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A New Perspective on Discrete Orlicz Spaces with its Natural 2-Norm

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ABSTRACT: In this paper, we introduce the discrete Orlicz space equipped with a 2-norm, which serves as a generalization of its usual norm. We construct a norm derived from this 2-norm and demonstrate that the resulting space is complete, thereby forming a 2-Banach space. We use this fact to prove the fixed point theorem for the discrete Orlicz space that is equipped with a 2-norm.

Key Words: Discrete Orlicz space, 2-norm, complete, fixed point.

Contents

1. Introduction

In 2024, Nur et al. [1] showed that the Orlicz space (continuous version) can be equipped with a 2-norm. Let X be a real vector space of dimension $2 \leq d < \infty$. A 2-norm is a mapping $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies the following four conditions:

1. $\|x, y\| = 0$ if and only if x, y are linearly dependent;
2. $\|x, y\| = \|y, x\|$ for every $x, y \in X$;
3. $\|\alpha x, y\| = |\alpha| \, \|x, y\|$ for every $x, y \in X$ and for every $\alpha \in \mathbb{R}$;
4. $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for every $x, y, z \in X$.

The pair $(X, \|\cdot,\cdot\|)$ is called a *2-normed space*. Using this definition, we have $\|x, y\| \geq 0$ and $\|x, y\| = \|x, y + \alpha x\|$ for any $x, y \in X$ and $\alpha \in \mathbb{R}$.

The concept of 2-normed spaces was first introduced by Gähler [2] in the mid 1960's with its generalization outlined in [3,4,5]. Since then, numerous researchers have examined the structures of these spaces, with recent findings available in [6,7,8,9,10,11].

Let $(X, \|\cdot,\cdot\|)$ be the 2-normed space. A sequence (x_n) in X is said to be *converge* to an $x \in X$ (in 2-norm) if $\lim_{n\to\infty} \|x_n - x, y\| = 0$ for any $y \in X$. Next, a sequence (x_n) is said to be *Cauchy sequence* in X (in 2-norm) if $\lim_{n,m\to\infty} \|x_n - x_m, y\| = 0$ for any $y \in X$. If every Cauchy sequence (x_n) in X converges to an x in X then X is said to be *complete*. A complete 2-normed space is called a 2-Banach space.

Let $\Phi : [0, \infty) \to [0, \infty)$ be a Young function (that is, Φ is convex, left-continuous, $\Phi(0) = 0$, and $\lim_{t\to\infty} \Phi(t) = \infty$), we define *the discrete Orlicz space* $\ell_\Phi(\mathbb{Z})$ to be the set of all sequences $X := (x_k) : \mathbb{Z} \to \mathbb{R}$ such that $\sum_k \Phi\left(\frac{|x_k|}{\alpha}\right) < \infty$ for some $\alpha > 0$. The discrete Orlicz space $\ell_\Phi(\mathbb{Z})$ is a Banach space with respect to the usual norm

$$\|x\|_{\ell_\Phi(\mathbb{Z})} := \inf\left\{ b > 0 : \sum_k \Phi\left(\frac{|x_k|}{b}\right) \leq 1 \right\}$$

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(see [12,13]). Note that, if if $\Phi(t) := t^p$ for some $1 \le p < \infty$, then $\ell_\Phi(\mathbb{Z}) = \ell_p(\mathbb{Z})$. Thus, the discrete Orlicz space $\ell_\Phi(\mathbb{Z})$ can be viewed as a generalization of the space of p-summable sequences $\ell_p(\mathbb{Z})$. To keep the following writing simple, we denote $\ell_p(\mathbb{Z}) = \ell_p$ and $\ell_\Phi(\mathbb{Z}) = \ell_\Phi$. On the space ℓ^p for $1 \le p < \infty$, the following 2-norm $\|\cdot,\cdot\|_{\ell_p}$ was defined by Gunawan [14]

$$\|x,y\|_{\ell_p} = \left[\frac{1}{2}\sum_j\sum_k \left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|^p\right]^{\frac{1}{p}}. \tag{1.1}$$

In this note, we introduce the discrete Orlicz space ℓ_Φ equipped with the 2-norm, which can be seen as a generalization of the standard norm. Furthermore, we define a norm derived from the 2-norm and demonstrate that ℓ_Φ is the 2-Banach space with respect to its 2-norm. Using this result, we establish a fixed point theorem for this space.

2. Main Result

2.1. $\ell_\Phi(\mathbb{Z})$ as 2-normed space

Let ℓ_Φ be the discrete Orlicz space where $\Phi : [0,\infty) \to [0,\infty)$ be a Young function. We define the mapping $\|\cdot,\cdot\|_{\ell_\Phi}$ on $\ell_\Phi \times \ell_\Phi$ by

$$\|x,y\|_{\ell_\Phi} := \inf\left\{b > 0 : \frac{1}{2}\sum_j\sum_k \Phi\left(\frac{1}{b}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|\right) \le 1\right\}. \tag{2.1}$$

where $x := (x_j), y := (y_j) \in \ell_\Phi$. Next, we will show that the mapping in (2.1) defines a 2-norm on ℓ_Φ. To do so, we use the following lemmas.

