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Ideal Structure of Semigroup Crossed Products by Endomorphisms *

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ABSTRACT: Given a cyclically ordered free abelian group G, a C*-algebra A which is not necessarily unital and an action α of the positive cone $P(G)$ by endomorphisms of A. We consider ideals of A and how do they relate with the ideals of the semigroup crossed product $A \times_\alpha P(G)$.

Key Words: Free abelian group, cyclically ordered group, C*-algebra, semigroup crossed product, endomorphisms, ideal structure, dynamical system.

Contents

1. Introduction

An attempt to extend ideas of the established theory of semigroups crossed products by endomorphisms has been shown in [12]. There was discussed the notion of semigroup crossed products by endomorphisms related to cyclically ordered abelian groups instead of regular linearly ordered groups considered in [8,4,1,2,3,10,5] and references therein. Every linearly ordered group is cyclically ordered group but not vice versa, hence the class of cyclically ordered groups is larger than the class of linearly ordered groups. Therefore the crossed product discussed in [12] is a sort of generalisation of regular crossed product related to linearly ordered groups, for example those are considered in [8,4,1,2,3,10,5].

Unfortunately in general the positive cone $P(G)$ of a cyclically ordered group G is not necessarily a semigroup [12]. Therefore it is very crucial to clasify cyclically ordered groups in which the positive cones are semigroups. In [11] it was shown that free abelian groups are cyclically ordered and the positive cones are semigroups. Hence the authors in [12] focussed on more specific class of groups i.e. cyclically ordered free abelian groups which ensure that the positive cones are semigroups [11].

An interesting problem related to the theory of crossed products is to look at ideal structure of a C*-algebra and how does it relate to the ideal structure of the crossed product, as it was considered in [2], [10] and [5]. We are interested to translate a similar problem to our setting.

Since we are going to analise ideals, we need to deal with the notion of crossed product which is also able to handle a non unital case. Given a cyclically ordered free abelian group G and a C*-algebra A which is not necessary to be unital, and an action α of $P(G)$ by endomorphisms of A. We borrow from [12] the notion of crossed product $A \times_\alpha P(G)$ of the dynamical system $(A, P(G), \alpha)$. To describe the ideal structure of the crossed product, we need to discuss the notion of invariant ideals which is a main key to describe the ideal structure of the crossed product.

The paper is organised as follow. Some preliminary results on cyclically ordered groups and some of their properties are presented in section 2. In subsection 3.1 we discuss the notion of crossed products by semigroups of endomorphisms. In subsection 3.2 we discuss invariant ideals which gives an ideal structure of the crossed product, which is the main results of our investigation.

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2. Preliminaries on Cyclically Ordered Groups

In [7] Hutington introduced the concept of cyclic order, it is a relation (ternary) compatible with the orientation of a circle. A ternary relation R on a non empty set S is a subset R of S^3. It is written $R(u, v, w)$ to denote $(u, v, w) \in R$.

Definition 2.1 A ternary relation R on a non empty set G is called a partial cyclic order if the following properties are satisfied:

R1 : if $R(u, v, w)$ then $u \neq v \neq w \neq u$ (strict),

R2 : if $R(u, v, w)$ then $R(v, w, u)$ (cyclic),

R3 : if $R(u, v, w)$ and $R(v, a, w)$ then $R(u, a, w)$ (transitive),

the cyclic order is denoted by R, and (G, R) is called a cyclically ordered set.

In [9] Rieger introduced the theory of partial cyclically ordered groups, denoted by $(G, +, R)$. It is a group G with a cyclic order R which is agree with the group's operation:

R4 : if $R(u, v, w)$ then $R(a + u + b, a + v + b, a + w + b)$ (compatible).

When additional property is satisfied:

R5 : if $u, v, w \in G$ such that $u \neq v \neq w \neq u$ then $R(u, v, w)$ or $R(u, w, v)$,

$(G, +, R)$ is called a total cyclically ordered group, or just a cyclically ordered group when the context is clear.

