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Behavior and Solutions of the Transcendental Equation $x^p = p^x$ via the Lambert $\mathcal{W}$ Function

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ABSTRACT: The main objective of this work is to study the transcendental equation $x^p = p^x$, with $p \in \mathbb{N}$ and $p \neq 1$, using the Lambert $\mathcal{W}$ function. The equation can be rewritten in a form solvable by $\mathcal{W}$, enabling an explicit determination of its real solutions. We analyze the behavior of these solutions using differential calculus. The number of real solutions depends on the parity of p: there are three solutions for even p and two for odd p. This work extends prior results (e.g., [1]) by generalizing the analysis beyond prime p. Additionally, we examine the function $-p^{-1} \ln p$ to understand the asymptotic behavior of the solutions. Finally, we propose a method for obtaining parametrized solutions through two auxiliary equations, which facilitates their computation and provides further insight into the structure of the solutions.

Key Words: Lambert function, transcendental equations, multivalued functions.

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1. Introduction

The *Lambert $\mathcal{W}$ function*, or simply the $\mathcal{W}$ function, is a transcendental function defined as the solution of the equation

$$\mathcal{W}(x)e^{\mathcal{W}(x)} = x.$$

Formally introduced in the context of complex analysis by Corless et al. [3], the $\mathcal{W}$ function plays a fundamental role in solving equations that combine exponential and polynomial terms, and it appears in various areas such as applied mathematics, physics, engineering, and information theory. Its analytical and asymptotic properties have been the subject of detailed investigation, given its capacity to express solutions to transcendental equations that would otherwise be intractable through elementary methods.

A notable example of the application of the $\mathcal{W}$ function is the resolution of the transcendental equation

$$x^p = p^x,$$

where p is a natural number different from 1. This equation is of considerable interest in several fields, including number theory, algorithm analysis, and population dynamics, as it illustrates phenomena of competitive growth and dominance relations. The non-trivial solutions of $x^p = p^x$ are generally transcendental or irrational, and their study contributes to the understanding of the structure of real and transcendental numbers.

The classical approach to studying this equation involves techniques from number theory. For example, dos Santos and Galvão [2] investigated the solutions of the equation $x^p = p^x$, with $p \in \mathbb{N} \setminus \{0, 1\}$ and $x \in \mathbb{R} \setminus \{0, 1\}$, establishing its relation to transcendental numbers of the form p^T. Moreover, they proposed a transcendence criterion for such powers, highlighting that although useful, this criterion is not absolute:

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there exist transcendental numbers that do not satisfy the condition, as well as numbers of the form p^T that may be algebraic.

Complementing this perspective, Bastos [1] carried out a detailed study of the real solutions of the equation $x^p = p^x$ in the case where p is a prime number, using analytical methods to characterize the behavior of the real zeros. The author analyzed how the number and nature of the solutions vary with the value of p, emphasizing in particular the aspects of transcendence and irrationality involved.

In this work, we present a new approach to the study of the equation $x^p = p^x$, systematically employing the Lambert $\mathcal{W}$ function. We extend the analysis to all $p \in \mathbb{N}\backslash\{1\}$, allowing for the explicit determination of real solutions through appropriate transformations and the use of analytical properties of this special function. In particular, we demonstrate that if $p \in \mathbb{N}$ is even, then the equation $x^p = p^x$ has exactly three real solutions. In contrast, if p is odd, the equation admits precisely two real solutions. All these solutions can be expressed explicitly in terms of the Lambert $\mathcal{W}$ function. For even values of p, one of the solutions is negative.