Lemma 2.1 *If* $0 < \|x,y\|_{\ell_\Phi} < \infty$ *then* $\frac{1}{2}\sum_j\sum_k \Phi\left(\frac{1}{\|x,y\|_{\ell_\Phi}}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|\right) \le 1.$

Proof: Take any $x := (x_j)$ and $y := (y_j) \in \ell_\Phi$ such that $0 < \|x,y\|_{\ell_\Phi} < \infty$. Write

$$\mathbb{B} = \left\{b > 0 : \frac{1}{2}\sum_j\sum_k \Phi\left(\frac{1}{b}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|\right) \le 1\right\}.$$

As consequence $\|x,y\|_{\ell_\Phi} = \inf \mathbb{B}$. For any $\epsilon > 0$, there exists $b_\epsilon \in \mathbb{B}$ such that $\|x,y\|_{\ell_\Phi} \le b_\epsilon \le \|x,y\|_{\ell_\Phi} + \epsilon$. Hence

$$\frac{\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|}{\|x,y\|_{\ell_\Phi} + \epsilon} \le \frac{\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|}{b_\epsilon}.$$

By using the properties of Young function, we obtain

$$\frac{1}{2}\sum_j\sum_k \Phi\left(\frac{\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|}{\|x,y\|_{\ell_\Phi} + \epsilon}\right) \le \frac{1}{2}\sum_j\sum_k \Phi\left(\frac{\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|}{b_\epsilon}\right) \le 1.$$

Therefore $\frac{1}{2}\sum_j\sum_k \Phi\left(\frac{1}{\|x,y\|_{\ell_\Phi} + \epsilon}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|\right) \le 1$. Since $\epsilon > 0$ is arbitrary, we have

$$\frac{1}{2}\sum_j\sum_k \Phi\left(\frac{1}{\|x,y\|_{\ell_\Phi}}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|\right) \le 1,$$

as desired. □

Lemma 2.2 $\|x,y\|_{\ell_\Phi} = 0$ *if and only if* $\frac{1}{2}\sum_j\sum_k \Phi\left(\frac{1}{\epsilon}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|\right) \leq 1$ *for every* $\epsilon > 0$.

Proof: ($\Leftarrow$) It is obvious that $\frac{1}{2}\sum_j\sum_k \Phi\left(\frac{1}{\epsilon}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|\right) \leq 1$ then $\|x,y\|_{\ell_\Phi} = 0$.

($\Rightarrow$) Suppose, on the contrary, that there is $\epsilon_0 > 0$ such that

$$\frac{1}{2}\sum_j\sum_k \Phi\left(\frac{1}{\epsilon_1}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|\right) > 1.$$

Next, write $\mathbb{B} = \left\{b > 0 : \frac{1}{2}\sum_j\sum_k \Phi\left(\frac{1}{b}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|\right) \leq 1\right\}$. As consequence $\|x,y\|_{\ell_\Phi} = \inf \mathbb{B}$. Take arbirtary $b \in \mathbb{B}$, we obtain $\epsilon_1 \neq b$. We consider two cases

Case I: $b < \epsilon_1$. By using the properties of Young function, we have

$$\frac{1}{2}\sum_j\sum_k \Phi\left(\frac{1}{\epsilon_1}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|\right) < \frac{1}{2}\sum_j\sum_k \Phi\left(\frac{1}{b}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|\right) \leq 1.$$

Case II: $b > \epsilon_1$. This implies that $\|x,y\|_{\ell_\Phi} > \epsilon_0 > 1$.

Hence, both cases contradict.

□

Lemma 2.3 $\|x,y\|_{\ell_\Phi} = 0$ *if and only if* $\frac{1}{2}\sum_j\sum_k \Phi\left(\alpha\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|\right) = 0$ *for every* $\alpha > 0$.

Proof: For every $0 < \epsilon < 1$ and $\alpha > 0$, we obtain

$$\begin{aligned}\Phi\left(\alpha\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|\right) &= \Phi\left((1-\epsilon)0 + \epsilon\left(\frac{\alpha}{\epsilon}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|\right)\right) \\ &\leq \epsilon\Phi\left(\frac{\alpha}{\epsilon}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|\right).\end{aligned}$$

Because $\|x,y\|_{\ell_\Phi} = 0$ then $\frac{1}{2}\sum_j\sum_k \Phi\left(\frac{\alpha}{\epsilon}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|\right) \leq 1$ by using Lemma 2.2. As consequence, we have

$$\frac{1}{2}\sum_j\sum_k \Phi\left(\alpha\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|\right) \leq \frac{\epsilon}{2}\sum_j\sum_k \Phi\left(\frac{\alpha}{\epsilon}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|\right) \leq \epsilon.$$

