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Geometric Interpretation and Some Analytical Properties of First Type S-Convex Function *

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This paper is dedicated to Professor Wono Setya Budhi for the occasion of his seventieth birthday.

Abstract: There are at least two types of s-convex functions. This article will only discuss the properties related to the first type of s-convex function, consist of the geometric meaning, monotonicity, continuity, inclusion properties, and Jensen's inequality. The novelties of this research are the geometric meaning of first type s-convex function, the continuity at 0, inclusion properties of two s-convex function classes on an interval, and necessary and sufficient condition for Jensen's type equality. All the properties will be proved analitically.

Key Words: s-convex functions, geometric interpretation, analytical properties.

Contents

1. Introduction

The s-convex function is a generalization of the classical convex function. There are at least two known types of s-convex function: the first and the second type. Hudzik *et al.* [1] and Pinheiro [2,3] examined the fundamental properties of these types, and provide examples of functions that satisfy both or that do not satisfy one of them. Ole [4] provided a geometric meaning and some properties of second type s-convex function. One of the well-known inequalities related to convex functions is Jensen's inequality. Several articles that provide further studies on Jensen's inequality for convex or strongly convex functions include [5–11]. Meanwhile, several researchers examined the validity of Jensen's inequality related to the second type of s-convex function, including [12]. Pinheiro [13] proved that the first type s-convex function satisfied Jensen's inequality, although the proof is not provided in detail, we will provide it at Section 4. In addition, we provide the condition so that the equality holds.

Moreover, we provide the monotonicity and continuity of the first type s-convex function at Section 2, the novelty is that if a first type s-convex function defined on an interval containing a neighborhood of zero, then it is continuous at zero.

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We also examine the inclusion properties between two first type s-convex function classes at Section 3. Our analysis is inspired by the study of function classes with geometric and convexity-related properties, such as strongly close-to-convex functions, which have been explored in [14].

At this Section 1, we start to understand the geometric meaning of the first type s-convex function.

Definition 1.1 [1–3, 13] Suppose that I is an interval on $\mathbb{R}$, and $s \in (0,1]$. A function $f : I \to \mathbb{R}$ is called the first type s-convex function if for any $x, y \in I$ and $\alpha, \beta \in [0,1]$, which $a^s + b^s = 1$, satisfy

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y). \tag{1.1}$$

The set of all first type s-convex functions on I is denoted by $K_{1,s}(I)$.

Let $s = 1$, then a function $f \in K_{1,1}(I)$ if and only if f is a convex function on I. On the other hand, let $0 < s < 1$ and $f \in K_{1,s}(I)$. If $x \in I,\ x \neq 0$ and $\alpha = \beta = 2^{-\frac{1}{s}}$, then

$$f\left(2^{\left(1-\frac{1}{s}\right)}x\right) \leq f(x). \tag{1.2}$$

Since x is arbitrary,

$$f\left(2^{n\left(1-\frac{1}{s}\right)}x\right) \leq f(x)$$

for any $n \in \mathbb{N}$. Since $2^{n\left(1-\frac{1}{s}\right)}x \to 0$ when $n \to \infty$, then f should be identifiable at infinitely many points near 0. To overcome this, it is reasonable that the interval I, discussed in this article, is an interval that satisfies

$$I \supseteq (0,r) \quad \text{or} \quad I \supseteq (-r,0) \tag{1.3}$$

for some $r > 0$.

Let $s \in (0,1)$, and $\alpha, \beta \in [0,1]$ such that $\alpha^s + \beta^s = 1$. Define

$$g_{x,y}(\alpha) = \alpha x + \beta y. \tag{1.4}$$

By observing its first derivative on $(0,1)$, that is

$$g'_{x,y}(\alpha) = x - \left(\frac{1-\alpha^s}{\alpha^s}\right)^{\frac{1-s}{s}} y,$$

we consider the following two cases.

(i) If both x, y either greater or less than 0, then

$$g'_{x,y}(\alpha) = 0 \iff \frac{1-\alpha^s}{\alpha^s} = \left(\frac{x}{y}\right)^{\frac{s}{1-s}} \tag{1.5}$$

on $(0,1)$. Notice that $0 < \frac{x}{y} < \infty$, $\lim_{\alpha \to 1} \frac{1-\alpha^s}{\alpha^s} = 0$, and $\lim_{\alpha \to 0} \frac{1-\alpha^s}{\alpha^s} = \infty$. Since $\frac{1-\alpha^s}{\alpha^s}$ is strictly decreasing continuous function on $(0,1)$, by intermediate value theorem, there is a unique $\bar{\alpha} \in (0,1)$ such that (1.5) holds. In case $0 < x \leq y$, $g'_{x,y} < 0$ on $(0,\bar{\alpha})$, and $g'_{x,y} > 0$ on $(\bar{\alpha},1)$. It means $g_{x,y}(\alpha)$ strictly decreasing on $(0,\bar{\alpha})$ and strictly increasing on $(\bar{\alpha},1)$, therefore

$$g_{x,y}(\bar{\alpha}) = \min\left\{\alpha x + \beta y \mid \alpha \in [0,1],\ \alpha^s + \beta^s = 1\right\}.$$

Similarly, if $y \leq x < 0$, then $g'_{x,y} > 0$ on $(0,\bar{\alpha})$, and $g'_{x,y} < 0$ on $(\bar{\alpha},1)$. It means $g_{x,y}(\alpha)$ strictly increasing on $(0,\bar{\alpha})$ and strictly decreasing on $(\bar{\alpha},1)$. Therefore

$$g_{x,y}(\bar{\alpha}) = \max\left\{\alpha x + \beta y \mid \alpha \in [0,1],\ \alpha^s + \beta^s = 1\right\}. \tag{1.6}$$