The concept of cyclically ordered groups generalised the theory of linearly ordered groups, because in general every linearly ordered group can be converted to be a cyclically ordered group, but not vice versa. If $(L, +, \leq)$ is a linearly ordered group, we can induce a cyclic order $R_{\leq}$ on L through the following way

$$R_{\leq}(u, v, w) \Leftrightarrow u < v < w \text{ or } v < w < u \text{ or } w < u < v. \tag{2.1}$$

The group $(L, +, R_{\leq})$ is a cyclically ordered group induced from $(L, +, \leq)$.

The set $P(G) := \{u \in G |\ R(0, u, 2u)\} \cup \{0\}$ is the positive cone of $(G, +, R)$. In contrast to linearly ordered groups, the positive cone of a cyclically ordered group is not necessarily a semigroup.

Remark 2.1 When G is a finite dimensional free abelian group, Theorem 1.1 of [6] implies that G is isomorphic to a direct sum of finite copies of the additive group of integers $\mathbb{Z}$. In [11] it was shown that there is a cyclic order which is induced from a usual linear lexicographic order of the direct sum. Under this cyclic order, G is then cyclically ordered and the positive cone $P(G)$ is a semigroup. A similar approach can be applied to any free abelian group, and hence any free abelian group is cyclically ordered and the positive cone is a semigroup.

It will be very advantageous if we are dealing with cyclically ordered groups whose positive cone is a semigroup. When the positive cone of a cyclically ordered abelian group is a semigroup, ones are able to view the elements as operators acting on a certain Hilbert space. In [13] it was formulated a representation of a positive cone of a cyclically ordered abelian group acting on a certain Hilbert space. Given a cyclically ordered abelian group G in which the positive cone $P(G)$ is a semigroup. For $y \in P(G)$, it was defined a bounded linear operator

$$V_y : \ell^2 (P(G), \mathbb{C}) \longrightarrow \ell^2 (P(G), \mathbb{C})$$

such that

$$(V_y (f))(x) = \begin{cases} f(x - y), & \text{if } x - y \in P(G) \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Since

$$\|V_y (f)\|^2 = \sum_{x - y \in P(G)} |f(x - y)|^2 = \sum_{z \in P(G)} |f(z)|^2 = \|f\|^2,$$

V_y is an isometry. A routine computation shows that $V : y \longmapsto V_y$ is a semigroup homomorphism of $P(G)$ into the C^*-algebra $B\left(\ell^2\left(P\left(G\right),\mathbb{C}\right)\right)$ of bounded operators on the Hilbert space $\ell^2\left(P\left(G\right),\mathbb{C}\right)$. Therefore, V is an isometric representation of the semigroup $P(G)$ on $\ell^2\left(P\left(G\right),\mathbb{C}\right)$.

3. Main Results

3.1. Crossed Product by Semigroups of Endomorphisms

In [1] and [5] it was discussed the notion of extensible endomorphisms. Suppose A is a C^*-algebra which is not necessarily unital, and endomorphism ϕ on A is called extensible if it extends uniquely to a strictly continuous endomorphism $\bar{\phi}$ of the multiplier algebra $M(A)$, i.e. there is an approximate identity (e_λ) for A and a projection p_ϕ in $M(A)$ such that $\phi(e_\lambda)$ converges strictly to p_ϕ in $M(A)$. Every nondegenerate homomorphism is clearly extensible [15, Corollary 2.51]. If ϕ is an extensible endomorphism and $(e_\lambda) \subset A$ is an approximate identity, then $\phi(e_\lambda)$ converges to $\bar{\phi}(1_{M(A)})$. It is clear that any endomorphism on a unital C^*-algebra is extensible.

Suppose G is a free abelian group. Example 2.1 assures that G is cyclically ordered, moreover the order is also total. Suppose $P(G)$ is the positive cone of $P(G)$. A *dynamical system* which is denoted by $(A, P(G), \alpha)$, is a system consists of a C^*algebra A, an action $\alpha : P(G) \to \mathrm{Endo}(A)$ of $P(G)$ on A by endomorphisms such that every α_t is extensible. A *covariant representation* of $(A, P(G), \alpha)$ is a pair (π, V) in which π is a nondegenerate representation of A on a Hilbert space H, V is an isometric representation of $P(G)$ on H and the covariant condition $\pi(\alpha_t(a)) = V_t\pi(a)V_t^*$ for $a \in A$ and $s, t \in P(G)$ is satisfied.