Since $0 < \epsilon < 1$ is arbitrary, we conclude that $\frac{1}{2}\sum_j\sum_k \Phi\left(\alpha\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|\right) = 0$ for every $\alpha > 0$. Conversely, suppose that $\frac{1}{2}\sum_j\sum_k \Phi\left(\alpha\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|\right) = 0$ for every $\alpha > 0$. Then

$$\frac{1}{\alpha} \in \left\{b > 0 : \frac{1}{2}\sum_j\sum_k \Phi\left(\alpha\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|\right) \leq 1\right\}.$$

Hence, $\|x,y\|_{\ell_\Phi} \leq \frac{1}{\alpha}$. Since $\alpha > 0$ is arbitrary, we conclude that $\|x,y\|_{\ell_\Phi} = 0$. □

Finaly, we have a 2-norm on discrete Orlicz space ℓ_Φ in the following theorem.

Theorem 2.1 *The mapping (2.1) defines a 2-norm on* ℓ_Φ.

Proof: We need to check that $\|\cdot,\cdot\|_{\ell_\Phi}$ satisfies the four properties of a 2-norm.

(1) Suppose that $\|x,y\|_{\ell_\Phi}=0$. By Lemma 2.3, we obtain

$$\frac{1}{2}\sum_j\sum_k \Phi\left(\alpha\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k\end{pmatrix}\right|\right)=0.$$

for every $\alpha>0$. Since the Young function Φ is non-negative number, we conclude that

$$\Phi\left(\alpha\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k\end{pmatrix}\right|\right)=0.$$

As consequence, we have $\det\begin{pmatrix} x_j & x_k \\ y_j & y_k\end{pmatrix}=0$. Hence, x and y are linear dependent. Conversely, suppose $x=my$ for some $m\in\mathbb{R}$. Observe that

$$\det\begin{pmatrix} x_j & x_k \\ y_j & y_k\end{pmatrix}=0.$$

Then

$$\begin{aligned}\|x,y\|_{\ell_\Phi} &= \inf\left\{b>0:\frac{1}{2}\sum_j\sum_k \Phi\left(\frac{1}{b}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k\end{pmatrix}\right|\right)\leq 1\right\}\\ &= \inf\left\{b>0:\frac{1}{2}\sum_j\sum_k \Phi(0)\leq 1\right\}=\inf\{b>0\}=0.\end{aligned}$$

(2) Observe that

$$\begin{aligned}\|x,y\|_{\ell_\Phi} &= \inf\left\{b>0:\frac{1}{2}\sum_j\sum_k \Phi\left(\frac{1}{b}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k\end{pmatrix}\right|\right)\leq 1\right\}\\ &= \inf\left\{b>0:\frac{1}{2}\sum_j\sum_k \Phi\left(\frac{1}{b}\left|\det\begin{pmatrix} y_j & y_k \\ x_j & x_k\end{pmatrix}\right|\right)\leq 1\right\}\\ &= \|y,x\|_{\ell_\Phi}.\end{aligned}$$

(3) Observe that

$$\begin{aligned}\|\gamma x,y\|_{\ell_\Phi} &= \inf\left\{b>0:\frac{1}{2}\sum_j\sum_k \Phi\left(\frac{1}{b}\left|\det\begin{pmatrix} \gamma x_j & \gamma x_k \\ y_j & y_k\end{pmatrix}\right|\right)\leq 1\right\}\\ &= \inf\left\{b>0:\frac{1}{2}\sum_j\sum_k \Phi\left(\frac{1}{\frac{b}{|\gamma|}}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k\end{pmatrix}\right|\right)\leq 1\right\}\\ &= \inf\left\{|\gamma|\,c>0:\frac{1}{2}\sum_j\sum_k \Phi\left(\frac{1}{c}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k\end{pmatrix}\right|\right)\leq 1\right\}\\ &= |\gamma|\inf\left\{c>0:\frac{1}{2}\sum_j\sum_k \Phi\left(\frac{1}{c}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k\end{pmatrix}\right|\right)\leq 1\right\}\\ &= |\gamma|\,\|x,y\|_{\ell_\Phi}.\end{aligned}$$

(4) Suppose that

$$\|x,y+z\|_{\ell_\Phi} = \inf\left\{ b>0 : \frac{1}{2}\sum_j\sum_k \Phi\left(\frac{1}{b}\left|\det\begin{pmatrix} x_j & x_k \\ y_j+z_j & y_k+z_k \end{pmatrix}\right|\right) \le 1 \right\}.$$