In addition to this explicit formulation, we propose an alternative approach based on the parametrization of the real solutions through two auxiliary equations. This reformulation provides a more geometric interpretation of the solutions and facilitates their numerical determination. In summary, formalizing this method, we show that the positive solutions of the equation $x^p = p^x$ can be parametrized using the auxiliary equation $c^p = p^c$, where $c \in \mathbb{R}_+^*$. This representation establishes a symmetric correspondence between the solutions and demonstrates the existence of exactly two positive real values of c that satisfy this condition. Additionally, for even values of p, the negative solution can also be described parametrically using the relation $-t^p = -p^{-t}$, with $t > 0$. This formulation provides a direct characterization of the third real solution based on a negative exponential symmetry. The proofs of these theorems will be presented in Sections 4 and 5, supported by analytical techniques, series expansions, and a detailed study of the asymptotic behavior of the solutions involved.

2. The Lambert $\mathcal{W}$ Function

In this section, we present the formal definition of the Lambert $\mathcal{W}$ function, discussing its main characteristics, such as the fact that it is a multivalued function, and the behavior of its different branches [3]. We explore its relationship with the function $f(x) = xe^x$, its series expansions, and the formula for its higher-order derivatives. In addition, we apply the $\mathcal{W}$ function to the solution of specific transcendental equations, such as $x^2 = 2^x$, and more generally $x^p = p^x$, for different values of p, analyzing the existence and behavior of the real solutions.

The Lambert $\mathcal{W}$ function, or simply $\mathcal{W}$, is a transcendental function defined as the solution of the equation:

$$\mathcal{W}(x)e^{\mathcal{W}(x)} = x.$$

It can also be characterized as the inverse function of $f(x) = xe^x$. However, the function $f(x)$ is not injective on the entire real line $\mathbb{R}$, which prevents the definition of a conventional global inverse. To overcome this difficulty, the $\mathcal{W}$ function is defined as a multivalued function, meaning that it can assign more than one image to the same domain element. By considering different branches, it is possible to treat $\mathcal{W}$ as distinct real functions on specific intervals.

The function $f(x) = xe^x$ is strictly decreasing on $(-\infty, -1]$ and strictly increasing on $[-1, \infty)$, which allows for the definition of the following local inverses:

$$f_1 : [-1, \infty) \to [-e^{-1}, \infty), \quad \text{and} \quad f_2 : (-\infty, -1] \to [-e^{-1}, 0).$$

Thus, the $\mathcal{W}$ function can be defined as:

$$\mathcal{W}(x) = \begin{cases} f_1^{-1}(x), & \text{if } x \in [-e^{-1}, \infty), \\ f_2^{-1}(x), & \text{if } x \in [-e^{-1}, 0). \end{cases}$$

The $\mathcal{W}$ function has two main branches:

- The principal branch, denoted by $\mathcal{W}_0(x)$, corresponding to the inverse of $f_1(x)$, with image $\mathcal{W}_0(x) \geq -1$.

- The secondary branch, denoted by $\mathcal{W}_{-1}(x)$, corresponding to the inverse of $f_2(x)$, with image $\mathcal{W}_{-1}(x) \le -1$.

In the figure below, the two branches $\mathcal{W}_0$ and $\mathcal{W}_{-1}$ can be easily identified.

