intersection of maximal essential submodules of P. Therefore X has ample δ_{ss}-supplements in P by Lemma 2.2. Now assuming that $P = A + B$ for some submodule B of P, we obtain that $P = X + B$. Then B has a submodule T such that $P = X + T$, $X \cap T \ll_\delta T$ and $X \cap T$ is semisimple. Here P/A is singular since $i^{-1}(X) = A$ is an essential submodule of P for the inclusion map $i : A \longrightarrow X$ by [9, 17.3(3)]. Thus we infer from $P/A = X/A + (T + A)/A$ that $P/A = (T + A)/A$, and so $P = T + A$. Since $A \cap T \leq X \cap T$, then $A \cap T \ll_\delta T$ and $A \cap T$ is semisimple by [10, Lemma 1.3] and [4, Corollary 8.1.5]

(3) $\Longrightarrow$ (4) Let A be a submodule of P and B be a maximal submodule of P such that B does not include A and P/B is singular. Then $P = A + B$. By (3), A has a submodule X such that $P = B + X$, $B \cap X \ll_\delta X$ and $B \cap X$ is semisimple. According to [6, Proposition 3.4] X is either a strongly δ-local or a projective semisimple module. Moreover, X is not a submodule of B. Thus B does not include $Loc_\delta(A) \cup Soc_p(A)$. Hence $A\beta(Loc_\delta(A) \cup Soc_p(A))$.

(4) $\Longrightarrow$ (5) Obvious.

(5) $\Longrightarrow$ (3) Assume that A is any maximal submodule of P with singular P/A and B is a submodule of P such that $P = A + B$. Then B has an element b such that A does not include b. Thus Sb is not included in A. Therefore, $Loc_\delta(Sb) \cup Soc_p(Sb)$ is not included in A as $(Sb)\beta(Loc_\delta(Sb) \cup Soc_p(Sb))$. Suppose that X is a strongly δ-local submodule of Sb and so of B such that X is not included in A. Therefore, $P = A + X$, $A \cap X \ll_\delta X$ and $A \cap X$ is semisimple, implying that X is a δ_{ss}-supplement of A in P. If X is a projective semisimple submodule of Sb and so of B, then X is not included in A. This leads us once more to the conclusion that X is a δ_{ss}-supplement of A in P. As a result, A possesses ample δ_{ss}-supplements in P. □

Corollary 2.3 *Let P be a module such that for all submodules A of P $A\beta(Loc_\delta(A) \cup Soc_p(A))$. Then each maximal submodule A of P with singular P/A has ample δ_{ss}-supplements in P.*

Lemma 2.3 *Assume that $P = P_1 + P_2$ is a module where each of the submodules P_1, P_2 possesses ample δ_{ss}-supplements in P. Then the intersection $P_1 \cap P_2$ possesses ample δ_{ss}-supplements in P.*

Proof: Assuming that $P = (P_1 \cap P_2) + A$ for any submodule A of P, then we deduce $P = P_1 + P_2 = P_1 + (P_2 \cap P) = P_1 + (P_2 \cap ((P_1 \cap P_2) + A)) = P_1 + (P_1 \cap P_2) + (P_2 \cap A) = P_1 + (P_2 \cap A)$, and with similar arguments we also deduce $P = P_2 + (P_1 \cap A)$. Thus, by the assumption, there exist a δ_{ss}-supplement B_1 of P_1 in P with $B_1 \leq P_2 \cap A$ and a δ_{ss}-supplement B_2 of P_2 in P with $B_2 \leq P_1 \cap A$. Therefore, we conclude that $P = (B_2 + B_1) + (P_1 \cap P_2)$ and $(B_2 + B_1) \cap (P_1 \cap P_2) \leq B_2 \cap (P_1 \cap P_2) + B_1 \cap (P_1 \cap P_2) \leq (B_2 \cap P_2) + (B_1 \cap P_1)$. Thus $(B_2 + B_1) \cap (P_1 \cap P_2) \ll_\delta B_2 + B_1$ by [10, Lemma 1.3]. Moreover, since $B_2 \cap P_2$ and $B_1 \cap P_1$ are semisimple modules, then $(B_2 + B_1) \cap (P_1 \cap P_2)$ is semisimple by [4, Corollary 8.1.5]. Hence $P_1 \cap P_2$ has a δ_{ss}-supplement $B_2 + B_1$ that is included in A in P. □

A ring S is defined *left max* when each nonzero left S-module possesses at least one maximal submodule. A module P is called as *coatomic* when each submodule that is not equal to P itself is included in some maximal submodules of P (see [12]).