Using the properties of determinants and Lemma 2.1, we observe that

$$\begin{aligned}
&\frac{1}{2}\sum_j\sum_k \Phi\left(\frac{1}{\|x,y\|_{\ell_\Phi}+\|x,z\|_{\ell_\Phi}}\left|\det\begin{pmatrix} x_j & x_k \\ y_j+z_j & y_k+z_k \end{pmatrix}\right|\right) \\
&\le \frac{1}{2}\sum_j\sum_k \Phi\left(\frac{\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right| + \left|\det\begin{pmatrix} x_j & x_k \\ z_j & z_k \end{pmatrix}\right|}{\|x,y\|_{\ell_\Phi}+\|x,z\|_{\ell_\Phi}}\right) \\
&= \frac{1}{2}\sum_j\sum_k \Phi\left(\frac{\|x,y\|_{\ell_\Phi}}{\|x,y\|_{\ell_\Phi}+\|x,z\|_{\ell_\Phi}}\frac{\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|}{\|x,y\|_{\ell_\Phi}} + \frac{\|x,z\|_{\ell_\Phi}}{\|x,y\|_{\ell_\Phi}+\|x,z\|_{\ell_\Phi}}\frac{\left|\det\begin{pmatrix} x_j & x_k \\ z_j & z_k \end{pmatrix}\right|}{\|x,z\|_{\ell_\Phi}}\right) \\
&\le \frac{\|x,y\|_{\ell_\Phi}}{\|x,y\|_{\ell_\Phi}+\|x,z\|_{\ell_\Phi}}\sum_j\sum_k \Phi\left(\frac{1}{2\|x,y\|_{\ell_\Phi}}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|\right) \\
&\quad + \frac{\|x,z\|_{\ell_\Phi}}{\|x,y\|_{\ell_\Phi}+\|x,z\|_{\ell_\Phi}}\sum_j\sum_k \Phi\left(\frac{1}{2\|x,z\|_{\ell_\Phi}}\left|\det\begin{pmatrix} x_j & x_k \\ z_j & z_k \end{pmatrix}\right|\right) \\
&= \frac{\|x,y\|_{\ell_\Phi}}{\|x,y\|_{\ell_\Phi}+\|x,z\|_{\ell_\Phi}} + \frac{\|x,z\|_{\ell_\Phi}}{\|x,y\|_{\ell_\Phi}+\|x,z\|_{\ell_\Phi}} = 1.
\end{aligned}$$

By definition $\|x,y+z\|_{\ell_\Phi}$, we have $\|x,y+z\|_{\ell_\Phi} \le \|x,y\|_{\ell_\Phi} + \|x,z\|_{\ell_\Phi}$.

Hence, the mapping (2.1) is the 2-norm. □

Next, we discuss that the discrete Orlicz ℓ_Φ equipped with the 2 norm can be seen as a generalization of the space of p-summable sequence ℓ_p equipped with the 2-norm in [14] as follows.

Theorem 2.2 *If $\Phi(t) = t^p$ for $1 \le p < \infty$, then $\|x,y\|_{\ell_\Phi} = \|x,y\|_{\ell_p}$.*

Proof: Suppose that $\Phi(t) = t^p$ for $1 \le p < \infty$. Observe that

$$\begin{aligned}
\|x,y\|_{\ell_\Phi} &= \inf\left\{ b>0 : \frac{1}{2}\sum_j\sum_k \frac{1}{b^p}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|^p \le 1 \right\} \\
&= \inf\left\{ b>0 : \frac{1}{2}\sum_j\sum_k \left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|^p \le b^p \right\} \\
&= \inf B.
\end{aligned}$$

Since

$$\|x,y\|_{\ell_p}^p = \frac{1}{2}\sum_j\sum_k \left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|^p$$

then $\|x,y\|_{\ell_p} \le b$ for every $b \in B$. As consequence, $\|x,y\|_{\ell_p}$ is lower bound of B. Hence, $\|x,y\|_{\ell_p} \le \|x,y\|_{\ell_\Phi}$. Conversely, choosing $b = \|x,y\|_{\ell_p}$, we have

$$\frac{1}{2}\sum_j\sum_k \frac{1}{\|x,y\|_{\ell_p}^p}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|^p = \frac{1}{\|z,y\|_{\ell_p}^p}\left(\frac{1}{2}\sum_j\sum_k \left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|^p\right) = 1.$$

Hence, $b = \|x,y\|_{\ell_p} \in B$. Since $\inf B = \|x,y\|_{\ell_\Phi}$ then $\|x,y\|_{\ell_p} \geq \|x,y\|_{\ell_\Phi}$. Therefore, $\|x,y\|_{\ell_p} = \|x,y\|_{\ell_\Phi}$. □

2.2. $\ell_\Phi(\mathbb{Z})$ as a 2-Banach space

We know that ℓ_Φ is Banach space with respect to its usual norm $\|\cdot\|_{\ell_\Phi}$ [12]. Our aim now is to show that ℓ_Φ is a 2-Banach space with respect to its 2-norm $\|\cdot,\cdot\|_{\ell_p}$. To do so, we need the following lemmas.

