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$(\mathfrak{q}-\mathrm{h})$-Hermite-Hadamard, Midpoint, and Trapezoidal Inequalities for Convex Functions *

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ABSTRACT: This paper establishes transformative advances in quantum calculus by introducing a comprehensive framework for $(\mathfrak{q}-\mathrm{h})$-generalized integral inequalities. We pioneer three foundational contributions to the field: First, we derive a novel bilateral $(\mathfrak{q}-\mathrm{h})$-Hermite-Hadamard inequality through an innovative synthesis of left and right quantum integrals, substantially generalizing classical integral inequalities. Second, we develop sharp midpoint-type inequalities with explicit error bounds for $(\mathfrak{q}-\mathrm{h})$-differentiable convex functions, employing strategic applications of Hölder's inequality and power mean inequalities to quantify approximation accuracy. Third, we construct advanced trapezoidal estimators incorporating quantum-calculus modifications that systematically account for parameterized displacements h. Through rigorous constructive proofs and numerical validation, we demonstrate that our unified framework naturally contains classical calculus as the limiting case $\lim\limits_{\mathfrak{q}\to 1,\mathrm{h}\to 0}$ achieves tighter error bounds compared to existing $\mathfrak{q}$-calculus results through optimized $(\mathfrak{q},\mathrm{h})$-coupling and enables precision-tunable approximations via flexible parameterization of $\mathfrak{q}$ and h. These breakthroughs significantly extend the operational calculus for convex functions in non-Newtonian analysis, with immediate applications in quantum probability measures, fractional variational optimization, and deformed mathematical physics models requiring non-uniform discretization schemes.

Key Words: $(\mathfrak{q}-\mathrm{h})$-derivatives; $(\mathfrak{q}-\mathrm{h})$-integrals, convex function, Hölder's inequality, power mean inequality, Hermite Hadamard Inequality.

Contents

1. Introduction

The theory of inequalities plays a vital role in several mathematical disciplines, such as probability theory, integral calculus, and offers interesting and fascinating tools for estimation and approximation. For further information, one can see [1,2,3,4]. Another fascinating subfield of classical calculus is known as quantum calculus, sometimes called $\mathfrak{q}$-calculus, that deals with $\mathfrak{q}$-integrals and $\mathfrak{q}$-differences. In the nineteenth century, Euler [5] was the first to propose the idea of quantum calculus, "commonly known as calculus without limits". It can be asserted that $\mathfrak{q}$-calculus recovers the conclusions of classical calculus on taking the limit as $\mathfrak{q}\to 1$, and hence it generalizes the derivative and integration of classical calculus. Its function serves as a link between mathematics and physics. In 1910, Jackson [6] used it to define $\mathfrak{q}$-integration and $\mathfrak{q}$-derivatives for continuous functions over an interval $(0,\infty)$. Kac and Cheung [5] initially covered the fundamental ideas of $\mathfrak{q}$-calculus in their book. In 1966, Al-Salam [7] presented the ideas of $\mathfrak{q}$-fractional and $\mathfrak{q}$-Riemann-Liouville fractional equations. In 2013 and 2014, Tariboon and Ntouyas presented the $\mathfrak{q}_{\mathrm{a}}$-integral and $\mathfrak{q}_{\mathrm{a}}$-derivative of continuous functions over finite intervals and presented

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quantum integral inequalities. Whereas in 2020, the $\mathfrak{q}_b$-integral and $\mathfrak{q}_b$-derivative over finite intervals were introduced by Bermudo et al. [8], who also demonstrated some of their fundamental characteristics, an interesting direction was presented in [9] when presenting operators with variable q unlike previous studies, when it was always considered constant.

Quantum calculus has been used in the study of different types of inequalities such as Simpson and Newton-type inequalities, Hanh inequalities, Ostrowski inequalities, Fejer-type inequalities, and Hermite Hadamard-like inequalities, respectively [11,12,13,14]. For some details on quantum calculus, we refer the readers to [5,15,16].

This paper aims to establish the Hermite Hadamard, trapeziod, and midpoint types of inequalities for left and right $(\mathfrak{q}-\mathrm{h})$-integral operators. Using convex functions, we will demonstrate these inequalities. Additionally, we present examples of convex functions that satisfy the aforementioned inequalities (Some of the extensions and generalizations of the classical convex function can be found in [18,19]).

Definition 1.1 *A function $\mathfrak{T}: I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$ is known as a convex function if it satisfies the following inequality:*

$$\mathfrak{T}(\lambda\varkappa + (1-\lambda)\mathrm{y}) \leq \lambda\mathfrak{T}(\varkappa) + (1-\lambda)\mathfrak{T}(\mathrm{y}), \lambda \in [0,1], \varkappa, \mathrm{y} \in I. \tag{1.1}$$

If the inequality (1.1) holds in reverse order, then $\mathfrak{T}$ is called the concave function.

The following is one of the best-known inequalities, linked to the notion of a convex function:

Theorem 1.1 *[20,21] Let $\mathfrak{T}: [\mu_1, \mu_2] \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function with $\mu_1 < \mu_2$, then the subsequent inequality holds:*

$$\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right) \leq \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\mathfrak{T}(\mathrm{u})d\mathrm{u} \leq \frac{\mathfrak{T}(\mu_1)+\mathfrak{T}(\mu_2)}{2}. \tag{1.2}$$

This double inequality is known as the Hermite-Hadamard inequality in literature for convex functions. We can find the upper and lower bounds for a convex function's mean value due to this inequality. In recent years, researchers have focused on various variations of Hermite-Hadamard's inequalities and their applications (see, [22,23,24,25,26,27,28,29]).

Theorem 1.2 *[30] Let $\mathfrak{T}$ and $\mathfrak{G}$ be two real valued functions defined on $[\mu_1, \mu_2]$. If $\mathrm{s} > 1$ and $\frac{1}{\mathrm{s}} + \frac{1}{\mathrm{r}} = 1$ such that $|\mathfrak{T}|^{\mathrm{s}}$ and $|\mathfrak{G}|^{\mathrm{r}}$ are integrable functions on $[\mu_1, \mu_2]$, then the following Hölder inequality holds:*

$$\int_{\mu_1}^{\mu_2}|\mathfrak{T}(\mathrm{u})\mathfrak{G}(\mathrm{u})|d\mathrm{u} \leq \left(\int_{\mu_1}^{\mu_2}|\mathfrak{T}(\mathrm{u})|^{\mathrm{s}}d\mathrm{u}\right)^{\frac{1}{\mathrm{s}}}\left(\int_{\mu_1}^{\mu_2}|\mathfrak{G}(\mathrm{u})|^{\mathrm{r}}d\mathrm{u}\right)^{\frac{1}{\mathrm{r}}}.$$

Theorem 1.3 *[30] Let $\mathfrak{T}$ and $\mathfrak{G}$ be two real valued functions defined on $[\mu_1, \mu_2]$. If $\mathrm{r} \geq 1$ and $\frac{1}{\mathrm{s}} + \frac{1}{\mathrm{r}} = 1$ such that $|\mathfrak{T}|$ and $|\mathfrak{T}||\mathfrak{G}|^{\mathrm{r}}$ are integrable functions on $[\mu_1, \mu_2]$, then the following Power mean inequality holds:*

$$\int_{\mu_1}^{\mu_2}|\mathfrak{T}(\mathrm{u})\mathfrak{G}(\mathrm{u})|d\mathrm{u} \leq \left(\int_{\mu_1}^{\mu_2}|\mathfrak{T}(\mathrm{u})|d\mathrm{u}\right)^{1-\frac{1}{\mathrm{r}}}\left(\int_{\mu_1}^{\mu_2}|\mathfrak{T}(\mathrm{u})||\mathfrak{G}(\mathrm{u})|^{\mathrm{r}}d\mathrm{u}\right)^{\frac{1}{\mathrm{r}}}.$$

The remainder of this paper is organized as follows: In Section 2, we review necessary background material from $\mathfrak{q}$-calculus and define the relevant $(\mathfrak{q}-\mathrm{h})$-differential and integral operators. Section 3 presents the main theoretical results, where we establish $(\mathfrak{q}-\mathrm{h})$-analogues of Hermite-Hadamard, midpoint, and trapezoidal inequalities for convex functions. Each subsection in Section 3 focuses on a specific class of inequalities and is supported with rigorous proofs. In Section 4, we illustrate the theoretical findings through computational examples that validate the sharpness and applicability of our inequalities. Finally, Section 5 provides concluding remarks and outlines directions for future work.

2. Preliminaries

In this section, we provide a basic introduction to the $\mathfrak{q}$-calculus and summarize some definitions and theorems below. Among the relevant antecedents to the study of Hermite-Hadamard inequality in quantum computing are the studies by Aple et al. in 2018 and by Bermudo et al. in 2020 using left and right quantum integral operators for convex functions.

Theorem 2.1 *[31] Let $0<\mathfrak{q}<1$, and $\mathfrak{T}:[\mu_1,\mu_2]\to\mathbb{R}$ be a differentiable convex function on an interval (μ_1,μ_2). Then, the left $\mathfrak{q}$-Hermite Hadamard inequality holds:*

$$\mathfrak{T}\left(\frac{\mu_1\mathfrak{q}+\mu_2}{1+\mathfrak{q}}\right)\leq\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\mathfrak{T}(\mathrm{u})_{\mu_1}d_{\mathfrak{q}}\mathrm{u}\leq\frac{\mathfrak{q}\mathfrak{T}(\mu_1)+\mathfrak{T}(\mu_2)}{1+\mathfrak{q}}.$$

Theorem 2.2 *[8] Let $\mathfrak{T}:[\mu_1,\mu_2]\to\mathbb{R}$ be a differentiable convex function on an interval (μ_1,μ_2). Then, the right $\mathfrak{q}$-Hadamard inequality holds:*

$$\mathfrak{T}\left(\frac{\mu_1+\mu_2\mathfrak{q}}{1+\mathfrak{q}}\right)\leq\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{\mathfrak{q}}\mathrm{u}\leq\frac{\mathfrak{T}(\mu_1)+\mathfrak{q}\mathfrak{T}(\mu_2)}{1+\mathfrak{q}}. \tag{2.1}$$

For every $\mathrm{n}\in\mathbb{N}$, the quantum analog or $\mathfrak{q}$-analog is as follows:

$$[\mathrm{n}]_{\mathfrak{q}}=\frac{1-\mathfrak{q}^{\mathrm{n}}}{1-\mathfrak{q}}=\mathfrak{q}^{\mathrm{n}-1}+\mathfrak{q}^{\mathrm{n}-2}+\ldots+\mathfrak{q}+1.$$

Definition [32] Let $0<\mathfrak{q}<1$, $\mathrm{h}\in\mathbb{R}$ and $\mathrm{u}\in I$. For a continuous function $\mathfrak{T}:I\to\mathbb{R}$, the left and right $\mathfrak{q}$–h-derivatives on I denoted by $C_{\mathrm{h}}D_{\mathfrak{q}}^{\mu_1^+}\mathfrak{T}$ and $C_{\mathrm{h}}D_{\mathfrak{q}}^{\mu_2^-}\mathfrak{T}$ are defined with the following equations respectively:

$$C_{\mathrm{h}}D_{\mathfrak{q}}^{\mu_1^+}\mathfrak{T}(\mathrm{u}):=\frac{\mathfrak{T}((1-\mathfrak{q})\mu_1+\mathfrak{q}(\mathrm{u}+\mathrm{h}))-\mathfrak{T}(\mathrm{u})}{(1-\mathfrak{q})(\mu_1-\mathrm{u})+\mathfrak{q}\mathrm{h}},\quad \mathrm{u}\neq\frac{\mathfrak{q}\mathrm{h}+(1-\mathfrak{q})\mu_1}{1-\mathfrak{q}}:=\mathrm{m}, \tag{2.2}$$

$$C_{\mathrm{h}}D_{\mathfrak{q}}^{\mu_2^-}\mathfrak{T}(\mathrm{u}):=\frac{\mathfrak{T}((1-\mathfrak{q})\mathrm{u}+\mathfrak{q}(\mu_2+\mathrm{h}))-\mathfrak{T}(\mu_2)}{(1-\mathfrak{q})(\mathrm{u}-\mu_2)+\mathfrak{q}\mathrm{h}},\quad \mathrm{u}\neq\frac{-\mathfrak{q}\mathrm{h}+(1-\mathfrak{q})\mu_2}{1-\mathfrak{q}}:=\mathrm{n}, \tag{2.3}$$

provided that $(1-\mathfrak{q})\mu_1+\mathfrak{q}(\mathrm{u}+\mathrm{h})\in[\mu_1,x]$ and $(1-\mathfrak{q})\mathrm{u}+\mathfrak{q}(\mu_2+\mathrm{h})\in[\mathrm{u},\mu_2]$. Also, $C_{\mathrm{h}}D_{\mathfrak{q}}^{\mu_1^+}\mathfrak{T}(\mathrm{m})=\lim_{\mathrm{u}\to\mathrm{m}}C_{\mathrm{h}}D_{\mathfrak{q}}^{\mu_1^+}\mathfrak{T}(x)$ and $C_{\mathrm{h}}D_{\mathfrak{q}}^{\mu_2^-}\mathfrak{T}(\mathrm{n})=\lim_{\mathrm{u}\to\mathrm{n}}C_{\mathrm{h}}D_{\mathfrak{q}}^{\mu_2^-}\mathfrak{T}(\mathrm{u})$.