In [12] it was discussed the notion of semigroup crossed products of cyclically ordered abelian groups, which is an adaption of semigroup crossed products by totally ordered abelian groups of [10,5]. The definition of the crossed product in [12] is given through the following definition.

Definition 3.1 The *crossed product* of $(A, P(G), \alpha)$ is a triple $(B, i_A, i_{P(G)})$ of a C^*algebra B together with a nondegenerate homomorphism $i_A : A \to B$, a homomorphism $i_{P(G)} : P(G) \to M(B)$ of $P(G)$ into the multiplier algebra $M(B)$ of B as isometries such that $i_{P(G)}(s)i_{P(G)}(t) = i_{P(G)}(s+t)$ for all $s, t \in P(G)$, and that satisfies the following conditions:

1) $i_A(\alpha_t(a)) = i_{P(G)}(t)i_A(a)i_{P(G)}(t)^*$ for $a \in A$ and $t \in P(G)$;

2) for any covariant representation (π, V) of $(A, P(G), \alpha)$, there is a nondegenerate representation $\pi \times V$ of B such that $(\pi \times V) \circ i_A = \pi$ and $(\pi \times V) \circ i_{P(G)} = V$;

3) B is generated by elements of the form $\{i_A(a)i_{P(G)}(x) : a \in A, x \in P(G)\}$.

A similar approach to the proof of Lemma 2.1 in [5] is applicable to show that the C^*algebra B is the closure of

$$\mathrm{span}\{i_{P(G)}(x)^* i_A(a) i_{P(G)}(y) : x, y \in P(G), a \in A\}.$$

When a dynamical system $(A, P(G), \alpha)$ admits a nontrivial covariant representation, a similar argument to that of in [5, Proposition 2.2] it was shown that there is a crossed product for the system, it is unique up to isomorphism and we denote it as $A \times_\alpha P(G)$. A complete proof can be found in [14].

3.2. Invariant Ideals

Our discussion will deal with ideals of crossed products. For that we need to discuss the notion of extensible invariant ideal. Suppose A is a C^*-algebra which is not necessarily unital and α is an extensible endomorphism of A. Let I be be an ideal of A. If $\alpha(I) \subset I$, the ideal is called α-invariant. Suppose $\phi : A \to M(I)$ is the canonical nodegenerate homomorphism extending the inclusion $I \to M(I)$. It is clear that it satisfies $\phi(a)i = ai$, $\forall a \in A, i \in I$. An extensible α-invariant ideal is an α-invariant ideal I such that for an approximate identity $(u_\lambda) \subset I$, we have $\alpha(u_\lambda)$ converges strictly to $\bar{\phi}(\bar{\alpha}(1_{M(A)}))$ in $M(I)$.

Theorem 3.1 *Suppose $(A, P(G), \alpha)$ is a dynamical system in which the crossed product $A \times_\alpha P(G)$ exists. If I is an extensible α-invariant ideal of A, then the set*

$$D := \overline{\operatorname{span}}\{i_{P(G)}(x)^* i_A(i) i_{P(G)}(y) : i \in I, x, y \in P(G)\}$$

is an ideal of $(A \times_\alpha P(G), i_A, i_{P(G)})$.