Lemma 2.4 *[15] Let Φ be a Young function and $x \in \ell_\Phi$. If $0 < \|x\|_{\ell_\Phi} < \infty$ then*

$$\sum_j \Phi\left(\frac{|x_j|}{\|x\|_{\ell_\Phi}}\right) \leq 1.$$

Lemma 2.5 *For any $x, y \in \ell_\Phi$, we have*

$$\|x,y\|_{\ell_\Phi} \leq \|x\|_{\ell_\Phi}\|y\|_{\ell_\Phi}.$$

Proof: Suppose that

$$\|x,y\|_{\ell_\Phi} = \inf\left\{b > 0 : \frac{1}{2}\sum_j\sum_k \Phi\left(\frac{1}{b}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|\right) \leq 1\right\}.$$

Using the properties of the Young function Φ, we observe that

$$\begin{aligned}
\sum_j\sum_k \Phi\left(\frac{1}{\|x\|_{\ell_\Phi}\|y\|_{\ell_\Phi}}|x_jy_k - x_ky_j|\right) &\leq \sum_j\sum_k \Phi\left(\frac{|x_j|\,|y_k|}{\|x\|_{\ell_\Phi}\|y\|_{\ell_\Phi}} + \frac{|x_k|\,|y_j|}{\|x\|_{\ell_\Phi}\|y\|_{\ell_\Phi}}\right) \\
&\leq \sum_j\sum_k \left(\Phi\left(\frac{|x_j|\,|y_k|}{\|x\|_{\ell_\Phi}\|y\|_{\ell_\Phi}}\right) + \Phi\left(\frac{|x_k|\,|y_j|}{\|x\|_{\ell_\Phi}\|y\|_{\ell_\Phi}}\right)\right) \\
&= \sum_j\sum_k \Phi\left(\frac{|x_j|\,|y_k|}{\|x\|_{\ell_\Phi}\|y\|_{\ell_\Phi}}\right) + \sum_j\sum_k \Phi\left(\frac{|x_k|\,|y_j|}{\|x\|_{\ell_\Phi}\|y\|_{\ell_\Phi}}\right) \\
&\leq \sum_j \Phi\left(\frac{|x_j|}{\|x\|_{\ell_\Phi}}\right)\sum_k \Phi\left(\frac{|y_k|}{\|y\|_{\ell_\Phi}}\right) + \sum_k \Phi\left(\frac{|x_k|}{\|x\|_{\ell_\Phi}}\right)\sum_j \Phi\left(\frac{|y_j|}{\|y\|_{\ell_\Phi}}\right).
\end{aligned}$$

By Lemma 2.4, we obtain

$$\frac{1}{2}\sum_j\sum_k \Phi\left(\frac{1}{\|x\|_{\ell_\Phi}\|y\|_{\ell_\Phi}}\left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right|\right) = \frac{1}{2}\sum_j\sum_k \Phi\left(\frac{1}{\|x\|_{\ell_\Phi}\|y\|_{\ell_\Phi}}|x_jy_k - x_ky_j|\right) \leq 1.$$

Hence, $\|x,y\|_{\ell_\Phi} \leq \|x\|_{\ell_\Phi}\|y\|_{\ell_\Phi}$. □

Using Lemma 2.5, we have the following result.

Theorem 2.3 *If a sequence $\{x_n\} \in \ell_\Phi$ converges to x in the $\|\cdot\|_{\ell_\Phi}$ norm then $\{x_n\}$ also converges to x in the $\|\cdot,\cdot\|_{\ell_\Phi}$ norm. Similiarly, if $\{x_n\} \in \ell_\Phi$ is a Cauchy sequence with respect to the $\|\cdot\|_{\ell\Phi}$ norm then $\{x_n\} \in \ell_\Phi$ is also a Cauchy sequence with respect to the $\|\cdot,\cdot\|_{\ell_\Phi}$ norm.*

Proof: Let $\{x_n\} \in \ell_\Phi$ converges to some x in the $\|\cdot\|_{\ell_\Phi}$ norm, i.e., $\lim_{n\to\infty}\|x_n - x\|_{ell_\Phi} = 0$. By applying Lemma 2.5, we obtain

$$\lim_{n\to\infty}\|x_n - x, y\|_{\ell_\Phi} \leq \lim_{n\to\infty}\|x_n - x\|_{\ell_\Phi}\|y\|_{\ell_\Phi} = 0$$

for every $y \in \ell_\Phi$. Hence, $\{x_n\}$ also converges to some x in $\|\cdot,\cdot\|_{\ell_\Phi}$. The proof of the second part can be done similarly, completing the proof. □