Remark 2.1 *For $\mathrm{h}=0$, we get Definition 4 see [32].*

Definition 2.1 *[33] A function $\mathfrak{T}:[\mu_1,\mu_2]\to\mathbb{R}$ is called $(\mathfrak{q}-\mathrm{h})_{\mu_1}$-definite integral or left plank's quantum definite integral on $[\mu_1,\mu_2]$, where $0<\mathfrak{q}<1$ and $\mathrm{h}\in\mathbb{R}$, if the following expression holds:*

$$\int_{\mu_1}^{\mathrm{u}}\mathfrak{T}(\lambda)_{\mu_1}d\lambda_{(\mathfrak{q}-\mathrm{h})}=((1-\mathfrak{q})(\mathrm{u}-\mu_1)+\mathfrak{q}\mathrm{h})\sum_{\mathrm{n}=0}^{\infty}\mathfrak{q}^{\mathrm{n}}\mathfrak{T}(\mathfrak{q}^{\mathrm{n}}\mathrm{u}+(1-\mathfrak{q}^{\mathrm{n}})\mu_1+\mathrm{n}\mathfrak{q}^{\mathrm{n}}\mathrm{h}),\mathrm{u}>\mu_1. \tag{2.4}$$

Also, $(\mathfrak{q}-\mathrm{h})_{\mu_2}$-definite integral or right plank's quantum definite integral on $[\mu_1,\mu_2]$ is expressed as:

$$\int_{\mathrm{u}}^{\mu_2}\mathfrak{T}(\lambda)^{\mu_2}d\lambda_{(\mathfrak{q}-\mathrm{h})}=((1-\mathfrak{q})(\mu_2-\mathrm{u})+\mathfrak{q}\mathrm{h})\sum_{\mathrm{n}=0}^{\infty}\mathfrak{q}^{\mathrm{n}}\mathfrak{T}(\mathfrak{q}^{\mathrm{n}}\mathrm{u}+(1-\mathfrak{q}^{\mathrm{n}})\mu_2+\mathrm{n}\mathfrak{q}^{\mathrm{n}}\mathrm{h}),\mathrm{u}<\mu_2. \tag{2.5}$$

Remark 2.2 *Observe that, (i) if $\mathrm{h}=0$ in (2.4) and (2.5), we get left and right $\mathfrak{q}$ integral operators given in [34].*
(ii) if $\mu_1=0$ and $\mathrm{h}=0$ in (11), then we get $\mathfrak{q}$ integral or Jackson integral in [5] .

$$\int_0^{\mathrm{u}}\mathfrak{T}(\lambda)d\lambda_{\mathfrak{q}}=(1-\mathfrak{q})\mathrm{u}\sum_{\mathrm{n}=0}^{\infty}\mathfrak{q}^{\mathrm{n}}\mathfrak{T}(\mathfrak{q}^{\mathrm{n}}\mathrm{u}).$$

Now, we give some well-known results from the $(\mathfrak{q}-\mathrm{h})$-calculus ([17]).

Theorem 2.3 *If $\mathfrak{T}, \mathfrak{G} : \Omega \to \mathbb{R}$ are $(\mathfrak{q}-\mathrm{h})$-differentiable functions on $[\mu_1, \mu_2]$, then (i) Product rule and (ii) quotient rule holds:*

$$\text{(i)(a)}_{\mu_1} D_{\mathfrak{q}-\mathrm{h}}(\mathfrak{T}(\mathrm{u})\mathfrak{G}(\mathrm{u})) = \mathfrak{G}(\mathrm{u})_{\mu_1} D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}(\mathrm{u}) + \mathfrak{T}\left(\mathfrak{q}\mathrm{u} + (1-\mathfrak{q})\mu_1 + \mathfrak{q}\mathrm{h}\right)_{\mu_1} D_{\mathfrak{q}-\mathrm{h}}\mathfrak{G}(\mathrm{u}). \tag{2.6}$$

since $\mathfrak{T}(\mathrm{u})\mathfrak{G}(\mathrm{u}) = \mathfrak{G}(\mathrm{u})\mathfrak{T}(\mathrm{u})$, then

$$\text{(b)}_{\mu_1} D_{\mathfrak{q}-\mathrm{h}}(\mathfrak{T}(\mathrm{u})\mathfrak{G}(\mathrm{u})) = \mathfrak{T}(\mathrm{u})_{\mu_1} D_{\mathfrak{q}-\mathrm{h}}\mathfrak{G}(\mathrm{u}) + \mathfrak{G}\left(\mathfrak{q}\mathrm{u} + (1-\mathfrak{q})\mu_1 + \mathfrak{q}\mathrm{h}\right)_{\mu_1} D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}(\mathrm{u}). \tag{2.7}$$

From (2.6), we have

$$\text{(ii) (a)}_{\mu_1} D_{\mathfrak{q}-\mathrm{h}}\left(\frac{\mathfrak{T}(\mathrm{u})}{\mathfrak{G}(\mathrm{u})}\right) = \frac{\mathfrak{G}(\mathrm{u})_{\mu_1} D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}(\mathrm{u}) - \mathfrak{T}(\mathrm{u})_{\mu_1} D_{\mathfrak{q}-\mathrm{h}}\mathfrak{G}(\mathrm{u})}{\mathfrak{G}(\mathrm{u})\mathfrak{G}\left(\mathfrak{q}\mathrm{u} + (1-\mathfrak{q})\mu_1 + \mathfrak{q}\mathrm{h}\right)}.$$

From (2.7), we have

$$\begin{aligned}&\text{(b)}\ {}_{\mu_1} D_{\mathfrak{q}-\mathrm{h}}\left(\frac{\mathfrak{T}(\mathrm{u})}{\mathfrak{G}(\mathrm{u})}\right) = \\ &\frac{\mathfrak{G}\left(\mathfrak{q}\mathrm{u} + (1-\mathfrak{q})\mu_1 + \mathfrak{q}\mathrm{h}\right)_{\mu_1} D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}(\mathrm{u}) - \mathfrak{T}\left(\mathfrak{q}\mathrm{u} + (1-\mathfrak{q})\mu_1 + \mathfrak{q}\mathrm{h}\right)_{\mu_1} D_{\mathfrak{q}-\mathrm{h}}\mathfrak{G}(\mathrm{u})}{\mathfrak{G}(\mathrm{u})\mathfrak{G}\left(\mathfrak{q}\mathrm{u} + (1-\mathfrak{q})\mu_1 + \mathfrak{q}\mathrm{h}\right)}.\end{aligned}$$

Remark 2.3 *(i) By setting $\mathrm{h} = 0$, in (2.6) and (2.7), we get the correspondence concepts in [35].*

Lemma 2.1 *Let $\mathfrak{T}, \mathfrak{G} : [\mu_1, \mu_2] \to \mathbb{R}$, be continuous functions, then the following rule holds:*

$$\begin{aligned}&\int_0^{\mathrm{b}} \mathfrak{T}(\lambda)^{\mu_2} D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{G}\left(\lambda\mu_1 + (1-\lambda)\mu_2\right) d_{(\mathfrak{q}-\mathrm{h})}\lambda \\ &= \frac{1}{\mu_2-\mu_1}\int_0^{\mathrm{b}} D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\lambda)\mathfrak{G}\left(\mathfrak{q}\lambda\mu_1 + (1-\mathfrak{q}\lambda)\mu_2 + \mathfrak{q}\mathrm{h}\right) d_{(\mathfrak{q}-\mathrm{h})}\lambda \\ &- \left.\frac{\mathfrak{T}(\lambda)\mathfrak{G}\left(\lambda\mu_1 + (1-\lambda)\mu_2\right)}{\mu_2-\mu_1}\right|_0^{\mathrm{b}}.\end{aligned} \tag{2.8}$$

Proof: Since,

$$D_{(\mathfrak{q}-\mathrm{h})}[\mathfrak{T}(\mathrm{u})\mathfrak{G}(\mathrm{u})] = \mathfrak{T}(\mathrm{u})D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{G}(\mathrm{u}) + \mathfrak{G}(\mathfrak{q}(\mathrm{u}+\mathrm{h}))D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mathrm{u}). \tag{2.9}$$

$$\mathfrak{G}(\mathrm{u}) = \mathfrak{G}\left(\lambda\mu_1 + (1-\lambda)\mu_2\right)$$
$$\mathfrak{G}(\mathfrak{q}\mathrm{u}) = \mathfrak{G}\left(\mathfrak{q}\lambda\mu_1 + (1-\mathfrak{q}\lambda)\mu_2\right)$$
$$\mathfrak{G}(\mathrm{u}) = \mathfrak{G}(\lambda).$$

$(\mathfrak{q}-\mathrm{h})$-integrate (2.9) over the interval $[0, \mathrm{b}]$, and after rearranging, we get:

$$\begin{aligned}&\int_0^{\mathrm{b}} \mathfrak{T}(\lambda)D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{G}\left(\lambda\mu_1 + (1-\lambda)\mu_2\right) d_{(\mathfrak{q}-\mathrm{h})}\lambda \\ &= \int_0^{\mathrm{b}} D_{(\mathfrak{q}-\mathrm{h})}\left(\mathfrak{T}(\lambda)\mathfrak{G}\left(\lambda\mu_1 + (1-\lambda)\mu_2\right)\right) d_{(\mathfrak{q}-\mathrm{h})}\lambda \\ &- \int_0^{\mathrm{b}} \mathfrak{G}\left(\mathfrak{q}\lambda\mu_1 + (1-\mathfrak{q}\lambda)\mu_2 + \mathfrak{q}\mathrm{h}\right) D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\lambda)d_{(\mathfrak{q}-\mathrm{h})}\lambda.\end{aligned}$$

After, simplification we get (2.8). □

Lemma 2.2 *Let $\mathfrak{T}, \mathfrak{G} : [\mu_1, \mu_2] \to \mathbb{R}$, be continuous functions, then the following rule holds:*

$$\begin{aligned}&\int_0^{\mathrm{b}} \mathfrak{T}(\lambda)_{\mu_1} D_{(\mathfrak{q}-\mathrm{h})} \mathfrak{G}\left(\lambda\mu_2 + (1-\lambda)\mu_1\right) d_{(\mathfrak{q}-\mathrm{h})}\lambda \\ &= \left.\frac{\mathfrak{T}(\lambda)\mathfrak{G}\left(\lambda\mu_2 + (1-\lambda)\mu_1\right)}{\mu_2-\mu_1}\right|_0^{\mathrm{b}} - \frac{1}{\mu_2-\mu_1}\int_0^{\mathrm{b}} D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\lambda)\mathfrak{G}\left(\mathfrak{q}\lambda\mu_2 + (1-\mathfrak{q}\lambda)\mu_1 + \mathfrak{q}\mathrm{h}\right) d_{(\mathfrak{q}-\mathrm{h})}\lambda. \quad (2.10)\end{aligned}$$

Proof: The proof can be done similar to the proof of Lemma 2.1. □

Remark 2.4 *By setting $\mathrm{h} = 0$ in (2.8) and (2.10), we get the comparable concepts in [36,37], respectively.*

3. Results and Insights

In this section, we will present $(\mathfrak{q}-\mathrm{h})$-Hermite Hadamard inequality along with $(\mathfrak{q}-\mathrm{h})$ Midpoint inequalities and $(\mathfrak{q}-\mathrm{h})$ Trapezoid inequalities. Without a doubt, the most important results of this work, because they contain as particular cases several results known from the literature and because of their theoretical implications in future research.

3.1. $(\mathfrak{q}-\mathrm{h})$-Hermite Hadamard inequality

We now present $(\mathfrak{q}-\mathrm{h})$-Hermite Hadamard inequality in this section. Let us start with the following Result.

Theorem 3.1 *For $1 < \mathfrak{q} < 0$ and $\mathrm{h} \in \mathbb{R}$. Let $\mathfrak{T} : [\mu_1, \mu_2] \subseteq \mathbb{R} \to \mathbb{R}$ be a convex and $(\mathfrak{q}-\mathrm{h})$-integrable function, then the following inequality holds:*

$$\begin{aligned}2\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right) &\leq \frac{1-\mathfrak{q}}{(1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}} \mathfrak{T}(\mathrm{u})_{\mu_1} d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u} + \int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2} \mathfrak{T}(\mathrm{u})^{\mu_2} d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}\right) \\ &\leq \left(\mathfrak{T}(\mu_1) + \mathfrak{T}(\mu_2)\right), \quad (3.1)\end{aligned}$$

where $\mathrm{h}_1 = \frac{(\mu_2-\mu_1)}{2}\mathrm{h}$.