Lemma 2.4 *Suppose that S is a ring. Each left S-module is cofinitely δ_{ss}-supplemented if and only if each left S-module is the sum of all strongly δ-local submodules or all projective semisimple submodules.*

Proof: ($\Longrightarrow$) By the assumption, the left S-module $_SS$ is cofinitely δ_{ss}-supplemented, and so by Lemma 2.1 $_SS$ is δ_{ss}-supplemented. Based on [6, Theorem 5.3], we conclude that S is a left δ_{ss}-perfect ring. [6, Proposition 5.5] implies that S is a left max ring. Notably, each left S-module is coatomic, so [6, Proposition 4.10] provides that each such module is the sum of its strongly δ-local submodules or its projective semisimple submodules.

($\Longleftarrow$) For any left S-module P, the assumption together with [6, Proposition 4.10] ensures that P is coatomic, and each cofinite submodule of P possesses a δ_{ss}-supplement in P. □

Theorem 2.3 *Consider a ring S. Then the conditions stated below are all equivalent:*

1. *$_SS$ is an (amply) cofinitely δ_{ss}-supplemented module.*
2. *S is a left δ_{ss}-perfect ring.*
3. *Each projective left S-module is (amply) cofinitely δ_{ss}-supplemented.*
4. *Each left S-module is (amply) cofinitely δ_{ss}-supplemented.*
5. *Each left S-module is the sum of all strongly δ-local submodules or all projective semisimple submodules.*
6. *$_SS$ is a finite sum of all strongly δ-local submodules or all projective semisimple submodules.*
7. *Each maximal left ideal J of S with singular S/J has ample δ_{ss}-supplements in S.*

Proof: (1) $\Longrightarrow$ (2) Since $_SS$ is finitely generated amply cofinitely δ_{ss}-supplemented module, then by Lemma 2.1 we conclude that $_SS$ is a δ_{ss}-supplemented module. Thus S is a left δ_{ss}-perfect ring according to [6, Theorem 5.3].

(2) $\Longrightarrow$ (3) By [6, Theorem 5.3].

(3) $\Longrightarrow$ (4) By Proposition 2.1 and [9, 18.6].

(4) $\Longrightarrow$ (5) By Lemma 2.4.

(5) $\Longrightarrow$ (6) Obvious.

(6) $\Longrightarrow$ (7) By [6, Corollary 4.11].

(7) $\Longrightarrow$ (1) By Theorem 2.2. □

Lemma 2.5 *Let P be a module, A_i be a strongly δ-local submodule or projective semisimple submodule of P for each $i = 1, 2, ..., m$ and B be a submodule of P such that $B + A_1 + ... + A_m$ has a δ_{ss}-supplement X in P. Then there is a subset I of $\{1, 2, ..., m\}$ (may possible be empty) such that $X + \sum_{i\in I} A_i$ is a δ_{ss}-supplement of B in P.*

Proof: Let $m = 1$. Then for the submodule $W = (B + X) \cap A_1$ of A_1, if $W = A_1$, then 0 is a δ_{ss}-supplement of W in A_1 and we obtain that $X = X + 0$ is a δ_{ss}-supplement of B in P by the proof of Lemma 2.2. When $W \neq A_1$, A_1 is a δ_{ss}-supplement of W in A_1, and so $X + A_1$ is a δ_{ss}-supplement of B in P by using again the proof of Lemma 2.2. Hence the proof of the case $m = 1$ is completed. Let $m > 1$. By induction on m, we reach at the conclusion that there is a subset J of $\{2, ..., m\}$ such that $X + \sum_{j\in J} A_j$ is a δ_{ss}-supplement of $B + A_1$ in P. Hence we conclude from the case $m = 1$ that either $X + \sum_{j\in J} A_j$ or $X + A_1 + \sum_{j\in J} A_j$ is a δ_{ss}-supplement of B in P. □

Theorem 2.4 *For any ring S and for any S-module P, the conditions stated below are all equivalent:*

1. *P is an amply cofinitely δ_{ss}-supplemented module.*
2. *Each maximal submodule A of P with singular P/A has ample δ_{ss}-supplements in P.*
3. *Given any submodule A and any cofinite submodule B of P satisfying $P = A + B$, there exist either strongly δ-local or projective semisimple submodules $A_1, ..., A_m \leq A$ such that $P = B + A_1 + ... + A_m$ for each $m \in \mathbb{Z}^+$.*

4. $\mathcal{M}(A) = \mathcal{M}(Loc_\delta(A) \cup Soc_p(A))$ *for each submodule* A *of* P.