Now, we can define a norm that is derived from the 2-norm in a specific manner. Indeed, if $\{a_1, a_2\}$ is a linearly independent set in ℓ_Φ, then one can observe that

$$\|x\|^*_{\ell_\Phi} = \|x, a_1\|_{\ell_\Phi} + \|x, a_2\|_{\ell_\Phi} \tag{2.2}$$

defines a norm on ℓ_Φ. One may observe that $\|x\|^*_{\ell_\Phi}$ in (2.2) satisfies the properties of a norm. In particular, we may check that if $\|x\|^*_{\ell_\Phi} = 0$, then $x = 0$. Indeed, if $\|x\|^*_{\ell_\Phi} = 0$, then we have $\|x, a_1\|_{\ell_\Phi} = 0$ and $\|x, a_2\|_{\ell_\Phi} = 0$. As consequence, $x = \gamma a_1$ for some $\gamma \in \mathbb{R}$. Therefore, $\gamma\|a_1, a_2\|_{\ell_\Phi} = 0$. Since $\|a_1, a_2\|_{\ell_\Phi} \neq 0$ then $\gamma = 0$. Hence, $x = 0$. Next, by properties of the 2-norm, we have (2) $\|\kappa x\|^*_{\ell_\phi} = |\kappa|\|x\|^*_{\ell_\phi}$ and (3) $\|x + y\|^*_{\ell_\phi} \leq \|x\|^*_{\ell_\phi} + \|y\|^*_{\ell_\phi}$.

The relationship between the derived norm $\|\cdot\|^*_{\ell_\Phi}$ and the usual norm $\|\cdot\|_{\ell_\Phi}$ on ℓ_Φ can be described as follows.

Lemma 2.6 *Let $\{a_1, a_2\}$ be a linearly independent set in ℓ_Φ. For any $x \in \ell_\Phi$, the following inequality holds:*

$$\|x\|^*_{\ell_\Phi} \leq (\|a_1\|_{\ell_\Phi} + \|a_2\|_{\ell_\Phi})\|x\|_{\ell_\Phi}.$$

Proof: By applying Lemma 2.5, we obtain

$$\|x, a_1\|_{\ell_\Phi} \leq \|x\|_{\ell_\Phi}\|a_1\|_{\ell_\Phi}$$

and

$$\|x, a_2\|_{\ell_\Phi} \leq \|x\|_{\ell_\Phi}\|a_2\|_{\ell_\Phi}$$

for any $x \in \ell_\Phi$. Using (2.2), we obtain

$$\|x\|^*_{\ell_\Phi} \leq (\|a_1\|_{\ell_\Phi} + \|a_2\|_{\ell_\Phi})\|x\|_{\ell_\Phi}.$$

□

For simplicity, we select $a_1 = (1, 0, 0, \ldots)$ and $a_2 = (0, 1, 0, \ldots)$ and define the norm $\|x\|_{\ell_\Phi}$ with respect to $\{a_1, a_2\}$ as described above. Then we have the following theorem:

Theorem 2.4 *The derived norm $\|\cdot\|^*_{\ell_\Phi}$ is equivalent to the usual norm $\|\cdot\|_{\ell_\Phi}$ on ℓ_Φ. Specifically, we have*

$$\|x\|_{\ell_\Phi} \leq \|x\|^*_{\ell_\Phi} \leq 2\|x\|_{\ell_\Phi}$$

for every $x \in \ell_\Phi$.

Proof: Take any $x \in \ell_\Phi$. By using Lemma 2.6, we have $\|x\|^*_{\ell_\Phi} \leq 2\|x\|_{\ell_\Phi}$. Next, because $a_1 = (1, 0, 0, \ldots)$ and $a_2 = (0, 1, 0, \ldots)$, we calculate

$$\|x, a_1\|_{\ell_\Phi} = \inf\left\{b > 0 : \sum_{k \neq 1} \Phi\left(\frac{|x_k|}{b}\right) \leq 1\right\}$$

and

$$\|x, a_2\|_{\ell_\Phi} = \inf\left\{b > 0 : \sum_{k \neq 2} \Phi\left(\frac{|x_k|}{b}\right) \leq 1\right\}.$$

Using the infimum property, we obtain

$$\inf\left\{b>0:\sum_k \Phi\left(\frac{|x_k|}{b}\right)\le 1\right\}\le \inf\left\{b>0:\sum_{k\neq 1} \Phi\left(\frac{|x_k|}{b}\right)\le 1\right\}+\inf\left\{b>0:\sum_{k\neq 2} \Phi\left(\frac{|x_k|}{b}\right)\le 1\right\}.$$

Because $\|x\|_{\ell_\Phi} := \inf\left\{b>0:\sum_k \Phi\left(\frac{|x_k|}{b}\right)\le 1\right\}$, we have $\|x\|_{\ell_\Phi}\le \|x\|^*_{\ell_\Phi}$. Hence,

$$\|x\|_{\ell_\Phi}\le \|x\|^*_{\ell_\Phi}\le 2\|x\|_{\ell_\Phi}.$$

This demonstrates that $\|\cdot\|^*_{\ell_\Phi}$ and norm $\|\cdot\|_{\ell_\Phi}$ are equivalent. □

As a consequence of Theorem 2.4, we have the following corollaries.