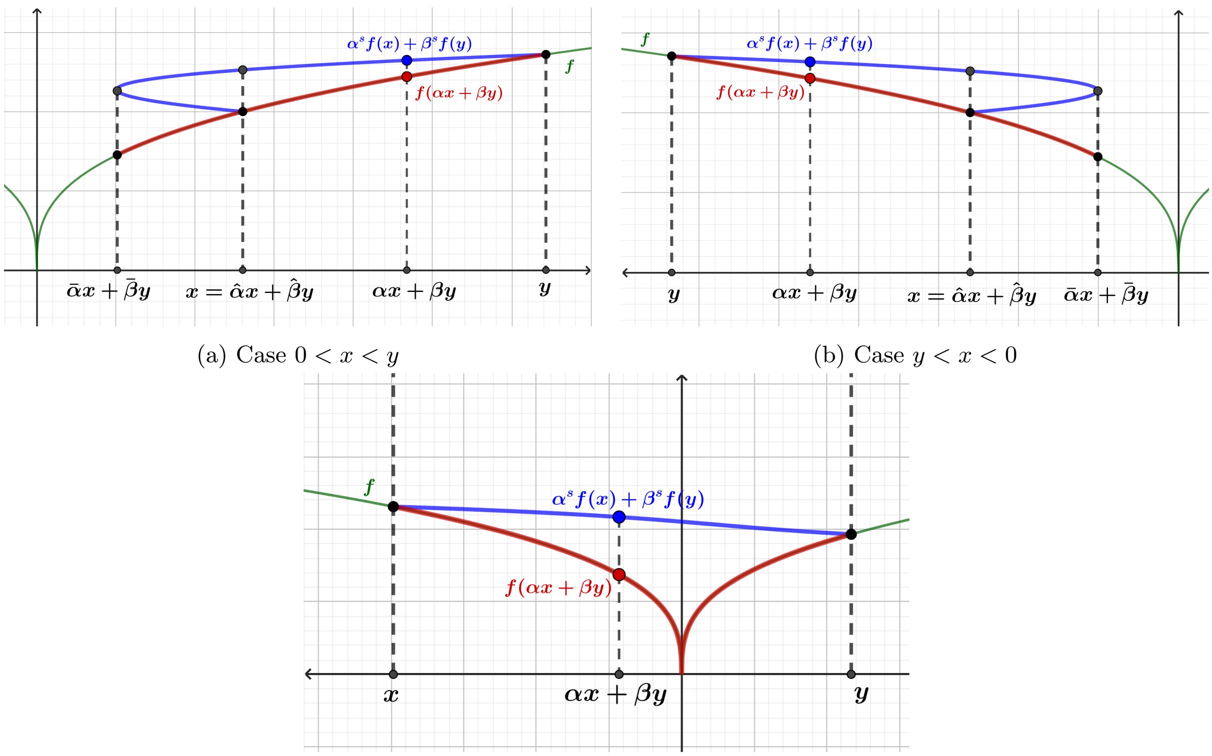
Furthermore, in case $0 < x < y$ (case $y < x < 0$ is similar), we have

$$g_{x,y}(0) > x > g_{x,y}(\bar{\alpha}).$$

By intermediate value theorem, there is a unique $\hat{\alpha} \in (0,\bar{\alpha})$ such that $g_{x,y}(\hat{\alpha}) = x = g_{x,y}(1)$. Therefore, $\alpha^s f(x) + \beta^s f(y)$ takes at most two different values for each $\alpha \in [\hat{\alpha},1]$, and exactly a value for each $\alpha \in [0,\hat{\alpha})$.

(ii) If $x < 0 < y$, then $g_{x,y}(\alpha) < 0$ on $(0,1)$, so that $g_{x,y}$ strictly decreasing on $[0,1]$. Therefore $\alpha^s f(x) + \beta^s f(y)$ takes exactly a value for each $\alpha \in [0,1]$.

Cases (i) and (ii) are the arguments to describe the first type s-convex function geometrically, see Figure 1.



(a) Case $0 < x < y$

(b) Case $y < x < 0$

(c) Case $x < 0 < y$

Figure 1: Illustration of the first type s-convex function

Theorem 1.1 *Let $s \in (0,1)$, and I is an interval satisfying condition (1.3). If $f \in K_{1,s}(I)$ and $c \in \mathbb{R}$, then g, defined as $g(x) = f(x) + c$ on I, is also element of $K_{1,s}(I)$.*

Proof: Let $x, y \in I$ and $\alpha, \beta \in [0,1]$ such that $\alpha^s + \beta^s = 1$. then

$$\begin{aligned} g(\alpha x + \beta y) &= f(\alpha x + \beta y) + c \\ &\leq (\alpha^s f(x) + \beta^s f(y)) + c \\ &= \alpha^s (f(x) + c) + \beta^s (f(y) + c) \\ &= \alpha^s g(x) + \beta^s g(y). \end{aligned}$$

It shows that $g \in K_{1,s}(I)$. □

Theorem 1.1 states that translation along y-axis preserves s-convexity. Another trivial consequences of Definition 1.1 are addition and scalar multiplication also preserving s-convexity. Theorem 6 at [1] shows that for a certain condition the composition of first and second type s-convex function could be a first type s-convex function.