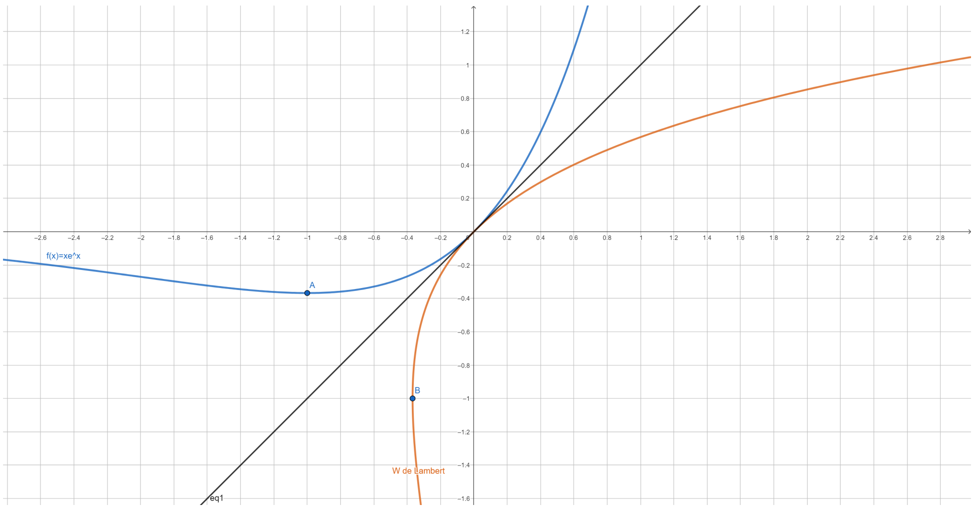

Figure 1: Function xe^x and its inverses $\mathcal{W}_0$ and $\mathcal{W}_{-1}$.

From this, several important observations arise:

(a) If $x < -e^{-1}$, the $\mathcal{W}$ function is not defined.

(b) If $-e^{-1} < x < 0$, there are two corresponding images, one in $\mathcal{W}_0(x)$ and the other in $\mathcal{W}_{-1}(x)$.

(c) If $x \ge 0$, there is only one image, belonging to the principal branch $\mathcal{W}_0(x)$.

Once the function is defined and some of its properties are known, it is possible to analyze its series expansion. In their article, Corless et al. [3] presented the following series expansion for $\mathcal{W}(x)$ near 0:

$$\mathcal{W}(x) = x - x^2 + \frac{3}{2}x^3 - \frac{8}{3}x^4 + \cdots$$

To derive this series, consider $f(x) = xe^x$. Then:

$$e^{sx} = \sum_{k=0}^{\infty} \frac{s^k x^k}{k!},$$

and the associated sequence of polynomials is:

$$P_k(s) = \frac{s^k}{k!}. \tag{2.1}$$

As defined by Labelle [4], the series for $f^{-1}(x)$, in general, is:

$$f^{-1}(x) = \sum_{n=0}^{\infty} \frac{P_n(-n-1)}{n+1} x^{n+1}. \tag{2.2}$$

Substituting (2.1) into (2.2), we obtain:

$$\mathcal{W}(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n,$$

which, according to [3], is a convergent series for $x \in [-e^{-1}, e^{-1}]$.

In addition to these approximations for the expansion of the $\mathcal{W}$ function, when studying its applications—especially in solving transcendental equations such as $x^p = p^x$—it becomes clear that understanding not only the behavior of the function itself but also its rates of change is essential. Analyzing the derivative of $\mathcal{W}$ is crucial for investigating, for example, the monotonicity, concavity, and asymptotic behavior of the involved solutions. Furthermore, in many practical problems, such as equation fitting or numerical methods involving successive approximations, it is necessary to know higher-order derivatives to ensure accuracy and stability in calculations. In this context, Corless et al. [3] established a general formula for the ith derivative of the $\mathcal{W}$ function, which allows for the explicit calculation of these derivatives for any order $i > 1$, using polynomials defined by a recurrence relation. This result also confirms that $\mathcal{W}$ belongs to the class C^∞, i.e., it admits continuous derivatives of all orders. More specifically, in the work by Corless et al., we have:

$$\frac{d^i\mathcal{W}(x)}{dx^i} = \frac{e^{i\mathcal{W}(x)}p_i(\mathcal{W}(x))}{(1+\mathcal{W}(x))^{2i-1}},$$

where the polynomials p_i satisfy the recurrence relation:

$$p_{i+1}(w) = -(iw + 3i - 1)p_i(w) + (1+w)p_i'(w),$$

with $p_1(w) = 1$. Therefore, $\mathcal{W}$ is a C^∞ function.

3. Solutions of the Equation $x^2 = 2^x$

After the definition and detailed analysis of the Lambert $\mathcal{W}$ function, as well as the exploration of its main properties and applications in solving transcendental equations, we now investigate the specific case of the equation $x^p = p^x$, which is the central focus of this work. This equation, whose solutions are generally transcendental or irrational, provides a concrete example of the usefulness of the $\mathcal{W}$ function in solving problems that do not admit a direct approach via elementary methods. In this section, we use the structure of the $\mathcal{W}$ function to characterize the real solutions of the equation $x^p = p^x$ for different values of $p \in \mathbb{N}$, analyzing the existence, number, and nature of these solutions based on the parity of p and properties of the involved function.