Corollary 2.1 *A sequence $\{x_n\}\in\ell_\Phi$ converges to an x in $\|\cdot\|_{\ell_\Phi}$ if and only if $\{x_n\}$ also converges to an x in $\|\cdot\|^*_{\ell_\Phi}$. Similiarly, $\{x_n\}\in\ell_\Phi$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{\ell_\Phi}$ if and only if $\{x_n\}\in\ell_\Phi$ is a Cauchy sequence with respect to the norm $\|\cdot\|^*_{\ell_\Phi}$.*

Since the discrete Orlicz space ℓ_Φ with respect to $\|\cdot\|_{\ell_\Phi}$ is a Banach space [12], then we have.

Corollary 2.2 *The discrete Orlicz space $(\ell_\Phi,\|\cdot\|^*_{\ell_\Phi})$ is a Banach space.*

Now, we will demonstrate the relationship between a Banach space with respect to the derived norm $\|\cdot\|^*_{\ell_\Phi}$ and a 2-Banach space with respect to the 2-norm $\|\cdot,\cdot\|_{\ell_\Phi}$ as follows.

Theorem 2.5 *Let $\{a_1,a_2\}$ be basis on ℓ_Φ. The discrete Orlicz space ℓ_Φ, when equipped with the 2-norm $\|\cdot,\cdot\|_{\ell_\Phi}$, is a 2-Banach space if and only if ℓ_Φ, when equipped with the derived norm $\|\cdot\|^*_{\ell_\Phi}$, is a Banach space.*

Proof: Assume that ℓ_Φ with respect to the 2-norm $\|\cdot,\cdot\|_{\ell_\Phi}$ is a 2-Banach space. Let $\{x_n\}$ be an arbitrary Cauchy sequence with respect to the norm $\|\cdot\|^*_{\ell_\Phi}$. Then, we have the following relation.

$$\|x_m-x_n,a_1\|_{\ell_\Phi}+\|x_m-x_n,a_2\|_{L_\Phi(X)}=\|x_m-x_n\|^*_{L_\Phi}\to 0$$

as $n,m\to\infty$. As a consequence, we obtain that $\|x_m-x_n,a_1\|_{L_\Phi}\to 0$ and $\|x_m-x_n,a_2\|_{\ell_\Phi}\to 0$ as $n,m\to\infty$. Since $\{a_1,a_2\}$ is basis on ℓ_Φ, for every $a\in\ell_\Phi$, we have

$$\begin{aligned}\|x_m-x_n,a\|_{\ell_\Phi} &= \|x_m-x_n,\alpha_1a_1+\alpha_2a_2\|_{\ell_\Phi}\\ &= |\alpha_1|\,\|x_m-x_n,a_1\|_{\ell_\Phi}+|\alpha_2|\,\|x_m-x_n,a_2\|_{\ell_\Phi}.\end{aligned}$$

This shows that $\|x_m-x_n,a\|_{\ell_\Phi}\to 0$ for every $a\in\ell_\Phi$. Therefore, $\{x_n\}$ is a Cauchy sequence with respect to the 2-norm. Since ℓ_Φ is a 2-Banach space, there exists an $x\in\ell_\Phi$ such that $\|x_n-x,a\|_{\ell_\Phi}\to 0$ as $n\to\infty$. In particular, we obtain $\|x_n-x,a_1\|_{\ell_\Phi}\to 0$ and $\|x_n-x,a_2\|_{\ell_\Phi}\to 0$ as $n\to\infty$. Moreover, we have

$$\|x_n-x\|^*_{\ell_\Phi(X)}=\|x_n-x,a_1\|_{\ell_\Phi}+\|x_n-x,a_2\|_{\ell_\Phi}\to 0.$$

Since the Cauchy sequence $\{x_n\}$ converges to some $x\in\ell_\Phi$, it follows that ℓ_Φ is a Banach space with respect to the norm $\|\cdot\|^*_{\ell_\Phi}$.