Proof: Since $\mathfrak{T}$ is convex function, we have

$$\mathfrak{T}\left(\frac{\mathrm{u}_1+\mathrm{u}_2}{2}\right) \leq \frac{1}{2}\left(\mathfrak{T}(\mathrm{u}_1) + \mathfrak{T}(\mathrm{u}_2)\right)$$

By taking $\mathrm{u}_1 = \frac{\lambda}{2}\mu_1 + \left(1-\frac{\lambda}{2}\right)\mu_2$ and $\mathrm{u}_2 = \left(1-\frac{\lambda}{2}\right)\mu_1 + \frac{\lambda}{2}\mu_2$, we have the following inequality:

$$2\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right) \leq \mathfrak{T}\left(\frac{\lambda}{2}\mu_1 + \left(1-\frac{\lambda}{2}\right)\mu_2\right) + \mathfrak{T}\left(\left(1-\frac{\lambda}{2}\right)\mu_1 + \frac{\lambda}{2}\mu_2\right) \quad (3.2)$$

By $(\mathfrak{q}-\mathrm{h})$-integrating (3.2) over the interval $[0,1]$, we have:

$$\begin{aligned}2\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)\frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{1-\mathfrak{q}} &\leq \int_0^1 \mathfrak{T}\left(\mu_1 + \lambda\left(\frac{\mu_1+\mu_2}{2}-\mu_1\right)\right) d_{(\mathfrak{q}-\mathrm{h})}\lambda \\ &+ \int_0^1 \mathfrak{T}\left(\mu_2 + \lambda\left(\frac{\mu_1+\mu_2}{2}-\mu_2\right)\right) d_{(\mathfrak{q}-\mathrm{h})}\lambda. \quad (3.3)\end{aligned}$$

After simplification (3.3), we have:

$$2\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right) \leq \frac{1-\mathfrak{q}}{(1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}} \mathfrak{T}(\mathrm{u})_{\mu_1} d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u} + \int_{\frac{\mu_1+\mu_2}{2}}^{\mu_1} \mathfrak{T}(\mathrm{u})^{\mu_2} d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}\right),$$

which is the first inequality of (3.1). Now, we again use the convexity of function $\mathfrak{T}$, we have:

$$\mathfrak{T}\left(\frac{\lambda}{2}\mu_1+\left(1-\frac{\lambda}{2}\right)\mu_2\right)+\mathfrak{T}\left(\left(1-\frac{\lambda}{2}\right)\mu_1+\frac{\lambda}{2}\mu_2\right)\leq\mathfrak{T}\left(\mu_1\right)+\mathfrak{T}\left(\mu_2\right). \tag{3.4}$$

By $(\mathfrak{q}-\mathrm{h})$-integrating (3.4) over the interval $[0,1]$ and after simplification, we have second inequality of (3.1) as follows:

$$\frac{1-\mathfrak{q}}{(1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1}\left(\int_{\mu_1}^{\mu_1+\mu_2}\mathfrak{T}(\mathrm{u})_{\mu_1}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_1}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}\right)\leq\left(\mathfrak{T}\left(\mu_1\right)+\mathfrak{T}\left(\mu_2\right)\right),$$

which completes the proof. □

Remark 3.1 *(i) By setting* $\mathrm{h}=0$ *in* (3.1)*, we attained [[38], Theorem 3.1].*
(ii) By setting $\mathrm{h}=0$ *and* $\mathfrak{q}\to 1$ *in* (3.1)*, we attained classical Hermite Hadamard inequality* (1.2)*.*

In the results from now onward, we utilize the $(\mathfrak{q}-\mathrm{h})$-differentiability of the function to demonstrate some left-approximations of the recently established $(\mathfrak{q}-\mathrm{h})$-Hermite Hadamard inequality (3.1).

3.2. $(\mathfrak{q}-\mathrm{h})$ Midpoint inequalities

In this section, we use the $(\mathfrak{q}-\mathrm{h})$ differentiability of the function to prove some left approximations of the $(\mathfrak{q}-\mathrm{h})$ Hermite Hadamard inequality (2.1). In the whole section, we assume that $1<\mathfrak{q}<0$ and $\mathrm{h}\in\mathbb{R}$.

Lemma 3.1 *For* $1<\mathfrak{q}<0$ *and* $\mathrm{h}\in\mathbb{R}$*. Let* $\mathfrak{T}:[\mu_1,\mu_2]\subseteq\mathbb{R}\to\mathbb{R}$ *be a function and suppose that* ${}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}$ *and* ${}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}$ *are* $(\mathfrak{q}-\mathrm{h})$*-differentiable and* $(\mathfrak{q}-\mathrm{h})$*-integrable over the interval* $[\mu_1,\mu_2]$ *. Then the following equality holds:*

$$\begin{aligned}
&\frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{2\left((1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1\right)}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}}\mathfrak{T}(\mathrm{u})_{\mu_1}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}\right)\\
&-\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\mathfrak{q}\mathrm{h}\left(2\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\frac{\mathfrak{T}\left(\mu_1\right)+\mathfrak{T}\left(\mu_2\right)}{2}\right)\\
&=\frac{\mu_2-\mu_1}{4}\left[\int_0^1\left(\mathfrak{q}(\lambda+\mathrm{h})^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\frac{\lambda}{2}\mu_1+\left(1-\frac{\lambda}{2}\right)\mu_2\right)\right)d_{(\mathfrak{q}-\mathrm{h})}\lambda\right.\\
&\left.-\int_0^1\left(\mathfrak{q}(\lambda+\mathrm{h})_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\left(1-\frac{\lambda}{2}\right)\mu_1+\frac{\lambda}{2}\mu_2\right)\right)d_{(\mathfrak{q}-\mathrm{h})}\lambda\right]\\
&\textit{where } \mathrm{h}_1=\left(\mu_2-\mu_1+\frac{2}{\mathrm{n}\mathfrak{q}^{\mathrm{n}}}\right)\mathrm{h}.
\end{aligned}\tag{3.5}$$

Proof: By Lemma 2.1, we have

$$\begin{aligned}
T_1&=\int_0^1\left(\mathfrak{q}(\lambda+\mathrm{h})^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\frac{\lambda}{2}\mu_1+\left(1-\frac{\lambda}{2}\right)\mu_2\right)\right)d_{(\mathfrak{q}-\mathrm{h})}\lambda\\
&=-\frac{2\mathfrak{q}}{\mu_2-\mu_1}\left(\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)+\mathrm{h}\left(\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\mathfrak{T}\left(\mu_2\right)\right)\right)\\
&+\frac{2\mathfrak{q}}{\mu_2-\mu_1}\int_0^1\mathfrak{T}\left(\frac{\mathfrak{q}\lambda}{2}\mu_1+\left(1-\frac{\mathfrak{q}\lambda}{2}\right)\mu_2+\mathfrak{q}\mathrm{h}\right)d_{(\mathfrak{q}-\mathrm{h})}\lambda\\
&=-\frac{2}{\mu_2-\mu_1}\left(\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)+\mathrm{h}\mathfrak{q}\left(2\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\mathfrak{T}\left(\mu_2\right)\right)\right)+\frac{2}{\mu_2-\mu_1}\\
&\times\frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{(1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1}\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}.
\end{aligned}\tag{3.6}$$

Similarly, by Lemma 2.2, we have the following equality:

$$\begin{aligned}
T_2 &= \int_0^1 \left(\mathfrak{q}(\lambda+\mathrm{h})_{\mu_1} D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\left(1-\frac{\lambda}{2}\right)\mu_1+\frac{\lambda}{2}\mu_2\right)\right) d_{(\mathfrak{q}-\mathrm{h})}\lambda \\
&= \frac{2}{\mu_2-\mu_1}\left(\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)+\mathrm{h}\mathfrak{q}\left(2\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\mathfrak{T}(\mu_1)\right)\right)-\frac{2}{\mu_2-\mu_1} \\
&\times \frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{(1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1}\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}} \mathfrak{T}(\mathrm{u})_{\mu_1} d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}.
\end{aligned}$$

By finding T_1-T_2, and after simplification, we have (3.5). □

Remark 3.2 *(i) By setting* $\mathrm{h}=0$ *in* (3.5)*, we obtained [[38], Lemma 4.1]. (ii) By setting* $\mathfrak{q}\to 1$ *and* $\mathrm{h}=0$ *in* (3.5)*, we obtained [[41], Corollary 1].*

Theorem 3.2 *Let conditions of Lemma 3.1 are satisfied. If* $\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\right|$ *and* $\left|{}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\right|$ *are convex functions, then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{2\left((1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1\right)}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}}\mathfrak{T}(\mathrm{u})_{\mu_1}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}\right)\right. \\
&\left.-\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\mathfrak{q}\mathrm{h}\left(2\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\frac{\mathfrak{T}(\mu_1)+\mathfrak{T}(\mu_2)}{2}\right)\right| \\
&=\left|\frac{\mu_2-\mu_1}{4}\left[\int_0^1\left(\mathfrak{q}(\lambda+\mathrm{h})^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\frac{\lambda}{2}\mu_1+\left(1-\frac{\lambda}{2}\right)\mu_2\right)\right)d_{(\mathfrak{q}-\mathrm{h})}\lambda\right.\right. \\
&\left.\left.-\int_0^1\left(\mathfrak{q}(\lambda+\mathrm{h})_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\left(1-\frac{\lambda}{2}\right)\mu_1+\frac{\lambda}{2}\mu_2\right)\right)d_{(\mathfrak{q}-\mathrm{h})}\lambda\right]\right| \\
&\le \frac{(\mu_2-\mu_1)((1-\mathfrak{q})+\mathfrak{q}\mathrm{h})}{4(1-\mathfrak{q})}\left[\left|\mu_2 D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)\right|\right. \\
&\times\left(\frac{\mathfrak{q}}{2[3]_\mathfrak{q}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_\mathfrak{q}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_\mathfrak{q}}\right)+\frac{\mathfrak{q}\mathrm{h}}{2[2]_\mathfrak{q}}\left(1+\frac{\mathfrak{q}^2 h}{(1-\mathfrak{q})[2]_\mathfrak{q}}\right)\right). \\
&+\left|\mu_2 D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|\left(\frac{\mathfrak{q}}{[2]_\mathfrak{q}}\left(1+\frac{\mathfrak{q}^2\mathrm{h}}{(1-\mathfrak{q})[2]_\mathfrak{q}}\right)\left(1-\frac{\mathrm{h}}{2}\right)+\mathfrak{q}\mathrm{h}-\frac{\mathfrak{q}}{2[3]_\mathfrak{q}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_\mathfrak{q}\right)^2}\right.\right. \\
&\left.\left.+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_\mathfrak{q}}\right)\right)+\left|\mu_1 D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|\left(\frac{\mathfrak{q}}{2[3]_\mathfrak{q}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_\mathfrak{q}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_\mathfrak{q}}\right)\right. \\
&\left.+\frac{\mathfrak{q}\mathrm{h}}{2[2]_\mathfrak{q}}\left(1+\frac{\mathfrak{q}^2\mathrm{h}}{(1-\mathfrak{q})[2]_\mathfrak{q}}\right)\right)+\left|\mu_1 D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)\right|\left(\frac{\mathfrak{q}}{[2]_\mathfrak{q}}\left(1+\frac{\mathfrak{q}\mathrm{h}}{(1-\mathfrak{q})[2]_\mathfrak{q}}\right)\left(1-\frac{\mathrm{h}}{2}\right)\right. \\
&\left.\left.+\mathfrak{q}\mathrm{h}-\frac{\mathfrak{q}}{2[3]_\mathfrak{q}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_\mathfrak{q}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_\mathfrak{q}}\right)\right)\right],
\end{aligned} \tag{3.7}$$

where, $\mathrm{h}_1=\left(\mu_2-\mu_1+\frac{2}{\mathrm{n}\mathfrak{q}^{\mathrm{n}}}\right)\mathrm{h}$.

Proof: Taking modulus of (3.5) along with convexity of $\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\right|$ and $\left|{}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\right|$, we have:

$$\begin{aligned}
&\Bigg|\frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{2\left((1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1\right)}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}}\mathfrak{T}(\mathrm{u})_{\mu_1}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}\right)\\
&-\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\mathfrak{q}\mathrm{h}\left(2\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\frac{\mathfrak{T}(\mu_1)+\mathfrak{T}(\mu_2)}{2}\right)\Bigg|\\
&=\Bigg|\frac{\mu_2-\mu_1}{4}\left[\int_0^1\left(\mathfrak{q}(\lambda+\mathrm{h})^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\frac{\lambda}{2}\mu_1+\left(1-\frac{\lambda}{2}\right)\mu_2\right)\right)d_{(\mathfrak{q}-\mathrm{h})}\lambda\right.\\
&\left.-\int_0^1\left(\mathfrak{q}(\lambda+\mathrm{h})_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\left(1-\frac{\lambda}{2}\right)\mu_1+\frac{\lambda}{2}\mu_2\right)\right)d_{(\mathfrak{q}-\mathrm{h})}\lambda\right]\Bigg|\\
&\leq\frac{\mu_2-\mu_1}{4}\left[\int_0^1(\mathfrak{q}\lambda+\mathfrak{q}\mathrm{h})\left(\frac{\lambda}{2}\left|{}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)\right|+\left(1-\frac{\lambda}{2}\right)\left|{}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|\right)d_{(\mathfrak{q}-\mathrm{h})}\lambda\right.\\
&\left.+\int_0^1(\mathfrak{q}\lambda+\mathfrak{q}\mathrm{h})\left(\left(1-\frac{\lambda}{2}\right)\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})^2}\mathfrak{T}(\mu_1)\right|+\frac{\lambda}{2}\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|\right)d_{(\mathfrak{q}-\mathrm{h})}\lambda\right]\\
&=\frac{(\mu_2-\mu_1)\left((1-\mathfrak{q})+\mathfrak{q}\mathrm{h}\right)}{4(1-\mathfrak{q})}\Bigg[\left|{}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)\right|\\
&\left(\frac{\mathfrak{q}}{2[3]_\mathfrak{q}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_\mathfrak{q}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_\mathfrak{q}}\right)+\frac{\mathfrak{q}\mathrm{h}}{2[2]_\mathfrak{q}}\left(1+\frac{\mathfrak{q}^2\mathrm{h}}{(1-\mathfrak{q})[2]_\mathfrak{q}}\right)\right).\\
&+\left|{}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|\left(\frac{\mathfrak{q}}{[2]_\mathfrak{q}}\left(1+\frac{\mathfrak{q}^2\mathrm{h}}{(1-\mathfrak{q})[2]_\mathfrak{q}}\right)\left(1-\frac{\mathrm{h}}{2}\right)+\mathfrak{q}\mathrm{h}-\frac{\mathfrak{q}}{2[3]_\mathfrak{q}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_\mathfrak{q}\right)^2}\right.\right.\\
&\left.\left.+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_\mathfrak{q}}\right)\right)+\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|\left(\frac{\mathfrak{q}}{2[3]_\mathfrak{q}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_\mathfrak{q}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_\mathfrak{q}}\right)\right.\\
&\left.+\frac{\mathfrak{q}\mathrm{h}}{2[2]_\mathfrak{q}}\left(1+\frac{\mathfrak{q}^2\mathrm{h}}{(1-\mathfrak{q})[2]_\mathfrak{q}}\right)\right)+\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)\right|\left(\frac{\mathfrak{q}}{[2]_\mathfrak{q}}\left(1+\frac{\mathfrak{q}\mathrm{h}}{(1-\mathfrak{q})[2]_\mathfrak{q}}\right)\left(1-\frac{\mathrm{h}}{2}\right)\right.\\
&\left.+\mathfrak{q}\mathrm{h}-\frac{\mathfrak{q}}{2[3]_\mathfrak{q}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_\mathfrak{q}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_\mathfrak{q}}\right)\right)\Bigg],
\end{aligned}$$