Definition 1.2 Let I be an interval that contains 0, and $s \in (0,1]$. The set of all first type s-convex functions on I, which are continuous at 0, is denoted by $K^0_{1,s}$.

It is obvious that $K^0_{1,s}(I) \subseteq K_{1,s}(I)$. These two classes may or may not be equal.

2. Monotonicity and Continuity

Theorem 2.1 *Let $s \in (0,1)$ and $f \in K_{1,s}(I)$.*

1. *If an interval $I' \subseteq (0,\infty) \cap I$ then f is nondecreasing on I'.*
2. *If an interval $I' \subseteq (-\infty,0) \cap I$ then f is nonincreasing on I'.*

Proof: It is enough to prove 1, the rest is similar. Suppose that $I' \subseteq (0,\infty) \cap I$. Let $x, y \in I'$, $x < y$, and $\alpha, \beta \in (0,1)$ which $\alpha^s + \beta^s = 1$. If $f \in K_{1,s}(I)$, then

$$\begin{aligned} &f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y) \iff \\ &f(\alpha x + \beta y) - f(x) \leq \beta^s \left(f(y) - f(x)\right). \end{aligned} \tag{2.1}$$

Observe that

$$\alpha x + \beta y = x \iff \alpha x + (1-\alpha^s)^{\frac{1}{s}} y = x \iff \frac{1-\alpha^s}{(1-\alpha)^s} = \left(\frac{x}{y}\right)^s. \tag{2.2}$$

Since $0 < x < y$, then $0 < \left(\frac{x}{y}\right)^s < 1$. Suppose that $g(\alpha) = \frac{1-\alpha^s}{(1-\alpha)^s}$. Since $\lim_{\alpha \to 1} g(\alpha) = 0$, $\lim_{\alpha \to 0} g(\alpha) = 1$, and g is continuous on $(0,1)$, by intermediate value theorem, there is an $a_0 \in (0,1)$ such that

$$g(\alpha_0) = \frac{x^s}{y^s}.$$

By (2.2) and (2.1), we have $\alpha_0 x + \beta_0 y = x$ which $\beta_0 = (1-\alpha_0^s)^{\frac{1}{s}}$, and

$$0 = f\left(\alpha_0 x + \beta_0 y\right) - f(x) \leq \beta_0^s \left(f(y) - f(x)\right).$$

Since $0 < \beta_0 < 1$, then $f(y) - f(x) \geq 0$. Since $x < y$ is arbitrary on I', then f is nondecreasing on I'. □

As a consequence of Theorem 2.1, both $\lim_{x\to 0^+} f(x)$ and $\lim_{x \to 0^-} f(x)$ must be exist or $-\infty$. But, the following lemma states that it is impossible to have divergent limit.

Lemma 2.1 *Let $s \in (0,1)$ and $I = (0,r)$ (or $I = (-r,0)$). If $f \in K_{1,s}(I)$ then $\lim_{x\to 0^+} f(x)$ (or $\lim_{x\to 0^-} f(x)$) exists.*

Proof: Let $f \in K_{1,s}(I)$ which $s \in (0,1)$ and $I = (0,r)$. By contradiction, suppose that $\lim_{x\to 0^+} f(x) = -\infty$. Let $r > y > x > 0$. Observe that

$$2^{-1}\left(f(x) + f(y)\right) \to -\infty \text{ as } x \to 0^+$$

Therefore, there is some $x_0 \in (0,y)$ such that

$$f\left(2^{-\frac{1}{s}} y\right) > 2^{-1}\left(f(x_0) + f(y)\right)$$

By Theorem 2.1, f is nondecreasing. As consequence, $f\left(2^{-\frac{1}{s}}(x_0 + y)\right) \geq f\left(2^{-\frac{1}{s}} y\right)$. Therefore, $f\left(2^{-\frac{1}{s}}(x_0 + y)\right) > 2^{-1}\left(f(x_0) + f(y)\right)$, contradict with $f \in K_{1,s}(I)$.

□

Lemma 2.2 *Let $s \in (0,1)$ and I be an interval that contains a neighborhood of 0. If $f \in K_{1,s}(I)$ then $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$.*

Proof: Lemma 2.1 ensures the existence of $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$. By contradiction, suppose that $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$. Without loss of generality, suppose that $\lim_{x\to 0^+} f(x) > \lim_{x\to 0^-} f(x)$. Let $y > 0$, and let (x_n) be an increasing sequence of negative numbers that converges to 0. Also, let a pair $\alpha_n, \beta_n \in [0,1]$, which $\alpha_n^s + \beta_n^s = 1$, is a solution of

$$\alpha_n x_n + \beta_n y = 0.$$

Since

$$x_n + \beta_n y < \alpha_n x_n + \beta_n y < \beta_n y,$$

by taking limit to the both sides, then

$$\lim_{n\to\infty} \beta_n y = 0.$$

As consequence,

$$\lim_{n\to\infty} \beta_n = 0 \text{ and } \lim_{n\to\infty} \alpha_n = 1.$$

Since $f \in K_{1,s}(I)$, then $f(0) = f(\alpha_n x_n + \beta_n y) \leq \alpha^n f(x_n) + \beta^n f(y)$ for each $n \in \mathbb{N}$. Let $n \to \infty$, we have

$$\begin{aligned} f(0) &\leq \lim_{n\to\infty} (\alpha_n^s f(x_n) + \beta_n^s f(y)) \\ &= \lim_{n\to\infty} f(x_n) \\ &= \lim_{x\to 0^-} f(x) \\ &< \lim_{y\to 0^+} f(y). \end{aligned}$$