The equation $x^2 = 2^x$ is transcendental and can be solved using the $\mathcal{W}$ function as follows:

$$\begin{aligned} x^2 = 2^x &\Rightarrow |x| = \sqrt{2^x} \\ &\Rightarrow \ln|x| = \ln 2^{x/2} \\ &\Rightarrow \frac{\ln|x|}{x} = \frac{\ln 2}{2} \\ &\Rightarrow x^{-1}\ln|x| = \frac{\ln 2}{2} \end{aligned}$$

Assuming $x > 0$:

$$\begin{aligned} x^{-1}\ln x = \frac{\ln 2}{2} &\Rightarrow \ln x \cdot e^{-\ln x} = \frac{\ln 2}{2} \\ &\Rightarrow -\ln x \cdot e^{-\ln x} = -\frac{\ln 2}{2} \\ &\Rightarrow -\ln x = \mathcal{W}\left(-\frac{\ln 2}{2}\right) \\ &\Rightarrow x = e^{-\mathcal{W}\left(-\frac{\ln 2}{2}\right)} \end{aligned}$$

Since the argument $-\frac{\ln 2}{2} \in [-e^{-1}, 0]$, there are two real solutions, associated with the branches $\mathcal{W}_0$ and $\mathcal{W}_{-1}$. The solutions are:

$$x_0 = e^{-\mathcal{W}_0\left(-\frac{\ln 2}{2}\right)}, \quad x_{-1} = e^{-\mathcal{W}_{-1}\left(-\frac{\ln 2}{2}\right)}.$$

Thus,

$$\begin{aligned} -\mathcal{W}_0\left(-\tfrac{1}{2}\ln 2\right) &= -\mathcal{W}_0\left(-\ln 2 \cdot e^{-\ln 2}\right) \\ &= -(-\ln 2) = \ln 2 \end{aligned}$$

Note that $-\ln 2 > -1$, so this solution belongs to the domain of $\mathcal{W}_0$. Now, to find the other solution, we proceed as follows:

$$\begin{aligned} -\mathcal{W}_{-1}\left(-\tfrac{1}{2}\ln 2\right) &= -\mathcal{W}_{-1}\left(-\tfrac{1}{2}\ln 2 \cdot 2 \cdot \tfrac{1}{2}\right) \\ &= -\mathcal{W}_{-1}\left(-\tfrac{1}{4}\ln 2^2\right) \\ &= -\mathcal{W}_{-1}\left(-\tfrac{1}{4}\ln 4\right) \\ &= -\mathcal{W}_{-1}\left(\ln 4 \cdot e^{-\ln 4}\right) \\ &= -(-\ln 4) = \ln 4 \end{aligned}$$

Since $-\ln 4 < -1$, this solution belongs to the domain of $\mathcal{W}_{-1}$, so we have:

$$\begin{aligned} x_0 &= e^{\mathcal{W}_0\left(-\frac{1}{2}\ln 2\right)} = e^{\ln 2} \Rightarrow x_0 = 2, \\ x_{-1} &= e^{\mathcal{W}_{-1}\left(-\frac{1}{2}\ln 4\right)} = e^{\ln 4} \Rightarrow x_{-1} = 4. \end{aligned}$$

Now, considering $x < 0$, we find:

$$x = -e^{-\mathcal{W}\left(\frac{\ln 2}{2}\right)}.$$

The value of $\mathcal{W}\left(\frac{\ln 2}{2}\right)$ can be approximated by truncating the Taylor series of Corless. Substituting $x = \frac{\ln 2}{2}$ into the expansion of the $\mathcal{W}$ function, we have:

$$\begin{aligned} \mathcal{W}\left(\tfrac{1}{2}\ln 2\right) &= \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{2n!}(\ln 2)^n \approx \sum_{n=1}^{50} \frac{(-n)^{n-1}}{2n!}(\ln 2)^n \\ &\approx 0.2656784901448456969500323411671836355932 = l \end{aligned}$$

Thus, if $x < 0$, we obtain the following solution:

$$x = -e^{-l} \approx -0.766.$$

In this way, we find the three desired solutions.