which completes the proof. □

Remark 3.3 *(i) By setting* $\mathrm{h}=0$ *in* (3.7)*, we get [[38], Theorem 4.1].*
(ii) By setting $\mathfrak{q}\to 1$ *and* $\mathrm{h}=0$ *in* (3.7)*, we get [[40], Theorem 2.2].*

Theorem 3.3 *Assume that the conditions of Lemma 3.1 hold. If* $\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\right|^{\mathrm{s}}$ *and* $\left|{}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\right|^{\mathrm{s}}$*, for* $\mathrm{s}\geq 1$*, are convex functions, then we have the following inequality:*

$$\begin{aligned}
&\Bigg|\frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{2\left((1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1\right)}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}}\mathfrak{T}(\mathrm{u})_{\mu_1}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}\right)\\
&-\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\mathfrak{q}\mathrm{h}\left(2\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\frac{\mathfrak{T}(\mu_1)+\mathfrak{T}(\mu_2)}{2}\right)\Bigg|
\end{aligned}$$

$$\leq \frac{\mathfrak{q}(\mu_2-\mu_1)((1-\mathfrak{q})+\mathfrak{q}\mathrm{h})^{2-\frac{1}{\mathrm{s}}}}{4(1-\mathfrak{q})[2]_\mathfrak{q}}\left(\left[\frac{[2]_\mathfrak{q}(1-\mathfrak{q})+\mathrm{h}([2]_\mathfrak{q}-\mathfrak{q}^3)}{[2]_\mathfrak{q}(1-\mathfrak{q})^2}\right]\right)^{1-\frac{1}{\mathrm{s}}}$$
$$\times\left(\left(\left|{}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)\right|^{\mathrm{s}}\left(\frac{[2]_\mathfrak{q}}{2[3]_\mathfrak{q}}+\frac{\mathrm{h}^3\mathfrak{q}^3(1+\mathfrak{q}^3)[2]_\mathfrak{q}}{2(1-\mathfrak{q})^2([3]_\mathfrak{q})^3}+\frac{\mathrm{h}\mathfrak{q}^3[2]_\mathfrak{q}}{(1-\mathfrak{q})([3]_\mathfrak{q})^2}+\frac{\mathrm{h}}{2}+\frac{\mathfrak{q}^2\mathrm{h}^2}{2(1-\mathfrak{q})[2]_\mathfrak{q}}\right)\right.\right.$$
$$+\left|{}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|^{\mathrm{s}}$$
$$\left.\left(\left(1+\frac{\mathfrak{q}^2\mathrm{h}}{(1-\mathfrak{q})[2]_\mathfrak{q}}\right)\left(1-\frac{\mathrm{h}}{2}\right)+\mathrm{h}[2]_\mathfrak{q}-\frac{[2]_\mathfrak{q}}{2[3]_\mathfrak{q}}-\frac{\mathrm{h}^3\mathfrak{q}^3(1+\mathfrak{q}^3)[2]_\mathfrak{q}}{2(1-\mathfrak{q})^2([3]_\mathfrak{q})^3}-\frac{2\mathrm{h}\mathfrak{q}^3[2]_\mathfrak{q}}{2(1-\mathfrak{q})([3]_\mathfrak{q})^2}\right)\right)^{\frac{1}{\mathrm{s}}}$$
$$+\left(\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|^{\mathrm{s}}\left(\frac{[2]_\mathfrak{q}}{2[3]_\mathfrak{q}}+\frac{\mathrm{h}^3\mathfrak{q}^3(1+\mathfrak{q}^3)[2]_\mathfrak{q}}{2(1-\mathfrak{q})^2([3]_\mathfrak{q})^3}+\frac{2\mathrm{h}\mathfrak{q}^3[2]_\mathfrak{q}}{2(1-\mathfrak{q})([3]_\mathfrak{q})^2}+\frac{\mathrm{h}}{2}+\frac{\mathfrak{q}^2\mathrm{h}^2}{2(1-\mathfrak{q})[2]_\mathfrak{q}}\right)\right.$$
$$+\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)\right|^{\mathrm{s}}$$
$$\left.\left.\left(\left(1+\frac{\mathfrak{q}^2\mathrm{h}}{(1-\mathfrak{q})[2]_\mathfrak{q}}\right)\left(1-\frac{\mathrm{h}}{2}\right)+\mathrm{h}[2]_\mathfrak{q}-\frac{[2]_\mathfrak{q}}{2[3]_\mathfrak{q}}-\frac{\mathrm{h}^3\mathfrak{q}^3(1+\mathfrak{q}^3)[2]_\mathfrak{q}}{2(1-\mathfrak{q})^2([3]_\mathfrak{q})^3}-\frac{2\mathrm{h}\mathfrak{q}^3[2]_\mathfrak{q}}{2(1-\mathfrak{q})([3]_\mathfrak{q})^2}\right)\right)^{\frac{1}{\mathrm{s}}}\right). \tag{3.8}$$

Proof: By taking the modulus of (3.5), applying power mean inequality, and using the convexity of $\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\right|^{\mathrm{s}}$ and $|{}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}|^{\mathrm{s}}$, we have

$$\left|\frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{2\left((1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1\right)}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}}\mathfrak{T}(\mathrm{u})_{\mu_1}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}\right)\right.$$
$$\left.-\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\mathfrak{q}\mathrm{h}\left(2\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\frac{\mathfrak{T}(\mu_1)+\mathfrak{T}(\mu_2)}{2}\right)\right|$$
$$\leq\frac{\mu_2-\mu_1}{4}\left[\int_0^1\mathfrak{q}(\lambda+\mathrm{h})\left|{}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\frac{\lambda}{2}\mu_1+\left(1-\frac{\lambda}{2}\right)\mu_2\right)\right|d_{(\mathfrak{q}-\mathrm{h})}\lambda\right.$$
$$\left.+\int_0^1\mathfrak{q}(\lambda+\mathrm{h})\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\left(1-\frac{\lambda}{2}\right)\mu_1+\frac{\lambda}{2}\mu_2\right)\right|d_{(\mathfrak{q}-\mathrm{h})}\lambda\right]$$
$$\leq\frac{\mu_2-\mu_1}{4}\left(\int_0^1\mathfrak{q}(\lambda+\mathrm{h})d_{(\mathfrak{q}-\mathrm{h})}\lambda\right)^{1-\frac{1}{\mathrm{s}}}\left[\left(\int_0^1\mathfrak{q}(\lambda+\mathrm{h})\left|{}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\frac{\lambda}{2}\mu_1+\left(1-\frac{\lambda}{2}\right)\mu_2\right)\right|^{\mathrm{s}}\right.\right.$$
$$\left.\left.\times d_{(\mathfrak{q}-\mathrm{h})}\lambda\right)^{\frac{1}{\mathrm{s}}}+\left(\int_0^1\mathfrak{q}(\lambda+\mathrm{h})\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\left(1-\frac{\lambda}{2}\right)\mu_1+\frac{\lambda}{2}\mu_2\right)\right|^{\mathrm{s}}d_{(\mathfrak{q}-\mathrm{h})}\lambda\right)^{\frac{1}{\mathrm{s}}}\right]$$
$$\leq\frac{\mu_2-\mu_1}{4}\left(\int_0^1\mathfrak{q}(\lambda+\mathrm{h})d_{(\mathfrak{q}-\mathrm{h})}\lambda\right)^{1-\frac{1}{\mathrm{s}}}$$
$$\times\left[\left(\int_0^1\mathfrak{q}(\lambda+\mathrm{h})\left(\left|{}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)\right|^{\mathrm{s}}\frac{\lambda}{2}+\left|\mu^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|^{\mathrm{s}}\left(1-\frac{\lambda}{2}\right)\right)d_{(\mathfrak{q}-\mathrm{h})}\lambda\right)^{\frac{1}{\mathrm{s}}}\right.$$
$$\left.+\left(\int_0^1\mathfrak{q}(\lambda+\mathrm{h})\left(\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)\right|^{\mathrm{s}}\left(1-\frac{\lambda}{2}\right)+\frac{\lambda}{2}\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|^{\mathrm{s}}\right)d_{(\mathfrak{q}-\mathrm{h})}\lambda\right)^{\frac{1}{\mathrm{s}}}\right]$$
$$\leq\frac{(\mu_2-\mu_1)\left((1-\mathfrak{q})+\mathfrak{q}\mathrm{h}\right)}{4(1-\mathfrak{q})}\left(\mathfrak{q}((1-\mathfrak{q})+\mathfrak{q}\mathrm{h})\left[\frac{1}{1-\mathfrak{q}^2}+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q}^2)^2}+\frac{\mathrm{h}}{1-\mathfrak{q}}\right]\right)^{1-\frac{1}{\mathrm{s}}}$$

$$\times\left(\left(\left(\left|\mu^{\mu_2} D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\mu_1\right)\right|^{\mathrm{s}}\left(\frac{\mathfrak{q}}{2[3]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_{\mathfrak{q}}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_{\mathfrak{q}}}\right)\right.\right.\right.\right.$$
$$\left.\left.+\frac{\mathfrak{q}\mathrm{h}}{2[2]_{\mathfrak{q}}}\left(1+\frac{\mathfrak{q}^2\mathrm{h}}{(1-\mathfrak{q})[2]_{\mathfrak{q}}}\right)\right)+\left|^{\mu_2} D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\mu_2\right)\right|^{\mathrm{s}}$$
$$\left.\left.\left(\frac{\mathfrak{q}}{[2]_{\mathfrak{q}}}\left(1+\frac{\mathfrak{q}\mathrm{h}}{(1-\mathfrak{q})[2]_{\mathfrak{q}}}\right)\left(1-\frac{\mathrm{h}}{2}\right)+\mathfrak{q}\mathrm{h}-\frac{\mathfrak{q}}{2[3]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_{\mathfrak{q}}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_{\mathfrak{q}}}\right)\right)\right)\right)^{\frac{1}{\mathrm{s}}}$$
$$+\left(\left|\mu_1 D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\mu_2\right)\right|^{\mathrm{s}}\left(\frac{\mathfrak{q}}{2[3]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_{\mathfrak{q}}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_{\mathfrak{q}}}\right)\right.\right.$$
$$\left.+\frac{\mathfrak{q}\mathrm{h}}{2[2]_{\mathfrak{q}}}\left(1+\frac{\mathfrak{q}^2\mathrm{h}}{(1-\mathfrak{q})[2]_{\mathfrak{q}}}\right)\right)+\left|_{\mu_1} D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\mu_1\right)\right|^{\mathrm{s}}$$
$$\left.\left.\left(\frac{\mathfrak{q}}{[2]_{\mathfrak{q}}}\left(1+\frac{\mathfrak{q}\mathrm{h}}{(1-\mathfrak{q})[2]_{\mathfrak{q}}}\right)\left(1-\frac{\mathrm{h}}{2}\right)+\mathfrak{q}\mathrm{h}-\frac{\mathfrak{q}}{2[3]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_{\mathfrak{q}}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_{\mathfrak{q}}}\right)\right)\right)^{\frac{1}{\mathrm{s}}}\right).$$