Proof: It is enough to show that D is closed under multiplication. Suppose $x, y, u, v \in P(G), a \in A, i \in I$ and consider $i_{P(G)}(x)^* i_A(i) i_{P(G)}(y) \in D$, $i_{P(G)}(u)^* i_A(a) i_{P(G)}(v) \in A \times_\alpha P(G)$. Then

$$i_{P(G)}(x)^* i_A(i) i_{P(G)}(y) i_{P(G)}(u)^* i_A(a) i_{P(G)}(v)$$

$$= \begin{cases} i_{P(G)}(x)^* i_A(i) i_{P(G)}(u-y)^* i_A(a) i_{P(G)}(v) & \text{if } u - y \in P(G), u \neq y, \\ i_{P(G)}(x)^* i_A(i) i_{P(G)}(y-u) i_A(a) i_{P(G)}(v) & \text{otherwise,} \end{cases}$$

$$= \begin{cases} i_{P(G)}(x+u-y)^* i_A(\alpha_{u-y}(i)a) i_{P(G)}(v) & \text{if } u - y \in P(G), u \neq y, \\ i_{P(G)}(x)^* i_A(i\alpha_{y-u}(a)) i_{P(G)}(y-u+v) & \text{otherwise,} \end{cases}$$

which are in D, because $\alpha_{u-y}(i)a$ and $i\alpha_{y-u}(a)$ are all in I from the α-invariance of I. Moreover D is closed under taking adjoint and the multiplication is continuous, therefore D is an ideal of $A \times_\alpha P(G)$ □

Let $(A, P(G), \alpha)$ be a dynamical system, and I be an extensible α-invariant ideal of A. The α-invariance of I implies that for each $x \in P(G)$, $\alpha_x(I) \subset I$, in other word the restriction $\alpha_x \in \operatorname{Endo}(A)$ to I gives an endomorphism of I. We denote by $\alpha|_I$ for such restriction. Therefore we can consider the dynamical system $(I, P(G), \alpha|_I)$.

Theorem 3.2 *Suppose $(A, P(G), \alpha)$ is a dynamical system in which the crossed product $A \times_\alpha P(G)$ exists. If I is an extensible α-invariant ideal of A then the ideal D in Lemma 3.1 is the crossed product of the dynamical system $(I, P(G), \alpha|_I)$. Therefore $I \times_{\alpha|_I} P(G)$ is an ideal of $A \times_\alpha P(G)$.*

Proof: Since D is an ideal of $A \times_\alpha P(G)$, there is a nondegenerate homomorphism $r : A \times_\alpha P(G) \to M(D)$ extending the inclusion $D \to M(D)$, and hence it satisfies $r(\xi)d = \xi d$ for $\xi \in A \times_\alpha P(G)$ and $d \in D$. We define $j_I : I \to D$ by

$$j_I(i) = r \circ i_A|_I(i) = i_A|_I(i) = i_A(i), \tag{3.1}$$

and $j_{P(G)} : P(G) \to M(D)$ by

$$j_{P(G)}(x) = \bar{r} \circ i_{P(G)}(x). \tag{3.2}$$

We show that $(D, j_I, j_{P(G)})$ is the crossed product $I \times_{\alpha|_I} P(G)$. Firstly, extensibility of I implies that j_I is nodegenerate, and a routine computation shows that $(j_I, j_{P(G)})$ satisfies the covariance property. Secondly, suppose (π, V) is a covariant representation of $(I, P(G), \alpha|_I)$, we want to get the representation $\pi \times V$ which satisfies property 2) in Definition 3.1. If $\psi : A \to M(I)$ is the canonical homomorphism, then $(\bar{\pi} \circ \psi, V)$ is a covariant representation of $(A, P(G), \alpha)$. Hence there is a nondegenerate representation ρ of $A \times_\alpha P(G)$ such that $\rho \circ i_A = \bar{\pi} \circ \psi$ and $\bar{\rho} \circ i_{P(G)} = V$. A routine computation shows that the restriction of ρ on D is nondegenerate, and hence $\rho|_D$ extends to a unital representation $\overline{\rho|_D}$ of $M(D)$ which satisfies $\rho|_D \circ j_I = \pi$ and $\overline{\rho|_D} \circ j_{P(G)} = V$. Therefore $\rho|_D$ is the representation $\pi \times V$ that we are seeking. Finally we need to show that D is $\overline{\operatorname{span}}\{j_{P(G)}(x)^* j_I(i) j_{P(G)}(y) : x, y \in P(G), i \in I\}$. But this is clear, since for $x, y \in P(G)$ and $i \in I$ we have

$$\begin{aligned} i_{P(G)}(x)^* i_A(i) i_{P(G)}(y) &= (\bar{r} \circ i_{P(G)}(x)^*)(r \circ i_A|_I(i))(\bar{r} \circ i_{P(G)}(y)) \\ &= j_{P(G)}(x)^* j_I(i) j_{P(G)}(y). \end{aligned}$$