4. Real Solutions of the Equation $x^p = p^x$ and the Asymptotic Analysis of the Solutions

After the detailed analysis of the equation $x^2 = 2^x$ using the Lambert $\mathcal{W}$ function, we now consider the more general case, namely the equation $x^p = p^x$, with $p \in \mathbb{N}$. This generalization is essential for the main objective of this work, which is to investigate the structure and behavior of the real solutions of this type of transcendental equation. Using the properties of the $\mathcal{W}$ function, we aim to characterize the existence and nature of the solutions as a function of different values of p, focusing in particular on the role of the parity of p and the dependence of the solutions on the analytical structure of the function. In summary, we prove the following result:

Theorem 4.1 *Let $p \in \mathbb{N}$. If p is even, then the equation $x^p = p^x$ admits three real solutions, which can be expressed as:*

$$x_1 = e^{-\mathcal{W}_0(-p^{-1}\ln p)} = p \tag{4.1}$$

$$x_2 = e^{-\mathcal{W}_{-1}(-p^{-1}\ln p)} = e^{-\mathcal{W}_{-1}(-(cp)^{-1}\ln p^c)} = \exp\left(\sum_{n=1}^{\infty}\frac{(-n)^{n-1}}{n!}(-(cp)^{-1}\ln p^c)^n\right) \tag{4.2}$$

$$x_3 = -e^{-\mathcal{W}_0(p^{-1}\ln p)} = -\exp\left(\sum_{n=1}^{\infty}\frac{(-n)^{n-1}}{n!}(p^{-1}\ln p)^n\right) \tag{4.3}$$

for some $c \in \mathbb{R}$. If p is odd, then there are two real solutions, given by (4.1) and (4.2).

Proof: To prove this result, consider $p \in \mathbb{N}$. Analogously to the case $p = 2$, the equation

$$x^p = p^x$$

for even p becomes:

$$x^{-1}\ln|x| = p^{-1}\ln p.$$

If $x > 0$, then

$$x = e^{-\mathcal{W}\left(-p^{-1}\ln p\right)}.$$

If $x < 0$, then

$$x = -e^{-\mathcal{W}\left(p^{-1}\ln p\right)}.$$

Hence, using the Lambert function $\mathcal{W}$, we can express the solutions explicitly. To complete the proof, we analyze the functions $h(p) = \dfrac{-\ln p}{p}$ and $-h(p)$, which appear in the arguments of $\mathcal{W}$. Note that:

$$h(p) = \frac{-\ln p}{p} \Rightarrow h'(p) = \frac{\ln p - 1}{p^2}.$$

To find the critical points of h, we solve:

$$\begin{aligned} h'(p) = 0 &\Rightarrow \frac{\ln p - 1}{p^2} = 0 \\ &\Rightarrow \ln p = 1 \\ &\Rightarrow p = e \end{aligned}$$

So $p = e$ is a critical point of $h(p)$. To determine its nature, we compute the second derivative:

$$h''(p) = \frac{-2\ln p + 3}{p^3} \quad \Rightarrow \quad h''(e) = \frac{1}{e^3} > 0.$$

Thus, $p = e$ is a local minimum of $h(p)$, and the smallest value of $h(p)$ occurs at $p = e$, with

$$h(e) = -\frac{1}{e}.$$

Also, we observe that $h'(p) < 0$ for $p \in (0, e)$ and $h'(p) > 0$ for $p \in (e, \infty)$. Therefore, $h(p)$ is strictly decreasing on $(0, e)$ and strictly increasing on (e, ∞). Since $\lim_{p\to\infty} h(p) = 0$, we conclude that, for $p > 1$:

$$-\frac{1}{e} < h(p) < 0.$$

This analysis implies that, for $p > 1$, the equation $\mathcal{W}(-p^{-1}\ln p)$ has two real solutions, corresponding to the two real branches of the Lambert function.