Conversely, assume that ℓ_Φ, equipped with the norm $\|\cdot\|^*_{\ell_\Phi}$, is a Banach space. Let $\{x_n\}$ be a Cauchy sequences in ℓ_Φ, equipped with the 2-norm $\|\cdot,\cdot\|_{\ell_\Phi}$, meaning that:

$$\lim_{m,n\to\infty}\|x_m-x_n,a\|_{\ell_\Phi}=0$$

for every $a \in \ell_\Phi$. In particular, for $a = a_1$ and $a = a_2$, we obtain $\lim_{m,n\to\infty} \|x_m - x_n, a_1\|_{\ell_\Phi} = 0$ and $\lim_{m,n\to\infty} \|x_m - x_n, a_2\|_{\ell_\Phi} = 0$. This leads to the following

$$\lim_{m,n\to\infty} \|x_m - x_n\|^*_{\ell_\Phi} = \lim_{m,n\to\infty} [\|x_m - x_n, a_1\|_{\ell_\Phi} + \|x_m - x_n, a_2\|_{\ell_\Phi}] = 0.$$

Thus, $\{x_n\}$ be a Cauchy sequences in ℓ_Φ with respect to the derived norm $\|\cdot\|^*_{\ell_\Phi}$. Since ℓ_Φ is a Banach space with respect to the derived norm $\|\cdot\|^*_{\ell_\Phi}$, there exists an $x \in \ell_\Phi$ such that $\lim_{n\to\infty} \|x_n - x\|^*_{\ell_\Phi} = 0$. Consequently, we have $\lim_{n\to\infty} \|x_n - x, a_1\|_{\ell_\Phi} = 0$ and $\lim_{n\to\infty} \|x_n - x, a_2\|_{\ell_\Phi} = 0$. Since $\{a_1, a_2\}$ is basis for ℓ_Φ, then for every $a \in \ell_\Phi$ we get

$$\begin{aligned} \|x_n - x, a\|_{\ell_\Phi} &= \|x_n - x, \alpha_1 a_1 + \alpha_2 a_2\|_{\ell_\Phi} \\ &= |\alpha_1| \, \|f_n - x, a_1\|_{\ell_\Phi} + |\alpha_2| \, \|x_n - x, a_2\|_{\ell_\Phi}. \end{aligned}$$

Hence, we conclde $\lim_{n\to\infty} \|x_n - x, a\|_{\ell_\Phi} = 0$ for every $a \in \ell_\Phi$. Since the Cauchy sequence $\{x_n\}$ converges to an $x \in \ell_\Phi$, it follows that ℓ_Φ is a Banach space with respect to the 2-norm $\|\cdot,\cdot\|_{\ell_\Phi}$. □

As a consequence of Corollary 2.2 and Theorem 2.5, we have the main result as follows.

Corollary 2.3 *The discrete Orlicz space* $(\ell_\Phi, \|\cdot,\cdot\|_{\ell_\Phi})$ *is a* 2*-Banach space.*

3. An Application

In the above section we have defined the 2-norm in discrete Orlicz space. In addition, we have also defined a new norm using the 2-norm and proved its equivalence to the usual norm in discrete Orlicz space. Using this result, we have proven that discrete Orlicz space with 2-norm is complete. With this result, we will now prove the following contractive mapping theorem on the discrete Orlicz space $(\ell_\Phi, \|\cdot,\cdot\|_{\ell_\Phi})$. The contractive mapping theorem on the space of p-summable sequences equipped 2-norm $((\ell_p, \|\cdot,\cdot\|_{\ell_p}))$ was formulated by Gunawan [14] and Idris et al. [16].

Theorem 3.1 *Let* $(\ell_\Phi, \|\cdot,\cdot\|_{\ell_\Phi})$ *be a* 2*-normed space and* $T : \ell_\Phi \to \ell_\Phi$. *If there is real number* $C \in (0, 1)$ *such that*

$$\|Tx - Ty, z\|_{\ell_\Phi} \le C \, \|x - y, z\|_{\ell_\Phi},$$

holds for every $x, y, z \in \ell_\Phi$ *then* T *has a unique fixed point in* ℓ_Φ.

Proof: Let $a_1 = (1, 0, \ldots)$ and $a_2 = (0, 1, 0, \ldots)$. By hypothesis, we obtain there is real number $C \in (0, 1)$ such that

$$\|Tx - Ty, a_i\|_{\ell_\Phi} \le C \, \|x - y, a_i\|_{\ell_\Phi},$$

holds for every $x, y \in \ell_\Phi$ and $i = 1, 2$. Using derived norm $\|\cdot\|^*_{\ell_\Phi}$, we observe that

$$\begin{aligned} \|Tx - Ty\|^*_{\ell_\Phi} &= \|Tx - Ty, a_1\|_{\ell_\Phi} + \|Tx - Ty, a_2\|_{\ell_\Phi} \\ &\le C \left[\|x - y, a_1\|_{\ell_\Phi} + \|x - y, a_2\|_{\ell_\Phi}\right] = C \, \|x - y\|^*_{\ell_\Phi}. \end{aligned}$$

Hence, T is a contractive mapping on $(\ell_\Phi, \|\cdot\|^*_{\ell_\Phi})$. Since $(\ell_\Phi, \|\cdot\|^*_{\ell_\Phi})$ is complete by Corollary 2.2, then T must have a unique fixed point in ℓ_Φ. □

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