Let $f(0) < c < \lim_{y\to 0^+} f(y)$. By Theorem 2.1, $f(y) > c$ for any $y \in (0,\infty)\cap I$. Therefore

$$1^s \cdot f(0) + 0 \cdot f(y) = f(0) < c < 0^s \cdot f(0) + 1^s \cdot f(y).$$

Since $\alpha^s f(0) + \beta^s f(y)$ continuous for any $\alpha \in [0,1]$, by intermediate value theorem, there exists $\alpha \in (0,1)$ such that

$$\alpha_0^s f(0) + \beta_0^s f(y) = c. \tag{2.3}$$

On the other hand, $\alpha_0 \cdot 0 + \beta_0 y = \beta_0 y > 0$, then $f(\alpha_0 \cdot 0 + \beta_0 y) > c$. Together with (2.3),

$$f(\alpha_0 \cdot 0 + \beta_0 y) > \alpha_0^s f(0) + \beta_0^s f(y).$$

Contradict with $f \in K_{1,s}(I)$. □

Theorem 2.2 *Let $s \in (0,1)$ and I be an interval containing a neighborhood of 0. If $f \in K_{1,s}(I)$, then f is continuous at 0. Consequently, $f(x) \geq f(0)$ for any $x \in I$.*

Proof: Observe that

$$f(0) = f\left(2^{-\frac{1}{s}}(-x) + 2^{-\frac{1}{s}}x\right) \leq \frac{1}{2}(f(-x) + f(x)).$$

By taking limit to both sides and Lemma 2.2, we have $f(0) \leq \lim_{x\to 0} f(x)$. Next, it will be shown that $f(0) = \lim_{x\to 0} f(x)$. By contradiction, suppose that $f(0) < \lim_{x\to 0} f(x)$. The same argument, as last 10 lines at Lemma 2.2 proof, shows that this is impossible. So, f is continuous at 0. As consequence of Theorem 2.1, we have $f \geq f(0)$. □

Remember that any interval I must meet the condition (1.3), so 0 must be a limit point of I. So, we can generalize Lemma 2.1.

Corollary 2.1 *Let $s \in (0,1)$ and an interval I satisfies the condition (1.3). If $f \in K_{1,s}(I)$ then $\lim_{x\to 0} f(x)$ exists.*

Proof: Direct consequence of Theorem 2.2 and Lemma 2.1. □

Theorem 2.3 *Let $s \in (0,1)$, I be an interval containing a neighborhood of 0, and $f \in K_{1,s}(I)$.*

1. *If an interval $I' \subseteq (0,\infty) \cap I$ then $\frac{f(x)-f(0)}{x^s}$ is nondecreasing on I'.*
2. *If an interval $I' \subseteq (-\infty,0) \cap I$ then $\frac{f(x)-f(0)}{(-x)^s}$ is nonincreasing on I'.*

Proof: It is enough to prove (1), the rest is similar. Let $x, y \in I'$, $x < y$, and choose $\beta = \frac{x}{y}$ and $\alpha \in (0,1)$ such that $\alpha^s + \beta^s = 1$. Since $f \in K_{1,s}(I)$, then

$$f(x) = f(\alpha \cdot 0 + \beta y) \leq \alpha^s f(0) + \beta^s f(y).$$

This implies

$$f(x) - f(0) \leq \beta^s \left(f(y) - f(0)\right) = \frac{x^s}{y^s}\left(f(y) - f(0)\right).$$

This shows that

$$\frac{f(x) - f(0)}{x^s} \leq \frac{f(y) - f(0)}{y^s}.$$

Since $x < y$ is arbitrary on I', the conclusion is obtained. □

Remark 2.1 Suppose that $s \in (0,1]$ and $f(x) = |x|^s$ defined on $\mathbb{R}$. It will be shown that $f \in K_{1,s}(\mathbb{R})$. Let $\alpha, \beta \in [0,1]$ such that $\alpha^s + \beta^s = 1$, and $x, y \in \mathbb{R}$. Observe that

$$p^s + (1-p)^s \geq 1$$

for any $p \in [0,1]$. If $|x| \neq |y|$, then $\alpha|x| + \beta|y| > 0$. Therefore

$$\left(\frac{\alpha|x|}{\alpha|x| + \beta|y|}\right)^s + \left(\frac{\beta|y|}{\alpha|x| + \beta|y|}\right)^s \geq 1.$$

As consequence,

$$\begin{aligned} f(\alpha x + \beta y) = |\alpha x + \beta y|^s &\leq (\alpha|x| + \beta|y|)^s \\ &\leq \alpha^s |x|^s + \beta^s |y|^s \\ &= \alpha^s f(x) + \beta^s f(y). \end{aligned}$$

If $|x| = |y|$, then

$$\begin{aligned} f(\alpha x + \beta y) = |\alpha x + \beta y|^s &\leq (\alpha|x| + \beta|y|)^s \\ &= (\alpha + \beta)^s |x|^s \\ &\leq (\alpha^s + \beta^s)|x|^s \\ &= \alpha^s f(x) + \beta^s f(y). \end{aligned}$$

It shows the desired conclusion. It can also be verified that f satisfies the conclusions of Theorems 2.1, 2.2, and 2.3.