Analogously, the function $\dfrac{\ln p}{p}$ is positive for $p > 1$, so the argument $p^{-1}\ln p$ belongs only to the domain of the principal branch of the Lambert function. Therefore, there is only one real solution for $\mathcal{W}(p^{-1}\ln p)$.

If p is even, the equation admits the following three real solutions:

$$x = e^{-\mathcal{W}\left(-\frac{\ln p}{p}\right)}, \quad x = -e^{-\mathcal{W}\left(p^{-1}\ln p\right)}.$$

Moreover, since $-\dfrac{1}{e} < h(p) < -h(p) < \dfrac{1}{e}$, by using the truncated Corless series, we obtain for even p:

$$\begin{aligned}
x_1 &= e^{-\mathcal{W}_0(-p^{-1}\ln p)} = p,\\
x_2 &= e^{-\mathcal{W}_{-1}(-p^{-1}\ln p)} = e^{-\mathcal{W}_{-1}(-(cp)^{-1}\ln p^c)} = \exp\left(\sum_{n=1}^{\infty}\frac{(-n)^{n-1}}{n!}(-(cp)^{-1}\ln p^c)^n\right),\\
x_3 &= -e^{-\mathcal{W}_0(p^{-1}\ln p)} = -\exp\left(\sum_{n=1}^{\infty}\frac{(-n)^{n-1}}{n!}(p^{-1}\ln p)^n\right).
\end{aligned}$$

However, for odd p, the equation $x^p = p^x$ becomes:

$$x^{-1}\ln x = p^{-1}\ln p,$$

so there is no negative real number $x < 0$ that satisfies the equation. Therefore, when p is odd, there are only two positive real solutions, given by (4.1) and (4.2). □

As for the non-trivial positive solution (4.2), we can study its asymptotic behavior as $p \to \infty$. Then:

$$\begin{aligned}
\lim_{p\to\infty} e^{-\mathcal{W}\left(\frac{-\ln p}{p}\right)} &= \exp\left(\lim_{p\to\infty} -\mathcal{W}\left(\frac{-\ln p}{p}\right)\right)\\
&= \exp\left(-\mathcal{W}\left(\lim_{p\to\infty}\frac{-\ln p}{p}\right)\right)\\
&= e^{-\mathcal{W}(0)}\\
&= e^0 = 1.
\end{aligned}$$

For the negative solution, the same reasoning can be applied. Thus, if p is even and $x < 0$, then:

$$\lim_{p\to\infty} -e^{-\mathcal{W}\left(\frac{\ln p}{p}\right)} = -1.$$

5. Parametrized Solutions and Their Interpretations

Note that to find the second positive solution in the case $p = 2$, we used a strategy to ensure that the argument belonged to the second branch of the Lambert $\mathcal{W}$ function. Since the principal branch for $x > 0$ returns only the trivial solution, we now examine how this method can be generalized. For $c \in \mathbb{R}_+^*$, we have:

$$\begin{aligned}
x &= e^{-\mathcal{W}\left(-p^{-1}\ln p\right)}\\
&= e^{-\mathcal{W}\left(\frac{-c\ln p}{cp}\right)}\\
&= e^{-\mathcal{W}\left(-c\ln p\cdot e^{-\ln cp}\right)}\\
&= e^{-\mathcal{W}\left(-\ln p^c\cdot e^{-\ln cp}\right)}.
\end{aligned}$$