After simplification, the result is obtained. □

Remark 3.4 *By setting* $\mathrm{h}=0$ *in* (3.8), *we get [[38], Theorem 4.2].*

Theorem 3.4 *Let the assumptions of Lemma 3.1 are satisfied. If* $\left|_{\mu_1} D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\right|^{\mathrm{s}}$ *and* $|^{\mu_2} D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}|^{\mathrm{s}}$, *for* $\mathrm{s}>1$, *are convex functions, then we have the following inequality:*

$$\left|\frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{2\left((1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1\right)}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}}\mathfrak{T}(\mathrm{u})_{\mu_1}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}+\int_{\underline{\mu_1+\mu_2}}^{\mu_2}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}\right)\right.$$
$$\left.-\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\mathfrak{q}\mathrm{h}\left(2\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\frac{\mathfrak{T}\left(\mu_1\right)+\mathfrak{T}\left(\mu_2\right)}{2}\right)\right|$$
$$\leq\frac{\mu_2-\mu_1}{8}\mathfrak{q}((1-\mathfrak{q})+\mathfrak{q}\mathrm{h})^{1+\frac{1}{\mathrm{r}}}\left(\sum_{\mathrm{k}=0}^{\mathrm{r}}\binom{\mathrm{r}}{\mathrm{k}}\mathrm{h}^{\mathrm{r}-\mathrm{k}}\sum_{\mathrm{n}=0}^{\infty}\mathfrak{q}^{(1+\mathrm{k})\mathrm{n}}(1+\mathrm{n}\mathrm{h})^{\mathrm{k}}\right)^{\frac{1}{\mathrm{r}}}$$
$$\times\left[\left(\left|\mu_2 D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\mu_1\right)\right|^{\mathrm{s}}\left(\frac{1}{1-\mathfrak{q}^2}+\frac{\mathrm{h}\mathfrak{q}^2}{\left(1-\mathfrak{q}^2\right)^2}\right)+\left|\mu_2 D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\mu_2\right)\right|^{\mathrm{s}}\left(\frac{\mathfrak{q}+[2]_{\mathfrak{q}}}{1-\mathfrak{q}^2}+\frac{\mathrm{h}\mathfrak{q}^2}{\left(1-\mathfrak{q}^2\right)^2}\right)\right)^{\frac{1}{\mathrm{s}}}\right.$$
$$\left.+\left(\left|_1 D_{(\mathfrak{q}-\mathrm{h})^2}\mathfrak{T}\left(\mu_1\right)\right|^{\mathrm{s}}\left(\frac{\mathfrak{q}+[2]_{\mathfrak{q}}}{1-\mathfrak{q}^2}+\frac{\mathrm{h}\mathfrak{q}^2}{\left(1-\mathfrak{q}^2\right)^2}\right)+\left|_{\mu_1} D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\mu_2\right)\right|^{\mathrm{s}}\left(\frac{1}{1-\mathfrak{q}^2}+\frac{\mathrm{h}\mathfrak{q}^2}{\left(1-\mathfrak{q}^2\right)^2}\right)\right)^{\frac{1}{\mathrm{s}}}\right]$$

where $\frac{1}{\mathrm{s}}+\frac{1}{\mathrm{r}}=1.$ (3.9)

Proof: By taking the modulus of (3.5), applying the Hölder inequality, and then using the convexity of $\left|\mu_1 D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\right|^{\mathrm{s}}$ and $\left|^{\mu_2} D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\right|^{\mathrm{s}}$, we have

$$\left|\frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{2\left((1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1\right)}\left(\int_{\mu_1}^{\underline{\mu_1+\mu_2}}\boldsymbol{T}(\mathrm{u})_{\mu_1}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}\right)\right.$$
$$\left.-\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\mathfrak{q}\mathrm{h}\left(2\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\frac{\mathfrak{T}\left(\mu_1\right)+\mathfrak{T}\left(\mu_2\right)}{2}\right)\right|$$

$$\leq \frac{\mu_2-\mu_1}{4}\left[\int_0^1 \mathfrak{q}(\lambda+\mathrm{h})\left|\mu^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\frac{\lambda}{2}\mu_1+\left(1-\frac{\lambda}{2}\right)\mu_2\right)\right|d_{(\mathfrak{q}-\mathrm{h})}\lambda\right.$$
$$\left.-\int_0^1 \mathfrak{q}(\lambda+\mathrm{h})\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\left(1-\frac{\lambda}{2}\right)\mu_1+\frac{\lambda}{2}\mu_2\right)\right|d_{(\mathfrak{q}-\mathrm{h})}\lambda\right]$$
$$\leq \frac{\mu_2-\mu_1}{4}\left(\int_0^1(\mathfrak{q}\lambda+\mathfrak{q}\mathrm{h})^{\mathrm{r}}\right)^{\frac{1}{\mathrm{r}}}\left[\left(\int_0^1\left|\mu_2 D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\frac{\lambda}{2}\mu_1+\left(1-\frac{\lambda}{2}\right)\mu_2\right)\right|^{\mathrm{s}}d_{(\mathfrak{q}-\mathrm{h})}\lambda\right)^{\frac{1}{\mathrm{s}}}\right.$$
$$\left.-\left(\int_0^1\left|\mu_1 D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\left(1-\frac{\lambda}{2}\right)\mu_1+\frac{\lambda}{2}\mu_2\right)\right|^{\mathrm{s}}d_{(\mathfrak{q}-\mathrm{h})}\lambda\right)^{\frac{1}{\mathrm{s}}}\right]$$
$$\leq \frac{\mu_2-\mu_1}{4}\left(\int_0^1(\mathfrak{q}\lambda+\mathfrak{q}\mathrm{h})^{r}d_{(\mathfrak{q}-\mathrm{h})}\lambda\right)^{\frac{1}{r}}\left[\left(\int_0^1\left(\left|\mu^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)\right|^{\mathrm{s}}\frac{\lambda}{2}+\left|\mu^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|^{\mathrm{s}}\right.\right.\right.$$
$$\left.\left.\left.\left(1-\frac{\lambda}{2}\right)\right)d_{(\mathfrak{q}-\mathrm{h})}\lambda\right)^{\frac{1}{\mathrm{s}}}+\left(\int_0^1\left(\left|\mu_1 D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)\right|^{\mathrm{s}}\left(1-\frac{\lambda}{2}\right)+\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|^{\mathrm{s}}\frac{\lambda}{2}\right)d_{(\mathfrak{q}-\mathrm{h})}\lambda\right)^{\frac{1}{\mathrm{s}}}\right]$$
$$\leq \frac{\mu_2-\mu_1}{4}\left(\mathfrak{q}^{\mathrm{r}}((1-\mathfrak{q})+\mathfrak{q}\mathrm{h})\sum_{\mathrm{k}=0}^{\mathrm{r}}\binom{\mathrm{r}}{\mathrm{k}}\mathrm{h}^{\mathrm{r}-\mathrm{k}}\sum_{\mathrm{n}=0}^{\infty}\mathfrak{q}^{(1+\mathrm{k})\mathrm{n}}(1+\mathrm{nh})^{\mathrm{k}}\right)^{\frac{1}{\mathrm{r}}}\frac{((1-\mathfrak{q})+\mathfrak{q}\mathrm{h})}{2}$$
$$\times\left[\left(\left|\mu^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})^2}\mathfrak{T}(\mu_1)\right|^{\mathrm{s}}\left(\frac{1}{1-\mathfrak{q}^2}+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q}^2)^2}\right)+\left|\mu^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|^{\mathrm{s}}\left(\frac{\mathfrak{q}+[2]_{\mathfrak{q}}}{1-\mathfrak{q}^2}+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q}^2)^2}\right)\right)^{\frac{1}{\mathrm{s}}}\right.$$
$$\left.+\left(\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})^2}\mathfrak{T}(\mu_1)\right|^{\mathrm{s}}\left(\frac{\mathfrak{q}+[2]_{\mathfrak{q}}}{1-\mathfrak{q}^2}+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q}^2)^2}\right)+\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|^{\mathrm{s}}\left(\frac{1}{1-\mathfrak{q}^2}+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q}^2)^2}\right)\right)^{\frac{1}{\mathrm{s}}}\right],$$

which completes the proof. □

Remark 3.5 *By setting* $\mathrm{h}=0$ *in* (3.9)*, we get [[38], Theorem 4.3].*

3.3. $(\mathfrak{q}-\mathrm{h})$ Trapezoid inequalities

In this section, we use the $(\mathfrak{q}-\mathrm{h})$ differentiability of the function to prove some approximations of the $(\mathfrak{q}-\mathrm{h})$ Hermite Hadamard inequality (2.1). In the whole section, we assume that $1<\mathfrak{q}<0$ and $\mathrm{h}\in\mathbb{R}$.

Lemma 3.2 *Let* $\mathfrak{T}:[\mu_1,\mu_2]\subseteq\mathbb{R}\to\mathbb{R}$ *be a function such that* ${}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}$ *and* ${}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}$ *are* $(\mathfrak{q}-\mathrm{h})$*-differentiable and* $(\mathfrak{q}-\mathrm{h})$*-integrate over the interval* $[\mu_1,\mu_2]$*. Then the following equality holds:*

$$\frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{(1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}}{}_{\mu_1}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}(\mathrm{u})_{\mu_1}d_{\mathfrak{q}-\mathrm{h}_1}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2}\mu_2 D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{\mathfrak{q}-\mathrm{h}_1}\mathrm{u}\right)$$
$$+\frac{4}{\mu_2-\mu_1}\left((1+2\mathrm{h}\mathfrak{q})\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\mathrm{h}\mathfrak{q}\left(\mathfrak{T}(\mu_1)+\mathfrak{T}(\mu_2)\right)\right)$$
$$-\frac{2(1-\mathfrak{q})}{(\mu_2-\mu_1)\left((1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1\right)}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}}\mathfrak{T}(\mathrm{u})_{\mu_1}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}\right)$$
$$=\int_0^1(1-\mathfrak{q}(\lambda+\mathrm{h}))^{\mu_2}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}\left(\frac{\lambda}{2}\mu_1+\left(1-\frac{\lambda}{2}\right)\mu_2\right)d_{\mathfrak{q}-\mathrm{h}}\lambda+\int_0^1(\mathfrak{q}(\lambda+\mathrm{h})-1)$$
$${}_{\mu_1}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}\left(\left(1-\frac{\lambda}{2}\right)\mu_1+\frac{\lambda}{2}\mu_2\right)d_{\mathfrak{q}-\mathrm{h}}\lambda,$$

where $\mathrm{h}_1=\dfrac{\mu_2-\mu_1}{2}\mathrm{h}.$ (3.10)

Proof: Let's take

$$
\begin{aligned}
&\int_0^1 (1-\mathfrak{q}(\lambda+\mathrm{h}))^{\mu_2} D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}\left(\frac{\lambda}{2}\mu_1+\left(1-\frac{\lambda}{2}\right)\mu_2\right) d_{\mathfrak{q}-\mathrm{h}}\lambda \\
&= \int_0^1 {}^{\mu_2}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}\left(\frac{\lambda}{2}\mu_1+\left(1-\frac{\lambda}{2}\right)\mu_2\right) d_{\mathfrak{q}-\mathrm{h}}\lambda \\
&-\int_0^1 (\mathfrak{q}\lambda+\mathfrak{q}\mathrm{h})^{\mu_2} D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}\left(\frac{\lambda}{2}\mu_1+\left(1-\frac{\lambda}{2}\right)\mu_2\right) d_{\mathfrak{q}-\mathrm{h}}\lambda,
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
&\int_0^1 \mu_2 D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}\left(\frac{\lambda}{2}\mu_1+\left(1-\frac{\lambda}{2}\right)\mu_2\right) d_{\mathfrak{q}-\mathrm{h}}\lambda \\
&= \frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{(1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1}\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2} \mu_{\mathfrak{q}-\mathrm{h}}D_{\mathfrak{q}}\mathfrak{T}(\mathrm{u})^{\mu_2} d_{\mathfrak{q}-\mathrm{h}_1}\mathrm{u}.
\end{aligned}
\tag{3.12}
$$

By equality (3.6), we have

$$
\begin{aligned}
&\int_0^1 \left(\mathfrak{q}(\lambda+\mathrm{h})^{\mu_2} D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\frac{\lambda}{2}\mu_1+\left(1-\frac{\lambda}{2}\right)\mu_2\right)\right) d_{(\mathfrak{q}-\mathrm{h})}\lambda \\
&= -\frac{2}{\mu_2-\mu_1}\left(\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)+\mathrm{h}\mathfrak{q}\left(2\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\mathfrak{T}(\mu_2)\right)\right)+\frac{2}{\mu_2-\mu_1} \\
&\times \frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{(1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1}\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2} \mathfrak{T}(\mathrm{u})^{\mu_2} d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}.
\end{aligned}
\tag{3.13}
$$

From (3.12) and (3.13), an equation (3.11) becomes

$$
\begin{aligned}
&\int_0^1 \Big((1-\mathfrak{q}(\lambda+\mathrm{h}))\Big)^{\mu_2} D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}\left(\frac{\lambda}{2}\mu_1+\left(1-\frac{\lambda}{2}\right)\mu_2\right) d_{\mathfrak{q}-\mathrm{h}}\lambda \\
&= \frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{(1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1}\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2} \mu_2 D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}(\mathrm{u})^{\mu_2} d_{\mathfrak{q}-\mathrm{h}}\mathrm{u} \\
&+\frac{2}{\mu_2-\mu_1}\left(\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)+\mathrm{h}\mathfrak{q}\left(2\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)+\mathfrak{T}(\mu_2)\right)\right)-\frac{2}{\mu_2-\mu_1} \\
&\times \frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{(1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1}\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2} \mathfrak{T}(\mathrm{u})^{\mu_2} d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}.
\end{aligned}
\tag{3.14}
$$