Remark 2.2 Let $s \in (0,1)$, and $g : [0,\infty) \to (0,\infty)$ which

$$g(x) = \begin{cases} x^s & x > 0 \\ 1 & x = 0 \end{cases}$$

Similar to Remark 2.1, for any $x, y > 0$, the function g satisfies (1.1). On the other hand,

$$f(\alpha \cdot 0 + \beta y) = f(\beta y) = \beta^s y^s \leq \alpha^s f(0) + \beta^s f(y)$$

for any $\alpha, \beta \in [0,1], \alpha^s + \beta^s = 1$, and $y \geq 0$. This shows that $f \in K_{1,s}([0,\infty))$. Since f is discontinous at 0, then $f \notin K^0_{1,s}([0,\infty))$. So, $K^0_{1,s}([0,\infty)) \subsetneq K_{1,s}([0,\infty))$.

Remark 2.3 For $0 < s < 1$, a function $f \in K_{1,s}(I)$ may not be continuous on $I \cap (0, \infty)$, see example 2 at [1].

Theorem 2.4 *Let $s \in (0, 1)$.*

1. *If $f : (0, \infty) \to \mathbb{R}$ is nondecreasing on $(0, \infty)$, and satisfies*
$$\frac{f(z) - f(x)}{(z - x)^s} \leq \frac{f(y) - f(x)}{(y - x)^s}$$
for any $x < z < y$, then $f \in K_{1,s}((0, \infty))$.

2. *If $f : (-\infty, 0) \to \mathbb{R}$ is nonincreasing on $(-\infty, 0)$, and satisfies*
$$\frac{f(x) - f(z)}{(x - z)^s} \geq \frac{f(x) - f(y)}{(x - y)^s},$$
for any $x > z > y$, then $f \in K_{1,s}((-\infty, 0))$.

Proof: We only need to prove part 1, the rest is similar. Let $0 < x < y$, and $\alpha, \beta \in [0, 1]$ such that $\alpha^s + \beta^s = 1$.

- If $\alpha x + \beta y \leq x$, then
$$f(\alpha x + \beta y) \leq f(x) = (\alpha^s + \beta^s) f(x) \leq \alpha^s f(x) + \beta^s f(y).$$

- If $x < \alpha x + \beta y < y$, suppose that $z = \alpha x + \beta y$, then
$$\begin{aligned} z - x &= (\alpha + \beta - 1) x + \beta(y - x) \\ &\leq (\alpha^s + \beta^s - 1) x + \beta(y - x) \\ &= \beta(y - x). \end{aligned}$$
Therefore,
$$\begin{aligned} f(z) - f(x) &\leq \left(\frac{z - x}{y - x}\right)^s (f(y) - f(x)) \leq \beta^s (f(y) - f(x)) \\ \implies & f(z) \leq \alpha^s f(x) + \beta^s f(y). \end{aligned}$$

We conclude that $f \in K_{1,s}((0, \infty))$. □

Lets denote the restriction of function $f : A \to \mathbb{R}$ on B as $f|_B$.

Theorem 2.5 *Let $s \in (0, 1)$, and $f : \mathbb{R} \to \mathbb{R}$. If*

- *$f|_{(0,\infty)} \in K_{1,s}((0, \infty))$ and $f|_{(-\infty,0)} \in K_{1,s}((-\infty, 0))$, and*
- *f is continuous at 0,*

then $f \in K_{1,s}(\mathbb{R})$.

Proof: We only need to prove that (1.1) holds for $x < 0 < y$ and $\alpha, \beta \in [0, 1]$ which $\alpha^s + \beta^s = 1$. By Theorem 2.1 and continuity of f at 0, we have $f(z) \geq f(0)$ for any $z \in \mathbb{R}$. Define $h(z) = f(z) - f(0)$ on $\mathbb{R}$. By Theorem 1.1, $h|_{(0,\infty)} \in K_{1,s}((0, \infty))$ and $h|_{(-\infty,0)} \in K_{1,s}((-\infty, 0))$. Let $0 < z < y$, by Theorem 2.1,
$$h(\beta y) \leq h(\alpha z + \beta y) \leq \alpha^s h(z) + \beta^s h(y).$$
By taking limit $z \to 0$, we have $h(\beta y) \leq \alpha^s h(0) + \beta^s h(y) = \beta^s h(y)$. Similarly, $h(\alpha x) \leq \alpha^s h(x)$. Since $x < 0 < y$, then $x < \alpha x + \beta y < y$. By Theorem 2.1,
$$h(\alpha x + \beta y) \leq \sup\{h(\alpha x), h(\beta y)\}.$$
Since $h(z) \geq 0$ on $\mathbb{R}$, then $\sup\{h(\alpha x), h(\beta y)\} \leq h(\alpha x) + h(\beta y)$. Therefore
$$h(\alpha x + \beta y) \leq h(\alpha x) + h(\beta y) \leq \alpha^s h(x) + \beta^s h(y).$$
This is equivalent to $f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$. □

3. Inclusion Properties

In this section, we present the necessary and sufficient conditions for the inclusion property between K_{1,s_1} and K_{1,s_2} where $s_1, s_2 \in (0,1)$. The results established in the preceding sections enable us to identify the key parameters governing the inclusion properties of these spaces.