Hence D is the crossed product $I \times_{\alpha|_I} P(G)$. □

Theorem 3.3 *Suppose $(A, P(G), \alpha)$ is a dynamical system in which the crossed product $A \times_\alpha P(G)$ exists. If I is an extensible α-invariant ideal of A then the crossed product $(A/I \times_{\tilde{\alpha}} P(G), k_{A/I}, k_{P(G)})$ exists, where $\tilde{\alpha}_x : a + I \mapsto \alpha_x(a) + I$.*

Proof: The action α induces an action $\tilde{\alpha}$ of $P(G)$ on A/I by endomorphisms: for $x \in P(G)$, the mapping $\tilde{\alpha}_x : a + I \mapsto \alpha_x(a) + I$ defines an endomorphism of A/I, and hence $\tilde{\alpha}$ defines the action we wanted. Furthermore $\tilde{\alpha}$ is extensible, because α is. We then consider the dynamical system $(A/I, P(G), \tilde{\alpha})$.

Suppose ρ is a nondegenerate representation of $A \times_\alpha P(G)$ with kernel $D := \overline{\text{span}}\{i_{P(G)}(x)^* i_A(i) i_{P(G)}(y) : i \in I, x, y \in P(G)\}$ (Lemma 3.1 allows us to consider the quotient $(A \times_\alpha P(G))/(I \times_{\alpha|_I} P(G))$, and then D is the kernel of the representation of the quotient). Since $\ker(\rho \circ i_A) = I$, the fundamental theorem of homomorphisms implies that there is a nondegenerate representation η of A/I such that $\eta(a + I) = \rho \circ i_A(a)$. Therefore $(\eta, \bar{\rho} \circ i_{P(G)})$ is a covariant representation of $(A/I, P(G), \tilde{\alpha})$. Theorem 1 of [14] implies the existence of the crossed product $(A/I \times_{\tilde{\alpha}} P(G), k_{A/I}, k_{P(G)})$. □

Theorem 3.4 *Suppose $A \times_\alpha P(G)$ is the crossed product of a dynamical system $(A, P(G), \alpha)$, and suppose I is an extensible α-invariant ideal of A. Then there is a short exact sequence*

$$0 \to I \times_{\alpha|_I} P(G) \xrightarrow{\Psi} A \times_\alpha P(G) \xrightarrow{\Phi} A/I \times_{\tilde{\alpha}} P(G) \to 0.$$

Proof: Lemma 3.2 assures the existence of $(I \times_{\alpha|_I} P(G), j_I, j_{P(G)})$ and gives the injection Ψ, the remaining tasks are to produce a surjection Φ with $\ker \Phi = \Psi(I \times_{\alpha|_I} P(G)) = D$. We consider the crossed product $(A/I \times_{\tilde{\alpha}} P(G), k_{A/I}, k_{P(G)})$ we obtained in Lemma 3.3. Let q be the canonical surjection of A onto A/I. The embedding $k_{A/I} \circ q$ is nondegenerate and $(k_{A/I} \circ q, k_{P(G)})$ is a covariant representation of $(A, P(G), \alpha)$, hence condition 2) of Definition 3.1 implies that there is a nondegenerate homomorphism Φ of $A \times_\alpha P(G)$ to $M(A/I \times_{\tilde{\alpha}} P(G))$ such that $\Phi \circ i_A = k_{A/I} \circ q$ and $\bar{\Phi} \circ i_{P(G)} = k_{P(G)}$. It is clear that $D \subset \ker \Phi$, because $\Phi(i_{P(G)}(x)^* i_A(i) i_{P(G)}(y)) = k_{P(G)}(x)^* k_{A/I} \circ q(i) k_{P(G)}(y) = 0, \forall x, y \in P(G), i \in I$. To see that $\ker \Phi = D$, consider the representation ρ and η we obtained in the proof of Lemma 3.3. Since $(\eta, \bar{\rho} \circ i_{P(G)})$ is a covariant representation of $(A/I, P(G), \tilde{\alpha})$, property 2) of Definition 3.1 implies that there is a nondegenerate representation $\eta \times \bar{\rho} \circ j_{P(G)}$ of $A/I \times_{\tilde{\alpha}} P(G)$ such that $(\eta \times \bar{\rho} \circ j_{P(G)}) \circ \Phi = \rho$. Hence $\ker \Phi \subset D$. Finally , the range of Φ is $A/I \times_{\tilde{\alpha}} P(G)$, because for $x, y \in P(G)$ and $a \in A$ we have