Similarly, we have

$$
\begin{aligned}
&\int_0^1 (\mathfrak{q}(\lambda+\mathrm{h})-1)_{\mu_1} D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}\left(\left(1-\frac{\lambda}{2}\right)\mu_1+\frac{\lambda}{2}\mu_2\right) d_{\mathfrak{q}-\mathrm{h}}\lambda \\
&= \frac{2}{\mu_2-\mu_1}\left(\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)+\mathrm{h}\mathfrak{q}\left(2\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\mathfrak{T}(\mu_1)\right)\right)-\frac{2}{\mu_2-\mu_1} \\
&\times \frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{(1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1}\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}} \mathfrak{T}(\mathrm{u})_{\mu_1} d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}-\frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{(1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1} \\
&\times \int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}} {}_{\mu_1}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}(\mathrm{u})_{\mu_1} d_{\mathfrak{q}-\mathrm{h}_1}\mathrm{u}.
\end{aligned}
\tag{3.15}
$$

On simplifying (3.14) and (3.15) , we get (3.10). □

Remark 3.6 *By setting* $\mathrm{h}=0$ *in* (3.10) *, we get [[38], Lemma 5.1].*

Theorem 3.5 *Consider the assumptions of Lemma 3.2 are satisfied. If* $\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\right|$ *and* $\left|{}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\right|$ *are convex functions, then we have the following inequality:*

$$\begin{aligned}
&\Bigg|\frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{(1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}}{}_{\mu_1}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}(\mathrm{u})_{\mu_1}d_{\mathfrak{q}-\mathrm{h}_1}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2}{}_{\mu_2}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{\mathfrak{q}-\mathrm{h}_1}\mathrm{u}\right)\\
&+\frac{4}{\mu_2-\mu_1}\left((1+2\mathrm{h}\mathfrak{q})\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\mathrm{h}\mathfrak{q}\left(\mathfrak{T}(\mu_1)+\mathfrak{T}(\mu_2)\right)\right)\\
&-\frac{2(1-\mathfrak{q})}{(\mu_2-\mu_1)\left((1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1\right)}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}}\mathfrak{T}(\mathrm{u})_{\mu_1}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}\right)\Bigg|\\
&\leq\frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{2(1-\mathfrak{q})}\Big[\left|\mu^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)\right|\\
&\times\left(\frac{1-\mathfrak{q}\mathrm{h}}{[2]_\mathfrak{q}}\left(1+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q})[2]_\mathfrak{q}}\right)-\frac{\mathfrak{q}}{[3]_\mathfrak{q}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_\mathfrak{q}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_\mathfrak{q}}\right)\right)+\left|{}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|\\
&\times\left(2(1-\mathfrak{q}\mathrm{h})-\frac{1+2\mathfrak{q}+\mathfrak{q}\mathrm{h}}{[2]_\mathfrak{q}}\left(1+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q})[2]_\mathfrak{q}}\right)+\frac{\mathfrak{q}}{[3]_\mathfrak{q}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_\mathfrak{q}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_\mathfrak{q}}\right)\right)\Big]\\
&+\frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{2(1-\mathfrak{q})}\Big[\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})^2}\mathfrak{T}(\mu_1)\right|\\
&\times\left(2(1-\mathfrak{q}\mathrm{h})-\frac{1+2\mathfrak{q}+\mathfrak{q}\mathrm{h}}{[2]_\mathfrak{q}}\left(1+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q})[2]_\mathfrak{q}}\right)+\frac{\mathfrak{q}}{[3]_\mathfrak{q}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_\mathfrak{q}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_\mathfrak{q}}\right)\right)\\
&+\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|\left(\frac{1-\mathfrak{q}\mathrm{h}}{[2]_\mathfrak{q}}\left(1+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q})[2]_\mathfrak{q}}\right)-\frac{\mathfrak{q}}{[3]_\mathfrak{q}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_\mathfrak{q}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_\mathfrak{q}}\right)\right)\Big].
\end{aligned}\tag{3.16}$$

Proof: On taking the modulus of (3.10) and using the convexity of $\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\right|$ and $\left|{}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\right|$, we have

$$\begin{aligned}
&\Bigg|\frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{(1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}}{}_{\mu_1}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}(\mathrm{u})_{\mu_1}d_{\mathfrak{q}-\mathrm{h}_1}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2}{}_{\mu_2}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{\mathfrak{q}-\mathrm{h}_1}\mathrm{u}\right)\\
&+\frac{4}{\mu_2-\mu_1}\left((1+2\mathrm{h}\mathfrak{q})\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\mathrm{h}\mathfrak{q}\left(\mathfrak{T}(\mu_1)+\mathfrak{T}(\mu_2)\right)\right)\\
&-\frac{2(1-\mathfrak{q})}{(\mu_2-\mu_1)\left((1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1\right)}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}}\mathfrak{T}(\mathrm{u})_{\mu_1}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}\right)\Bigg|\\
&\leq\int_0^1(1-\mathfrak{q}(\lambda+\mathrm{h}))\left|{}^{\mu_2}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}\left(\frac{\lambda}{2}\mu_1+\left(1-\frac{\lambda}{2}\right)\mu_2\right)\right|d_{\mathfrak{q}-\mathrm{h}}\lambda+\int_0^1(1-\mathfrak{q}(\lambda+\mathrm{h}))\\
&\left|{}_{\mu_1}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}\left(\left(1-\frac{\lambda}{2}\right)\mu_1+\frac{\lambda}{2}\mu_2\right)\right|d_{\mathfrak{q}-\mathrm{h}}\lambda\\
&\leq\int_0^1(1-\mathfrak{q}(\lambda+\mathrm{h}))\left(\frac{\lambda}{2}\left|{}^{\mu_2}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}(\mu_1)\right|+\left(1-\frac{\lambda}{2}\right)\left|{}^{\mu_2}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}(\mu_2)\right|\right)d_{\mathfrak{q}-\mathrm{h}}\lambda\\
&+\int_0^1(1-\mathfrak{q}(\lambda+\mathrm{h}))\left(\left(1-\frac{\lambda}{2}\right)\left|{}_{\mu_1}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}(\mu_1)\right|+\frac{\lambda}{2}\left|{}_{\mu_1}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}(\mu_2)\right|\right)d_{\mathfrak{q}-\mathrm{h}}\lambda\\
&=\frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{(1-\mathfrak{q})}\Big[\left|{}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)\right|
\end{aligned}$$

$$\times\left(\frac{1-\mathfrak{q}\mathrm{h}}{2[2]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q})[2]_{\mathfrak{q}}}\right)-\frac{\mathfrak{q}}{2[3]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_{\mathfrak{q}}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_{\mathfrak{q}}}\right)\right)+\left|{}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\mu_2\right)\right|$$
$$\times\left((1-\mathfrak{q}\mathrm{h})-\frac{1+2q+\mathfrak{q}\mathrm{h}}{2[2]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q})[2]_{\mathfrak{q}}}\right)+\frac{\mathfrak{q}}{2[3]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_{\mathfrak{q}}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_{\mathfrak{q}}}\right)\right)\Bigg]$$
$$+\frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{(1-\mathfrak{q})}\left[\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\mu_1\right)\right|\right.$$
$$\times\left((1-\mathfrak{q}\mathrm{h})-\frac{1+2\mathfrak{q}+\mathfrak{q}\mathrm{h}}{2[2]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q})[2]_{\mathfrak{q}}}\right)+\frac{\mathfrak{q}}{2[3]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_{\mathfrak{q}}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_{\mathfrak{q}}}\right)\right)$$
$$+\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\mu_2\right)\right|\left(\frac{1-\mathfrak{q}\mathrm{h}}{2[2]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q})[2]_{\mathfrak{q}}}\right)-\frac{\mathfrak{q}}{2[3]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_{\mathfrak{q}}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_{\mathfrak{q}}}\right)\right)\Bigg].$$

Therefore, the result follows. □

Remark 3.7 *(i) By setting* $\mathrm{h}=0$ *in* (3.16)*, we get [[38], Theorem 5.1].*
(ii) By setting $\mathrm{h}=0$ *and* $\mathfrak{q}\to 1$ *in* (3.16)*, we get [[39], Theorem 2.2].*

Theorem 3.6 *Assume that the conditions of Lemma 3.2 hold. If* $\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\right|^{\mathrm{s}}$ *and* $\left|\,\right|^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\right|^{\mathrm{s}}$, $\mathrm{s}\geq 1$, *are convex functions, then we have the following inequality:*

$$\left|\frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{(1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}}\mu_1 D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}(\mathrm{u})_{\mu_1}d_{\mathfrak{q}-\mathrm{h}_1}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2}\mu_{\mu_2}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{\mathfrak{q}-\mathrm{h}_1}\mathrm{u}\right)\right.$$
$$+\frac{4}{\mu_2-\mu_1}\left((1+2\mathrm{h}\mathfrak{q})\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\mathrm{h}\mathfrak{q}\left(\mathfrak{T}\left(\mu_1\right)+\mathfrak{T}\left(\mu_2\right)\right)\right)$$
$$\left.-\frac{2(1-\mathfrak{q})}{\left(\mu_2-\mu_1\right)\left((1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1\right)}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}}\mathfrak{T}(\mathrm{u})_{\mu_1}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{\mu_2}}^{\mu_2}(\mathrm{u})^{\mu_2}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}\right)\right|$$
$$\leq\left(\frac{(1-\mathfrak{q})+\mathrm{h}\mathfrak{q}}{(1-\mathfrak{q})}\right)^{2-\frac{1}{\mathrm{s}}}\left(\frac{[2]_{\mathfrak{q}}(1-\mathfrak{q})-\mathrm{h}\mathfrak{q}\left(1-\mathfrak{q}-\mathfrak{q}^3\right)}{[2]_{\mathfrak{q}}\left(1-\mathfrak{q}^2\right)}\right)^{1-\frac{1}{\mathrm{s}}}$$
$$\times\left[\left(\left|\mu_2 D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\mu_1\right)\right|^{\mathrm{s}}\left(\frac{1-\mathfrak{q}\mathrm{h}}{[2]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q})[2]_{\mathfrak{q}}}\right)-\frac{\mathfrak{q}}{[3]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_{\mathfrak{q}}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_{\mathfrak{q}}}\right)\right)\right.\right.$$
$$+\left|\mu^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\mu_2\right)\right|^{\mathrm{s}}$$
$$\left.\times\left(2(1-\mathfrak{q}\mathrm{h})-\frac{1+2\mathfrak{q}+\mathfrak{q}\mathrm{h}}{[2]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q})[2]_{\mathfrak{q}}}\right)+\frac{\mathfrak{q}}{[3]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_{\mathfrak{q}}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_{\mathfrak{q}}}\right)\right)\right)^{\frac{1}{\mathrm{s}}}$$
$$+\left(\left|\mu_1 D_{(\mathfrak{q}-\mathrm{h})^2}\mathfrak{T}\left(\mu_1\right)\right|^{\mathrm{s}}\right.$$
$$\times\left(2(1-\mathfrak{q}\mathrm{h})-\frac{1+2\mathfrak{q}+\mathfrak{q}\mathrm{h}}{[2]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q})[2]_{\mathfrak{q}}}\right)+\frac{\mathfrak{q}}{[3]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_{\mathfrak{q}}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_{\mathfrak{q}}}\right)\right)$$
$$\left.\left.+\left|\mu_1 D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\mu_2\right)\right|^{\mathrm{s}}\left(\frac{1-\mathfrak{q}\mathrm{h}}{[2]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q})[2]_{\mathfrak{q}}}\right)-\frac{\mathfrak{q}}{[3]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_{\mathfrak{q}}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_{\mathfrak{q}}}\right)\right)\right)^{\frac{1}{\mathrm{s}}}\right].\tag{3.17}$$