Theorem 3.1 *If $s \in (0,1)$, and I be an interval that contains a neighborhood of 0, then $K_{1,s}(I) = K^0_{1,s}(I)$.*

Proof: Direct consequence of Theorem 2.2. □

Theorem 3.2 *For any $s \in (0,1)$, there is no inclusion between $K_{1,s}(I)$ and $K_{1,1}(I)$.*

Proof: Observe that $|x|^s \in K_{1,s}(I)/K_{1,1}(I)$, and exponential function e^{-x} or $e^x \in K_{1,1}(I)/K_{1,s}(I)$. □

Theorem 3.3 *Let I is an interval that satisfies condition (1.3). If $0 < s_1 < s_2 < 1$, then $K_{1,s_1}(I) \not\subseteq K_{1,s_2}(I)$.*

Proof: By Remark 2.1, $f(x) = |x|^{s_1} \in K_{1,s_1}(I)$. Let $\alpha, \beta \in (0,1)$ such that $\alpha^{s_1} + \beta^{s_1} = 1$, and $0 < |x| < |y| < 1$. Observe that

$$\begin{aligned}\lim_{x\to 0} f(\alpha x + \beta y) &= \lim_{x\to 0} |\alpha x + \beta y|^{s_1} \\ &= \beta^{s_1}|y|^{s_1} \\ &> \beta^{s_2}|y|^{s_2} \\ &= \lim_{x\to 0} (\alpha^{s_2}|x|^{s_2} + \beta^{s_2}|y|^{s_2}) \\ &= \lim_{x\to 0} (\alpha^{s_2} f(x) + \beta^{s_2} f(y)).\end{aligned}$$

This shows that $f \notin K_{1,s_2}(I)$. So, $K_{1,s_1}(I) \not\subseteq K_{1,s_2}(I)$. □

Theorem 3.4 *Let an interval I satisfies exactly one of the conditions (1.3). If $0 < s_1 < s_2 < 1$ then $K_{1,s_2}(I) \subset K_{1,s_1}(I)$.*

Proof: Let $I \supseteq (0,r)$ but $I \not\supseteq (-r,0)$ for some $r > 0$ (another case is similar), and $f \in K_{1,s_2}(I)$, it will be shown that $f \in K_{1,s_1}(I)$. Suppose that $\alpha, \beta \in [0,1]$ which $\alpha^{s_1} + \beta^{s_1} = 1$, and $x, y \in I$. Since

$$\left(\alpha^{\frac{s_1}{s_2}}\right)^{s_2} + \left(\beta^{\frac{s_1}{s_2}}\right)^{s_2} = 1,$$

then

$$f\left(\alpha^{\frac{s_1}{s_2}} x + \beta^{\frac{s_1}{s_2}} y\right) \le \left(\alpha^{\frac{s_1}{s_2}}\right)^{s_2} f(x) + \left(\beta^{\frac{s_1}{s_2}}\right)^{s_2} f(y) = \alpha^{s_1} f(x) + \beta^{s_1} f(y). \tag{3.1}$$

Since $p \le p^{\frac{s_1}{s_2}}$ for any $p \in [0,1]$, then $\alpha x + \beta y \le \alpha^{\frac{s_1}{s_2}} x + \beta^{\frac{s_1}{s_2}} y$. If $\alpha x + \beta y > 0$, by monotonicity of f (Theorem 2.1), then $f(\alpha x + \beta y) \le f\left(\alpha^{\frac{s_1}{s_2}} x + \beta^{\frac{s_1}{s_2}} y\right)$. Combine with (3.1),

$$f(\alpha x + \beta y) \le \alpha^{s_1} f(x) + \beta^{s_1} f(y). \tag{3.2}$$

If $\alpha x + \beta y = 0$, then there are three cases.

- If $x = y = 0$, then $f(\alpha \cdot 0 + \beta \cdot 0) = f(0) = \alpha^{s_1} f(0) + \beta^{s_1} f(0)$.
- If $x = \beta = 0$, then $\alpha = 1$ and $f(1 \cdot 0 + 0 \cdot y) = f(0) = 1^s f(0) + 0^{s_1} f(1)$.
- If $\alpha = y = 0$, then $\beta = 1$ and $f(0 \cdot x + 1 \cdot 0) = f(0) = 0^{s_1} f(x) + 1^{s_1} f(0)$.

We conclude that $f \in K_{1,s_1}(I)$. Since f is arbitrary, then $K_{1,s_2}(I) \subseteq K_{1,s_1}(I)$. By Theorem 3.3, $K_{1,s_2}(I) \subset K_{1,s_1}(I)$. □

Theorem 3.5 *Let $0 < s_1 < s_2 < 1$ and I is an interval that satisfies condition (1.3), then $K_{1,s_2}(I) \subset K_{1,s_1}(I)$.*

Proof: It is enough to prove when I satisfying both conditions at (1.3) (otherwise, it has been proved at Theorem 3.4). Let function f be any element of $K_{1,s_2}(I)$. Define $h(x) = f(x) - f(0)$ on I. By Theorem 1.1, $h \in K_{1,s_2}(I)$, and it is trivial that restriction $h|_{(I\cap[0,\infty))} \in K_{1,s_2}(I \cap [0,\infty))$, and $h|_{(I\cap(-\infty,0])} \in K_{1,s_2}(I \cap (-\infty,0])$, respectively. By Theorem 3.4,

$$h|_{(I\cap[0,\infty))} \in K_{1,s_1}(I \cap [0,\infty)) \text{ and } h|_{(I\cap(-\infty,0])} \in K_{1,s_1}(I \cap (-\infty,0]).$$

Next, let x, y be any elements on I.