$$\Phi(i_{P(G)}(x)^* i_A(a) i_{P(G)}(y)) = k_{P(G)}(x)^* k_{A/I}(a) k_{P(G)}(y)$$

which generates $A/I \times_{\tilde{\alpha}} P(G)$. □

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References

1. S. Adji, *Crossed Products of C^*- Algebras by Semigroups of Endomorphisms.* Ph.D. thesis, The University of Newcastle, 1995.
2. S. Adji, *Semigroup crossed products and the structure of Toeplitz algebras*, J. Op. Theory **44** (2000), 139–150.
3. S. Adji, *A remark on semigroup crossed products*, Vietnam Journal of Mathematics **31**:4 (2003), 429-435.
4. S. Adji, M. Laca, M. Nilsen and I. Raeburn, *Crossed products by semigroups of endomorphisms and the Toeplitz algebras of ordered groups*, Proc. Amer. Math. Soc. **122(4)** (1994), 1133-1141.
5. S. Adji and R. Rosjanuardi, *Twisted semigroup crossed products and the Toeplitz algebras of ordered groups*, Acta Mathematica Sinica, English Series **23(9)** (2007), 1639-1648.
6. T. W. Hungerford, *Algebra*, Springer-Verlag, GTM 73 (1974).
7. E. V. Hutington, *A set of independent postulates fot cyclic order*, *Proceedings of the National Academy of Sciences of the United States of America* **2 (11)**(1916), 630-631 .

8. G. J. Murphy, *Ordered group and Toeplitz algebras*, J. Operator Theory **18** (1987), 303-326.

9. L. Rieger, *On ordered and cyclically ordered groups* (Czech), Vestnik Kral. Cesk Spol. Nauk. **6** (1946), 1-26.

10. R. Rosjanuardi, *Structure Theory of Twisted Toeplitz Algebras and Ideal Structure of the (Untwisted) Toeplitz Algebras of Ordered Groups*. Ph.D. thesis, Institut Teknologi Bandung, Indonesia, 2005.

11. R. Rosjanuardi, S. M. Gozali, and I. N. Albania, *c-convex subgroups of finite dimensional cyclically ordered free abelian groups*, International Journal of Mathematics and Computer Science, **18(1)**, 37-45 (2023).

12. R. Rosjanuardi, S. M. Gozali, and I. N. Albania, *Faithful representations of semigroup crossed products by endomorphisms*, International Journal of Mathematics and Computer Science, **20(1)**, 307-312 (2025).

13. R. Rosjanuardi, I. Yusnitha, and S. M. Gozali, *Representations of cyclically ordered semigroups*, MATEC Web of Conferences. **197**, 01002 (2018), 1-3.

14. R. Rosjanuardi, I. Yusnitha, and S. M. Gozali, *Crossed Products by Cyclically Ordered Semigroups*, Preprint.

15. D.P. William and I. Raeburn, *Morita Equivalence and Continuous-Trace C^*-Algebras*. Amer. Math. Soc., Providence, RI, 1998.

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