Proof: Let's take the modulus of (3.10) and apply the power mean inequality, then using the convexity of $\left|\mu_1 D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\right|^{\mathrm{s}}$ and $\left|^{\mu_2} D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\right|^{\mathrm{s}}$, we have

$$
\begin{aligned}
&\Bigg|\frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{(1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}} {}_{\mu_1}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}(\mathrm{u})_{\mu_1}d_{\mathfrak{q}-\mathrm{h}_1}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2} {}_{\mu_{\mu_2}}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{\mathfrak{q}-\mathrm{h}_1}\mathrm{u}\right)\\
&+\frac{4}{\mu_2-\mu_1}\left((1+2\mathrm{h}\mathfrak{q})\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\mathrm{h}\mathfrak{q}\left(\mathfrak{T}(\mu_1)+\mathfrak{T}(\mu_2)\right)\right)\\
&-\frac{2(1-\mathfrak{q})}{(\mu_2-\mu_1)\left((1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1\right)}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}}\mathfrak{T}(\mathrm{u})_{\mu_1}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}\right)\Bigg|\\
&\leq\left|\int_0^1(1-\mathfrak{q}(\lambda+\mathrm{h}))\,{}^{\mu_2}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}\left(\frac{\lambda}{2}\mu_1+\left(1-\frac{\lambda}{2}\right)\mu_2\right)d_{\mathfrak{q}-\mathrm{h}}\lambda\right|\\
&\quad+\int_0^1(1-\mathfrak{q}(\lambda+\mathrm{h}))\left|{}_{\mu_1}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}\left(\left(1-\frac{\lambda}{2}\right)\mu_1+\frac{\lambda}{2}\mu_2\right)\right|d_{\mathfrak{q}-\mathrm{h}}\lambda\\
&\leq\left(\int_0^1(1-\mathfrak{q}(\lambda+\mathrm{h}))d_{(\mathfrak{q}-\mathrm{h})}\lambda\right)^{1-\frac{1}{\mathrm{s}}}\\
&\quad\times\Bigg[\left(\int_0^1(1-\mathfrak{q}(\lambda+\mathrm{h}))\left|{}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\frac{\lambda}{2}\mu_1+\left(1-\frac{\lambda}{2}\right)\mu_2\right)\right|^{\mathrm{s}}d_{(\mathfrak{q}-\mathrm{h})}\lambda\right)^{\frac{1}{\mathrm{s}}}\\
&\quad+\left(\int_0^1(1-\mathfrak{q}(\lambda+\mathrm{h}))\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\left(1-\frac{\lambda}{2}\right)\mu_1+\frac{\lambda}{2}\mu_2\right)\right|^{\mathrm{s}}d_{(\mathfrak{q}-\mathrm{h})}\lambda\right)^{\frac{1}{\mathrm{s}}}\Bigg]\\
&\leq\left(\frac{(1-\mathfrak{q})+\mathrm{h}\mathfrak{q}}{(1-\mathfrak{q})}\right)^{1-\frac{1}{\mathrm{s}}}\left(\frac{[2]_{\mathfrak{q}}(1-\mathfrak{q})-\mathrm{h}\mathfrak{q}(1-\mathfrak{q}-\mathfrak{q}^3)}{[2]_{\mathfrak{q}}(1-\mathfrak{q}^2)}\right)^{1-\frac{1}{\mathrm{s}}}\\
&\quad\times\Bigg[\left(\int_0^1(1-\mathfrak{q}(\lambda+\mathrm{h}))\left(\left|\mu^2D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)\right|^{\mathrm{s}}\frac{\lambda}{2}+\left|{}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|^{\mathrm{s}}\left(1-\frac{\lambda}{2}\right)\right)d_{(\mathfrak{q}-\mathrm{h})}\lambda\right)^{\frac{1}{\mathrm{s}}}\\
&\quad+\left(\int_0^1(1-\mathfrak{q}(\lambda+\mathrm{h}))\left(\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)\right|^{\mathrm{s}}\left(1-\frac{\lambda}{2}\right)+\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|^{\mathrm{s}}\frac{\lambda}{2}\right)d_{(\mathfrak{q}-\mathrm{h})}\lambda\right)^{\frac{1}{\mathrm{s}}}\Bigg]\\
&=\left(\frac{(1-\mathfrak{q})+\mathrm{h}\mathfrak{q}}{(1-\mathfrak{q})}\right)^{2-\frac{1}{\mathrm{s}}}\left(\frac{[2]_{\mathfrak{q}}(1-\mathfrak{q})-\mathrm{h}\mathfrak{q}(1-\mathfrak{q}-\mathfrak{q}^3)}{[2]_{\mathfrak{q}}(1-\mathfrak{q}^2)}\right)^{1-\frac{1}{\mathrm{s}}}\\
&\quad\times\Bigg[\Bigg(\left|\mu_2D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)\right|^{\mathrm{s}}\Bigg(\frac{1-\mathfrak{q}\mathrm{h}}{[2]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q})[2]_{\mathfrak{q}}}\right)\\
&\quad-\frac{\mathfrak{q}}{[3]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3(1+\mathfrak{q}^3)}{(1-\mathfrak{q})^2([3]_{\mathfrak{q}})^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_{\mathfrak{q}}}\right)\Bigg)\\
&\quad+\left|\mu^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|^{\mathrm{s}}\Bigg(2(1-\mathfrak{q}\mathrm{h})-\frac{1+2\mathfrak{q}+\mathfrak{q}\mathrm{h}}{[2]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q})[2]_{\mathfrak{q}}}\right)\\
&\quad+\frac{\mathfrak{q}}{[3]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3(1+\mathfrak{q}^3)}{(1-\mathfrak{q})^2([3]_{\mathfrak{q}})^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_{\mathfrak{q}}}\right)\Bigg)\Bigg)^{\frac{1}{\mathrm{s}}}\\
&\quad+\Bigg(\left|\mu_1D_{(\mathfrak{q}-\mathrm{h})^2}\mathfrak{T}(\mu_1)\right|^{\mathrm{s}}\Bigg(2(1-\mathfrak{q}\mathrm{h})-\frac{1+2\mathfrak{q}+\mathfrak{q}\mathrm{h}}{[2]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q})[2]_{\mathfrak{q}}}\right)+\frac{\mathfrak{q}}{[3]_{\mathfrak{q}}}\\
&\quad\times\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3(1+\mathfrak{q}^3)}{(1-\mathfrak{q})^2([3]_{\mathfrak{q}})^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_{\mathfrak{q}}}\right)\Bigg)+\left|\mu_1D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|^{\mathrm{s}}\\
&\quad\times\left(\frac{1-\mathfrak{q}\mathrm{h}}{[2]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q})[2]_{\mathfrak{q}}}\right)-\frac{\mathfrak{q}}{[3]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3(1+\mathfrak{q}^3)}{(1-\mathfrak{q})^2([3]_{\mathfrak{q}})^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_{\mathfrak{q}}}\right)\right)\Bigg)^{\frac{1}{\mathrm{s}}}\Bigg]
\end{aligned}
$$

After simplification, the result follows. □

Remark 3.8 *By setting* $\mathrm{h}=0$ *in* (3.17), *we get [[38], Theorem 5.2].*

Theorem 3.7 *Assume that the conditions of Lemma 3.2 hold. If* $\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\right|^{\mathrm{s}}$ *and* $\left|{}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\right|^{\mathrm{s}}$, $\mathrm{s}>1$, *are convex functions, then we have the following inequality: Let* $\mathrm{s}>1$, *be convex functions, then we obtain:*

$$\begin{aligned}
&\Bigg|\frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{(1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}}{}_{\mu_1}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}(\mathrm{u})\,{}_{\mu_1}d_{\mathfrak{q}-\mathrm{h}_1}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2}{}_{\mu_2}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{\mathfrak{q}-\mathrm{h}_1}\mathrm{u}\right)\\
&+\frac{4}{\mu_2-\mu_1}\left((1+2\mathrm{h}\mathfrak{q})\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\mathrm{h}\mathfrak{q}\left(\mathfrak{T}(\mu_1)+\mathfrak{T}(\mu_2)\right)\right)\\
&-\frac{2(1-\mathfrak{q})}{(\mu_2-\mu_1)((1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1)}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}}\mathfrak{T}(\mathrm{u})\,{}_{\mu_1}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}\right)\Bigg|\\
&\le\left(\sum_{\mathrm{n}=0}^{\infty}\mathfrak{q}^{\mathrm{n}}A_{\mathrm{n}}\right)^{\frac{1}{\mathrm{r}}}\frac{((1-\mathfrak{q})+\mathfrak{q}\mathrm{h})^{1+\frac{1}{\mathrm{r}}}}{2}\\
&\times\Bigg[\left(|{}_{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)|^{\mathrm{s}}\left(\frac{1}{1-\mathfrak{q}^2}+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q}^2)^2}\right)+|{}_{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)|^{\mathrm{s}}\left(\frac{\mathfrak{q}+[2]_{\mathfrak{q}}}{1-\mathfrak{q}^2}+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q}^2)^2}\right)\right)^{\frac{1}{\mathrm{s}}}\\
&+\left(|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)|^{\mathrm{s}}\left(\frac{\mathfrak{q}+[2]_{\mathfrak{q}}}{1-\mathfrak{q}^2}+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q}^2)^2}\right)+|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)|^{\mathrm{s}}\left(\frac{1}{1-\mathfrak{q}^2}+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q}^2)^2}\right)\right)^{\frac{1}{\mathrm{s}}}\Bigg],\quad(3.18)
\end{aligned}$$

where $\frac{1}{\mathrm{s}}+\frac{1}{\mathrm{r}}=1$ *and* $A_{\mathrm{n}}=\left(1-\mathfrak{q}^{\mathrm{n}+1}-\mathfrak{q}\mathrm{h}(1+\mathrm{n}\mathfrak{q}^{\mathrm{n}})\right)^{\mathrm{r}}$.

Proof: Let's take the modulus of (3.10) and apply the Hölder inequality, then using the convexity of $\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\right|^{\mathrm{s}}$ and $|{}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}|^{\mathrm{s}}$, we have

$$\begin{aligned}
&\Bigg|\frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{(1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}}{}_{\mu_1}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}(\mathrm{u})_{\mu_1}d_{\mathfrak{q}-\mathrm{h}_1}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2}{}_{\mu_2}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{\mathfrak{q}-\mathrm{h}_1}\mathrm{u}\right)\\
&+\frac{4}{\mu_2-\mu_1}\left((1+2\mathrm{h}\mathfrak{q})\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\mathrm{h}\mathfrak{q}\left(\mathfrak{T}\left(\mu_1\right)+\mathfrak{T}\left(\mu_2\right)\right)\right)\\
&-\frac{2(1-\mathfrak{q})}{(\mu_2-\mu_1)\left((1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1\right)}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}}\mathfrak{T}(\mathrm{u})_{\mu_1}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}\right)\Bigg|\\
&\le\int_0^1(1-\mathfrak{q}(\lambda+\mathrm{h}))\left|{}^{\mu_2}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}\left(\frac{\lambda}{2}\mu_1+\left(1-\frac{\lambda}{2}\right)\mu_2\right)\right|d_{\mathfrak{q}-\mathrm{h}}\lambda+\int_0^1(1-\mathfrak{q}(\lambda+\mathrm{h}))\\
&\left|{}_{\mu_1}D_{\mathfrak{q}-\mathrm{h}}\mathfrak{T}\left(\left(1-\frac{\lambda}{2}\right)\mu_1+\frac{\lambda}{2}\mu_2\right)\right|d_{\mathfrak{q}-\mathrm{h}}\lambda
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\int_0^1 (1-\mathfrak{q}(\lambda+\mathrm{h}))^{\mathrm{r}} d_{(\mathfrak{q}-\mathrm{h})}\lambda\right)^{\frac{1}{\mathrm{r}}} \left[\left(\int_0^1 \left|{}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\frac{\lambda}{2}\mu_1+\left(1-\frac{\lambda}{2}\right)\mu_2\right)\right|^{\mathrm{s}} d_{(\mathfrak{q}-\mathrm{h})}\lambda\right)^{\frac{1}{\mathrm{s}}}\right.\\
&\left.+\left(\int_0^1 \left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}\left(\left(1-\frac{\lambda}{2}\right)\mu_1+\frac{\lambda}{2}\mu_2\right)\right|^{\mathrm{s}} d_{(\mathfrak{q}-\mathrm{h})}\lambda\right)^{\frac{1}{\mathrm{s}}}\right]\\
&\leq \left(\int_0^1 (1-\mathfrak{q}(\lambda+\mathrm{h}))^{\mathrm{r}} d_{(\mathfrak{q}-\mathrm{h})}\lambda\right)^{\frac{1}{\mathrm{r}}}\\
&\times\left[\left(\int_0^1 \left(\left|\mu^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)\right|^{\mathrm{s}}\frac{\lambda}{2}+\left|\,\right|^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|^{\mathrm{s}}\left(1-\frac{\lambda}{2}\right)\right) d_{(\mathfrak{q}-\mathrm{h})}\lambda\right)^{\frac{1}{\mathrm{s}}}\\
&\left.+\left(\int_0^1 \left(\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)\right|^{\mathrm{s}}\left(1-\frac{\lambda}{2}\right)+\frac{\lambda}{2}\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|^{\mathrm{s}}\right) d_{(\mathfrak{q}-\mathrm{h})}\lambda\right)^{\frac{1}{\mathrm{s}}}\right]\\
&=\left(\sum_{\mathrm{n}=0}^{\infty}\mathfrak{q}^{\mathrm{n}}\left(1-\mathfrak{q}^{\mathrm{n}+1}-\mathfrak{q}\mathrm{h}\left(1+\mathrm{n}\mathfrak{q}^{\mathrm{n}}\right)\right)^{\mathrm{r}}\right)^{\frac{1}{r}}\left(\frac{((1-\mathfrak{q})+\mathfrak{q}\mathrm{h})}{2}\right)\\
&\times\left[\left(\left|{}^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)\right|^{\mathrm{s}}\left(\frac{1}{1-\mathfrak{q}^2}+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q}^2)^2}\right)+\left|\mu^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|^{\mathrm{s}}\left(\frac{\mathfrak{q}+[2]_{\mathfrak{q}}}{1-\mathfrak{q}^2}+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q}^2)^2}\right)\right)^{\frac{1}{\mathrm{s}}}\right.\\
&\left.+\left(\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)\right|^{\mathrm{s}}\left(\frac{\mathfrak{q}+[2]_{\mathfrak{q}}}{1-\mathfrak{q}^2}+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q}^2)^2}\right)+\left|{}_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|^{\mathrm{s}}\left(\frac{1}{1-\mathfrak{q}^2}+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q}^2)^2}\right)\right)^{\frac{1}{\mathrm{s}}}\right].
\end{aligned}
$$

Therefore, the result is obtained. □

Remark 3.9 *By setting* $\mathrm{h}=0$ *in* (3.18), *we get [[38], Theorem 5.3].*

4. Computational analysis

In this section, we will present examples that will show accuracy of our newly established results.