- Case 1 : $x, y \in I \cap [0,\infty)$. Since $h|_{(I\cap[0,\infty))} \in K_{1,s_1}(I \cap [0,\infty))$ then

$$h(\alpha x + \beta y) \leq \alpha^{s_1} h(x) + \beta^{s_1} h(y) \tag{3.3}$$

  for any $\alpha, \beta \in [0,1]$, $\alpha^{s_1} + \beta^{s_1} = 1$.

- Case 2 : $x, y \in I \cap (-\infty,0]$. Similarly, since

$$h|_{(I\cap(-\infty,0])} \in K_{1,s_1}(I \cap (-\infty,0]),$$

  then (3.3) also holds.

- Case 3 : $x, y \in I$, $x < 0 < y$. Since $h|_{(I\cap[0,\infty))} \in K_{1,s_1}(I \cap [0,\infty))$ and $h|_{(I\cap(-\infty,0])} \in K_{1,s_1}(I \cap (-\infty,0])$, then

$$h(\alpha x) = h(\alpha x + \beta \cdot 0) \leq \alpha^{s_1} h(x),$$

  and

$$h(\beta y) = h(\alpha \cdot 0 + \beta y) \leq \beta^{s_1} h(y)$$

  for any $\alpha, \beta \in [0,1]$, $\alpha^{s_1} + \beta^{s_1} = 1$. Since $h(0) = 0$ and $h \in K_{1,s_2}(I)$, by Theorem 2.2, then $h(x) \geq 0$ on I. Therefore

$$\sup\{h(\alpha x), h(\beta y)\} \leq h(\alpha x) + h(\beta y) \leq \alpha^{s_1} h(x) + \beta^{s_1} h(y).$$

  Since $\alpha x \leq \alpha x + \beta y \leq \beta y$, by monotonicity of h (Theorem 2.1), we have $f(\alpha x + \beta y) \leq \sup\{h(\alpha x), h(\beta y)\}$. Therefore, h holds (3.3).

We conclude that

$$\begin{aligned} & h(\alpha x + \beta y) \leq \alpha^{s_1} h(x) + \beta^{s_1} h(y) \\ \iff & f(\alpha x + \beta y) - f(0) \leq \alpha^{s_1}(f(x) - f(0)) + \beta^{s_1}(f(x) - f(0)) \\ & \qquad = \alpha^{s_1} f(x) + \beta^{s_1} f(x) - f(0) \\ \iff & f(\alpha x + \beta y) \leq \alpha^{s_1} f(x) + \beta^{s_1} f(y) \end{aligned}$$

for any $\alpha, \beta \in [0,1], \alpha^{s_1} + \beta^{s_1} = 1$. Since $x, y \in I$ are arbitraries, then $f \in K_{1,s_1}(I)$. □

Remark 3.1 Theorem 3.5 generalize Theorem 4(c) at [1], and the inclusion $K_{1,s_2}(I) \subset K_{1,s_1}(I)$ is properly.

4. Jensen's Inequality

In this section, the sufficient and necessary conditions for the equality of Jensen's inequality will be given as one of the novelty of this research.

Theorem 4.1 (Jensen's Inequality) *Suppose that $n \in \mathbb{N}/\{1\}$, $s \in (0,1]$, and I is an interval that satisfies condition (1.3). If $f \in K_{1,s}(I)$ then*

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \le \sum_{i=1}^{n} \alpha_i^s f(x_i) \tag{4.1}$$

for any $\alpha_i \in (0,1)$, $\sum_{i=1}^{n} \alpha_i^s = 1$ and $x_i \in I$.

Proof: It will be proved by using strong induction. If $n = 2$, then inequality (4.1) is trivial. Suppose that inequality (4.1) holds for $n = 2, 3, \cdots, k$. Let $\alpha_1, \cdots, \alpha_{k+1} \in (0,1)$ which $\sum_{i=1}^{k+1} \alpha_i^s = 1$, and $x_1, \cdots, x_{k+1} \in I$. Observe that,

$$\sum_{i=1}^{k+1} \alpha_i^s = \left(\sum_{i=1}^{k-1} \alpha_i^s\right) + \left(\alpha_k^s + \alpha_{k+1}^s\right) = \left(\sum_{i=1}^{k-1} \alpha_i^s\right) + \beta^s,$$

which $\beta = (\alpha_k^s + \alpha_{k+1}^s)^{\frac{1}{s}}$. On the other hand,

$$\begin{aligned}
\sum_{i=1}^{k+1} \alpha_i x_i &= \left(\sum_{i=1}^{k-1} \alpha_i x_i\right) + (\alpha_k x_k + \alpha_{k+1} x_{k+1}) \\
&= \left(\sum_{i=1}^{k-1} \alpha_i x_i\right) + \beta\left(\frac{\alpha_k}{\beta} x_k + \frac{\alpha_{k+1}}{\beta} x_{k+1}\right) \\
&= \left(\sum_{i=1}^{k-1} \alpha_i x_i\right) + \beta y_k,
\end{aligned}$$

which $y_k = \frac{\alpha_k}{\beta} x_k + \frac{\alpha_{k+1}}{\beta} x_{k+1}$. Since $\sum_{i=1}^{k-1} \alpha_i^s + \beta^s = 1$ and $\left(\frac{\alpha_k}{\beta}\right)^s + \left(\frac{\alpha_{k+1}}{\beta}\right)^s = 1$, by induction hypothesis, then