From the definition of the Lambert $\mathcal{W}$ function, we have $\mathcal{W}(x)e^{\mathcal{W}(x)} = x$. Considering

$$y(c) = -\ln(p^c) = -\ln(cp),$$

we obtain:

$$x = e^{-y(p)} \quad \Rightarrow \quad x = cp = p^c.$$

Observe that the function $y(c)$ is characterized by the intersection of the functions $-\ln(p^c)$ and $-\ln(cp)$. Furthermore, when a negative solution exists, the same approach is valid. Thus, if $t \in \mathbb{R}_+^*$, we have:

$$x = -\exp\left(-\mathcal{W}\left(\frac{\ln p}{p}\right)\right) \Rightarrow x = -\exp\left(-\mathcal{W}\left(\ln(p^t) \cdot e^{\ln(tp)^{-1}}\right)\right)$$
$$\Rightarrow x = -tp = -\frac{1}{p^t}.$$

This leads to the following theorem:

Theorem 5.1 *The equation $cp = p^c$, with $c \in \mathbb{R}_+^*$ and $p \in \mathbb{N}$, has two solutions given by:*

$$c_1 = 1 \tag{5.1}$$
$$c_2 = \exp\left(\sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} \left(-(cp)^{-1} \ln p^c\right)^n\right) \cdot p^{-1} \tag{5.2}$$

As already discussed, the positive solutions of the equation $x^p = p^x$ can be described in the form $x = cp = p^c$. We now prove that this auxiliary equation has exactly two real solutions, corresponding to each positive solution of the main equation.

Proof: Consider $g(c) = cp - p^c$. Then, differentiating:

$$g'(c) = p - p^c \cdot \ln p,$$

which yields a critical point at:

$$c = \frac{\ln p - \ln(\ln p)}{\ln p} = c_0.$$

Furthermore, we compute:

$$g''(c) = -p^c \cdot (\ln p)^2 < 0,$$

so c_0 is a global maximum. Evaluating the limits:

$$\lim_{c \to 0^+} cp - p^c = -1,$$
$$\lim_{c \to +\infty} cp - p^c = -\infty.$$

Since $g(c)$ is strictly increasing on $(0, c_0)$ and strictly decreasing on (c_0, ∞), and since it takes both positive and negative values, by the Intermediate Value Theorem there exists exactly one root in each interval. Therefore, $g(c)$ has exactly two real roots.

Thus, since $x = cp = p^c$, we obtain the two solutions:

$$c_1 = 1,$$
$$c_2 = \exp\left(\sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} \left(-(cp)^{-1} \ln p^c\right)^n\right) \cdot p^{-1}.$$

□

Theorem 5.2 *If p is even and $t \in \mathbb{R}_+^*$, then the equation $-tp = -p^{-t}$ has a unique solution, which corresponds to the negative solution of $x^p = p^x$, given by:*

$$t = \exp\left(\sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!}\left(p^{-1}\ln p\right)^n\right) \cdot p^{-1} \tag{5.3}$$

Proof: Analogously to the previous proof, consider the function

$$h(t) = p^t - tp \quad \Rightarrow \quad h'(t) = -p^{-t} \cdot \ln(p) - p.$$

Also, note:

$$\lim_{t\to 0^+} p^{-t} - tp = 1,$$
$$\lim_{t\to +\infty} p^{-t} - tp = -\infty.$$

Since h is strictly decreasing and assumes both positive and negative values, there exists exactly one root of $h(t)$ in the interval $(0, \infty)$. Given that $x = -tp = -p^{-t}$, we obtain:

$$t = \exp\left(\sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!}\left(p^{-1}\ln p\right)^n\right) \cdot p^{-1}.$$

□

The results of equations (5.1), (5.2), and (5.3) are important because they establish a connection between the solutions of the equation $x^p = p^x$ and two auxiliary equations, providing a parametrization of the solutions. This significantly simplifies the process of obtaining the solutions: instead of relying on the evaluation of infinite series (which typically requires numerical methods or advanced calculators), it is sufficient to use a graphing tool or numerical solver to find the roots of the functions $g(c)$ and $h(t)$ for a fixed value of p. This method was implicitly applied in Section 3 to find the second positive solution of the equation $x^2 = 2^x$.

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