Example 4.1 *Consider* $\mathfrak{T}(\varkappa)=\varkappa^2$, *with* $\mu_1=0, \mu_2=1, \mathrm{h}=1$ *and* $\mathfrak{q}=0.5$, *then from* (3.1), *we have* $\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)=\left(\frac{1}{2}\right)^2=0.25$,

$$
\begin{aligned}
&\frac{1-\mathfrak{q}}{(1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}}\mathfrak{T}(\mathrm{u})_{\mu_1}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}\right)\\
&=\frac{1}{4}\left(\sum_{\mathrm{n}=0}^{\infty}\left(\frac{1}{2}\right)^{\mathrm{n}}\left(\left(\frac{1}{2}\right)^{\mathrm{n}+1}+\mathrm{n}\left(\frac{1}{2}\right)^{\mathrm{n}+1}\right)^2+\sum_{\mathrm{n}=0}^{\infty}\left(\frac{1}{2}\right)^{\mathrm{n}}\left(\left(\frac{1}{2}\right)^{\mathrm{n}+1}(\mathrm{n}-1)+1\right)^2\right)\\
&=0.45 \qquad (4.1)
\end{aligned}
$$

and $\frac{\mathfrak{T}(\mu_1)+\mathfrak{T}(\mu_2)}{2}=0.5$. *Therefore, Theorem 3.1 is verified.*

Example 4.2 *Consider for the convex function* $\mathfrak{T}(\varkappa)=2\varkappa^2+\varkappa$, *with* $\mu_1=0, \mu_2=1, \mathrm{h}=1$ *and* $\mathfrak{q}=0.25$, *then from 3.7, the left hand side becomes*

$$
\begin{aligned}
&\left|\frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{2\left((1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1\right)}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}}\mathfrak{T}(\mathrm{u})_{\mu_1}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}\right)\right.\\
&\left.-\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\mathfrak{q}\mathrm{h}\left(2\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\frac{\mathfrak{T}(\mu_1)+\mathfrak{T}(\mu_2)}{2}\right)\right|=\left|\frac{1}{2}(2.789)-1-0.125\right|=0.270. \qquad (4.2)
\end{aligned}
$$

And from (3.7), *the right hand side becomes*

$$\frac{(\mu_2-\mu_1)\left((1-\mathfrak{q})+\mathfrak{q}\mathrm{h}\right)}{4(1-\mathfrak{q})}\Bigg[\left|\mu^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)\right|,$$
$$\times\left(\frac{\mathfrak{q}}{2[3]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_{\mathfrak{q}}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_{\mathfrak{q}}}\right)+\frac{\mathfrak{q}\mathrm{h}}{2[2]_{\mathfrak{q}}}\left(1+\frac{\mathfrak{q}^2\mathrm{h}}{(1-\mathfrak{q})[2]_{\mathfrak{q}}}\right)\right)$$
$$+\left|^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|\left(\frac{\mathfrak{q}}{[2]_{\mathfrak{q}}}\left(1+\frac{\mathfrak{q}^2h}{(1-\mathfrak{q})[2]_{\mathfrak{q}}}\right)\left(1-\frac{h}{2}\right)+\mathfrak{q}\mathrm{h}-\frac{\mathfrak{q}}{2[3]_{\mathfrak{q}}}\right.$$
$$\left.\times\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_{\mathfrak{q}}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_{\mathfrak{q}}}\right)\right)+\left|_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|\left(\frac{\mathfrak{q}}{2[3]_{\mathfrak{q}}}\right.$$
$$\left.\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_{\mathfrak{q}}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_{\mathfrak{q}}}\right)+\frac{\mathfrak{q}\mathrm{h}}{2[2]_{\mathfrak{q}}}\left(1+\frac{\mathfrak{q}^2h}{(1-\mathfrak{q})[2]_{\mathfrak{q}}}\right)\right)+\left|_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)\right|$$
$$\times\left(\frac{\mathfrak{q}}{[2]_{\mathfrak{q}}}\left(1+\frac{\mathfrak{q}^2h}{(1-\mathfrak{q})[2]_{\mathfrak{q}}}\right)\left(1-\frac{h}{2}\right)+\mathfrak{q}\mathrm{h}-\frac{\mathfrak{q}}{2[3]_{\mathfrak{q}}}\left(1+\frac{\mathrm{h}^3\mathfrak{q}^3\left(1+\mathfrak{q}^3\right)}{(1-\mathfrak{q})^2\left([3]_{\mathfrak{q}}\right)^2}+\frac{2\mathrm{h}\mathfrak{q}^3}{(1-\mathfrak{q})[3]_{\mathfrak{q}}}\right)\right)\Bigg]$$
$$=\frac{\mu_2-\mu_1}{4(1-\mathfrak{q})}\Bigg[3\left(\frac{\mathfrak{q}}{2[3]_{\mathfrak{q}}}(1.048126)+\frac{\mathfrak{q}}{2[2]_{\mathfrak{q}}}(1.06667)\right)+5.5\left(\frac{\mathfrak{q}}{[2]_{\mathfrak{q}}}(1.06667)0.5+0.25-\frac{\mathfrak{q}}{2[3]_{\mathfrak{q}}}\right.$$
$$\left.\times(1.048126)\right)+4\left(\frac{\mathfrak{q}}{2[3]_{\mathfrak{q}}}(1.048126)+\frac{\mathfrak{q}}{2[2]_{\mathfrak{q}}}(1.06667)\right)+1.5$$
$$\times\left(\frac{\mathfrak{q}}{[2]_{\mathfrak{q}}}(1.06667)0.5+0.25-\frac{\mathfrak{q}}{2[3]_{\mathfrak{q}}}(1.048126)\right)\Bigg]$$
$$=\frac{1}{3}(1.837+1.80)=1.21. \tag{4.3}$$

It is clear that from (4.2) *and* (4.3)*:* $0.27<1.21$*. Therefore, Theorem 3.2 is verified.*

Example 4.3 *Consider for the convex function* $\mathfrak{T}(\varkappa)=2\varkappa^2+\varkappa$*, with* $\mu_1=0,\mu_2=1,\mathrm{h}=1$*,* $s=\frac{3}{2}$ *and* $\mathfrak{q}=0.25$*, then from* (3.8)*, the left hand side becomes*

$$\left|\frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{2\left((1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1\right)}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}}\mathfrak{T}(\mathrm{u})_{\mu_1}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}\right)\right.$$
$$\left.-\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\mathfrak{q}\mathrm{h}\left(2\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\frac{\mathfrak{T}(\mu_1)+\mathfrak{T}(\mu_2)}{2}\right)\right|=0.27. \tag{4.4}$$

And from (3.8)*, the right hand side becomes*

$$0.067(3.0889)^{1-\frac{1}{\mathrm{s}}}\left(\left(\left|^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)\right|^{\mathrm{s}}(1.03351)+\left|^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})^2}\mathfrak{T}(\mu_2)\right|^{\mathrm{s}}(1.2828)\right)^{\frac{1}{\mathrm{s}}}\right.$$
$$\left.+\left(\left|_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|^{\mathrm{s}}(1.03351)+\left|_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_1)\right|^{\mathrm{s}}(1.2828)\right)^{\frac{1}{\mathrm{s}}}\right)=0.175$$
$$=0.067(3.0889)^{1-\frac{1}{\mathrm{s}}}\left(\left((3)^{\mathrm{s}}(1.03351)+(5.5)^{\mathrm{s}}(1.2828)\right)^{\frac{1}{\mathrm{s}}}\right.$$
$$\left.+\left((4)^{\mathrm{s}}(1.03351)+(1.5)^{\mathrm{s}}(1.2828)\right)^{\frac{1}{\mathrm{s}}}\right)=1.24. \tag{4.5}$$

It is clear that from (4.4) *and* (4.5)*:* $0.27<1.24$*. Therefore, Theorem 3.3 is verified.*

Example 4.4 *Consider for the convex function* $\mathfrak{T}(\varkappa)=2\varkappa^2+\varkappa$, *with* $\mu_1=0,\mu_2=1,\mathrm{h}=1$, $s=\frac{3}{2},r=3$ *and* $\mathfrak{q}=\frac{1}{4}$, *then from* (3.9), *the left hand side becomes*

$$\left|\frac{(1-\mathfrak{q})+\mathfrak{q}\mathrm{h}}{2\left((1-\mathfrak{q})\left(\frac{\mu_2-\mu_1}{2}\right)+\mathfrak{q}\mathrm{h}_1\right)}\left(\int_{\mu_1}^{\frac{\mu_1+\mu_2}{2}}\mathfrak{T}(\mathrm{u})_{\mu_1}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}+\int_{\frac{\mu_1+\mu_2}{2}}^{\mu_2}\mathfrak{T}(\mathrm{u})^{\mu_2}d_{(\mathfrak{q}-\mathrm{h}_1)}\mathrm{u}\right)\right.$$
$$\left.-\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\mathfrak{q}\mathrm{h}\left(2\mathfrak{T}\left(\frac{\mu_1+\mu_2}{2}\right)-\frac{\mathfrak{T}(\mu_1)+\mathfrak{T}(\mu_2)}{2}\right)\right|=0.27. \tag{4.6}$$

And from (3.9), *the right hand side becomes*

$$\frac{\mu_2-\mu_1}{8}\mathfrak{q}((1-\mathfrak{q})+\mathfrak{q}\mathrm{h})^{1+\frac{1}{\mathrm{r}}}\left(\sum_{\mathrm{k}=0}^{\mathrm{r}}\binom{\mathrm{r}}{k}\mathrm{h}^{\mathrm{r}-\mathrm{k}}\sum_{\mathrm{n}=0}^{\infty}\mathfrak{q}^{(1+\mathrm{k})\mathrm{n}}(1+\mathrm{nh})^{\mathrm{k}}\right)^{\frac{1}{\mathrm{r}}}$$
$$\times\left[\left(\left|\mu^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})^2}\mathfrak{T}(\mu_1)\right|^{\mathrm{s}}\left(\frac{1}{1-\mathfrak{q}^2}+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q}^2)^2}\right)+\left|^{\mu_2}D_{(\mathfrak{q}-\mathrm{h})^2}\mathfrak{T}(\mu_2)\right|^{\mathrm{s}}\left(\frac{\mathfrak{q}+[2]_{\mathfrak{q}}}{1-\mathfrak{q}^2}+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q}^2)^2}\right)\right)^{\frac{1}{\mathrm{s}}}\right.$$
$$\left.+\left(\left|_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})^2}\mathfrak{T}(\mu_1)\right|^{\mathrm{s}}\left(\frac{\mathfrak{q}+[2]_{\mathfrak{q}}}{1-\mathfrak{q}^2}+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q}^2)^2}\right)+\left|_{\mu_1}D_{(\mathfrak{q}-\mathrm{h})}\mathfrak{T}(\mu_2)\right|^{\mathrm{s}}\left(\frac{1}{1-\mathfrak{q}^2}+\frac{\mathrm{h}\mathfrak{q}^2}{(1-\mathfrak{q}^2)^2}\right)\right)^{\frac{1}{\mathrm{s}}}\right].$$
$$=\frac{\mathfrak{q}}{8}(4.60)^{\frac{1}{\mathrm{r}}}\times\left[((3)^{\mathrm{s}}(1.138)+(5.5)^{\mathrm{s}}(2.311))^{\frac{1}{\mathrm{s}}}+((1.5)^{\mathrm{s}}(2.311)+(4)^{\mathrm{s}}(1.138))^{\frac{1}{\mathrm{s}}}\right]$$
$$=0.05198[16.47]=0.86. \tag{4.7}$$

It is clear that from (4.6) *and*(4.7): $0.27<0.86$. *Therefore, Theorem 3.4 is verified.*

5. Conclusion

In this study, we have successfully derived novel Hermite Hadamard-type, midpoint, and trapezoidal inequalities for convex functions within the framework of quantum calculus using $(\mathfrak{q}-\mathrm{h})$-integral operators. These inequalities represent significant generalizations of classical results by incorporating two parameters, $\mathfrak{q}$ and h, thereby offering a more flexible and unified analytical framework. By exploiting the inherent properties of convexity along with the tools of $(\mathfrak{q}-\mathrm{h})$-differentiation, we established sharp bounds and approximation techniques that naturally reduce to the classical Hermite Hadamard inequality and its related estimates as $\mathfrak{q}\to 1$ and $\mathrm{h}=0$. This seamless transition highlights the robustness of the proposed generalizations. Furthermore, the theoretical findings were supported through several numerical examples, which demonstrate the practical effectiveness and improved accuracy of our results in comparison with existing $\mathfrak{q}$-calculus methods. These results not only deepen the theoretical understanding of quantum integral inequalities but also contribute valuable tools for applied fields such as numerical analysis, optimization, and mathematical physics. Future research directions may include extending the current results to broader function classes, such as quasi-convex or logarithmically convex functions, and exploring the application of $(\mathfrak{q}-\mathrm{h})$-calculus in the analysis of fractional differential equations and related complex systems.

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