5. $\mathcal{M}(Sp) = \mathcal{M}(Loc_\delta(Sp) \cup Soc_p(Sp))$ *for any element* p *of* $P - \delta(P)$.

Proof: The implications (1) $\Longrightarrow$ (2) and (4) $\Longrightarrow$ (5) are obvious.

(3) $\Longrightarrow$ (1) By Lemma 2.5.

(2) $\Longrightarrow$ (4) Let A be any submodule of P and B be a maximal submodule of P which does not include A with singular P/B. Then $P = A + B$. By the assumption, A has a submodule X such that X is a δ_{ss}-supplement of B in P. By [6, Proposition 3.4] X is a strongly δ-local or a projective semisimple module. This implies that $Loc_\delta(A) \cup Soc_p(A)$ is not a submodule of B and (4) holds.

(2) $\Longrightarrow$ (3) Assume that P has a cofinite submodule B such that $P = B + X$ for some $X \leq P$ and $P \neq B + Y$ for each submodule Y of X where Y is a finite sum of strongly δ-local or projective semisimple submodules. By Γ, we signify the collection of submodules C of P such that $B \leq C$ and $P \neq C + Y$ for each submodule Y of X where Y is a finite sum of strongly δ-local or projective semisimple submodules. Using Zorn's Lemma, Γ includes a maximal element M. Since M is a cofinite submodule of P and $P \neq M$, then P has a maximal submodule D such that $M \leq D$. Obviously, it implies that $P = D + X$. Here by the assumption, X has a submodule X' such that X' is a δ_{ss}-supplement of D in P. That is $P = X' + D$ and $X' \cap D \ll_\delta X'$ with semisimple $X' \cap D$. By [6, Proposition 3.4] X' is a strongly δ-local or a projective semisimple submodule of P. X' is obviously not a submodule of D, and also of M, i.e. $M \neq M + X'$. Since M is a maximal element of Γ, there exists a submodule W of X such that $P = (M + X') + W$ and W is a finite sum of strongly δ-local or projective semisimple submodules. However, $X' + W$ is a finite sum of strongly δ-local or projective semisimple submodules and a submodule of X. So that, $P = M + (X' + W)$. This is a contradiction. Thus (3) holds.

(5) $\Longrightarrow$ (2) Suppose that A is a maximal submodule of P with singular P/A and B is a submodule of P such that $P = A + B$. Then there exists $b \in B$ such that $b \notin A$, and so $P = A + Sb$. Note that $b \in P - \delta(P)$. Thus since $b \in P - \delta(P)$, by the hypothesis $A \notin \mathcal{M}(Sb) = \mathcal{M}(Loc_\delta(Sb) \cup Soc_p(Sb))$. By [6, Proposition 3.4] Sb has a submodule X which is a strongly δ-local or projective semisimple module such that X is not a submodule of A. Therefore, $P = A + X$, $A \cap X \ll_\delta X$ with semisimple $A \cap X$. Then X is a δ_{ss}-supplement of A in P. This provides (2). □

Before concluding the text, let us show with the next example that the modules defined in this paper is a proper generalization of cofinitely ss-supplemented modules.

Example 2.1 *(See [6, Example 4.4.(1)]) Consider the non-noetherian commutative ring* $R = \prod_{i\geq 1} \mathbb{Z}_2$ *and the subring* $S = \langle \bigoplus_{i\geq 1} \mathbb{Z}_2, 1_R \rangle$ *of* R. *Let* $P = {}_S S$. *Then* P *is a (an amply) cofinitely* δ_{ss}*-supplemented module but not a (an amply) cofinitely* ss*-supplemented module.*

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