$$\begin{aligned}
f\left(\sum_{i=1}^{k+1} \alpha_i x_i\right) &= f\left(\left(\sum_{i=1}^{k-1} \alpha_i x_i\right) + \beta y_k\right) \\
&\le \left(\sum_{i=1}^{k-1} \alpha_i^s f(x_i)\right) + \beta^s f(y_k) \\
&= \left(\sum_{i=1}^{k-1} \alpha_i^s f(x_i)\right) + \beta^s f\left(\frac{\alpha_k}{\beta} x_k + \frac{\alpha_{k+1}}{\beta} x_{k+1}\right) \\
&\le \left(\sum_{i=1}^{k-1} \alpha_i^s f(x_i)\right) + \beta^s \left(\frac{\alpha_k^s}{\beta^s} f(x_k) + \frac{\alpha_{k+1}^s}{\beta^s} f(x_{k+1})\right) \\
&\le \left(\sum_{i=1}^{k+1} \alpha_i^s f(x_i)\right).
\end{aligned}$$

This shows that (4.1) holds for $n = k + 1$. So, (4.1) holds for any $n \in \mathbb{N}/\{1\}$. □

Theorem 4.2 *Suppose that $n \in \mathbb{N}/\{1\}$, $s \in (0,1)$, and I be an interval that satisfies condition (1.3). If $f \in K_{1,s}(I)$, then equality (4.1) holds for any such α_i, x_i at Theorem 4.1 if and only if f is constant.*

Proof: Clearly, only the necessity part needs an argument. Let (4.1) holds, and $x \in I$ is arbitrary. Choose $\alpha_i = n^{-\frac{1}{s}}$ and $x_i = x$ for each i, then

$$f\left(n^{1-\frac{1}{s}}x\right) = f(x) \implies f\left(n^{m\left(1-\frac{1}{s}\right)}x\right) = f(x) \text{ for any } m \in \mathbb{N}.$$

Since $n^{m\left(1-\frac{1}{s}\right)}x \to 0$ as $m \to \infty$, by Corollary 2.1, then

$$\lim_{x\to 0} f(x) = \lim_{m\to\infty} f\left(n^{m\left(1-\frac{1}{s}\right)}x\right) = f(x).$$

This shows that f is constant. □

5. Conclusions

- Section 1.
  - If $s \in (0,1)$, $0 < x < y$ and $\alpha, \beta \in [0,1]$ which $\alpha^s + \beta^s = 1$, then there is unique stationary point $\bar{\alpha}$ of $g_{x,y}(\alpha) = \alpha x + \beta y$, such that $\bar{\alpha}x + \bar{\beta}y \le x$ and $\bar{\alpha}x + \bar{\beta}y \le \alpha x + \beta y \le y$. Also, there is a unique $\hat{\alpha} \in (0, \bar{\alpha})$ such that $\hat{\alpha}x + \hat{\beta}y = x$. As consequence, $\alpha^s f(x) + \beta^s f(y)$ takes at most two different values for each $\alpha \in [\hat{\alpha}, 1]$ and takes exactly a value for each $\alpha \in [0, \hat{\alpha})$. Based on these observations, first type s-convex function can be illustrated as figure 1.
  - First type s-convex function is translation invariant along y-axis.
- Section 2.
  - If $s \in (0,1)$ and $f \in K_{1,s}(I)$, then f is nondecreasing on $I \cap (0,\infty)$ and non increasing on $I \cap (-\infty, 0)$. If I contains a neighborhood of 0, then f is continuous at 0, but it still can be not continuous at another point. Also, $\frac{f(x)-f(0)}{|x|^s}$ is nondecreasing on $I \cap (0,\infty)$ and nonincreasing on $I \cap (-\infty, 0)$.
  - Sufficient conditions for $f \in K_{1,s}\left((0,\infty)\right)$ are f is nondecreasing on $(0,\infty)$ and

    $$\frac{f(z)-f(x)}{(z-x)^s} \le \frac{f(y)-f(x)}{(y-x)^s}$$

    for any $0 < x < z < y$. On the other hand, sufficient condition for $f \in K_{1,s}\left((-\infty,0)\right)$ are f is nonincreasing on $(0,\infty)$ and

    $$\frac{f(x)-f(z)}{(x-z)^s} \ge \frac{f(x)-f(y)}{(x-y)^s}$$

    for any $0 > x > z > y$. Furthermore, when $f|_{(0,\infty)} \in K_{1,s}\left((0,\infty)\right)$, $f|_{(-\infty,0)} \in K_{1,s}\left((-\infty,0)\right)$, and f is continuous at 0, then $f \in K_{1,s}\left(\mathbb{R}\right)$.
- Section 3.
  - If $s \in (0,1)$ and interval I containing neighborhood of 0, then $K_{1,s}(I) = K^0_{1,s}(I)$.
  - For any $s \in (0,1)$, there is no inclusion between $K_{1,s}$ and $K_{1,1}$.
  - For any $0 < s_1 < s_2 < 1$, the inclusion $K_{1,s_2} \subset K_{1,s_1}$ occured.
- Section 4.
  - If $s \in (0,1]$, then Theorem 4.1 shows that Jensen's Inequality holds for any $f \in K_{1,s}(I)$. If $0 < s < 1$, Theorem 4.2 says that the equality holds only for constant functions.

6. Application for Further Research

The properties of s-convex function in this article will be used as a foundation to generalize Orlicz Space. A way to do this is by changing Young's function with first type s-Young's function. We believe that this idea can contribute to developing the results that have been obtained at [15–23].

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