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The Cauchy problem for fractional m-evolution models with memory: Well-Posedness and blow-up time estimates

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ABSTRACT: In this paper, we investigate the well-posedness and establish lower and upper bounds for the blow-up time of solutions to a class of fractional Laplacian equations. The governing model includes a nonlinear source term and dissipative effects with variable exponents. The equation features a wave-like structure with a fractional diffusion term, a memory term involving a convolution with a relaxation kernel, a nonlinear damping term with a variable exponent, and a source term with another variable exponent. The kernel is assumed to be smooth, non-increasing, and satisfies a specific smallness condition on its total integral. We prove the existence of a solution and then derive lower and upper bounds for the blow-up time, which depend on the fractional exponent, the variable growth exponents in the damping and source terms, and the properties of the relaxation kernel.

Keywords: Nonlinear Higher-order wave equation, memory, lower and upper bounds for the blow-up, critical exponents, variable exponent nonlinearity

Contents

1. Introduction

In recent years, the viscoelastic equation with nonlinear frictional damping and a relaxation function involving exponent nonlinearities has increasingly attracted much attention, as it better describes the memory and heredity of some complex systems compared to the wave equation with nonlinear damping and source terms. So far, mathematical nonlinear models of hyperbolic, parabolic, and elliptic equations with variable exponents of nonlinearity have been applied in various research fields, such as flows of electro-rheological fluids, nonlinear viscoelasticity, filtration processes in porous media, and image processing equations with variable exponents. For additional relevant references, we direct readers to [7,8]. The study of variable exponent nonlinearities extends the analysis of constant exponents in Lebesgue and Sobolev spaces, first introduced in the literature in 1931 by [6]. In this concept, the order can continuously change as a function of either dependent or independent variables to more accurately describe the variation of memory properties over time or space [11]. The spaces of variable exponent nonlinearity have been employed to describe various physical phenomena, including chemical reactions, heat transfer, population dynamics, and biological sciences (e.g., [9,10]). Notable studies have focused on areas such as electrorheological fluids (e.g., [12]), porous media [15], and image processing (e.g., [13,14]). These topics are both novel and intriguing, originating from the theories of nonlinear elasticity and electrorheological fluids. These fluids possess the fascinating property of varying viscosity in response to an applied electric field. For a general overview of the underlying physics, refer to [28]; for mathematical perspectives, see [26]. A series of papers addressing issues related to rheological and electrorheological fluids, which have led to the study of spaces with variable exponents, has recently been published by Diening and Růžička [27]. The primary effects investigated typically involve $p(.)$-Laplacian operators, where the exponent $p(.)$

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varies with the spatial variable. Research has shown that spatial variations can disrupt the propagation of waves in nonlinear wave equations. In this paper, we investigate the well-posedness and establish upper and lower bounds for the blow-up time of solutions to a class of fractional Laplacian equations involving a nonlinear source and dissipative terms with variable exponents

$$\begin{array}{c} u_{tt} + (-\Delta)^m u - \int_0^t g(t-s)(-\Delta)^m u(x,s)\,\mathrm{d}s + |u_t|^{r(.)-2} u_t \\ = |u|^{p(.)-2} u \ \text{ in } \Omega\times(0,+\infty), \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \qquad x\in\Omega, \\ \frac{\partial^i u}{\partial\nu^i}=0, \ i=0,1,...,m-1, \ \text{on } \Gamma\times(0,+\infty), \end{array} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n (n\geq 1)$ with a smooth boundary $\partial\Omega$, ν is the unit outward normal vector on $\partial\Omega$, $\frac{\partial^i u}{\partial\nu^i}$ denotes the i-th order normal derivative of u, $u_0\in H_0^m(\Omega)$, $u_1\in L^2(\Omega)$, $m\geq 1$ is a natural number, $g(.)$ is a positive function satisfying certain conditions to be specified later, and T is the maximal existence time of solutions. In the higher-order case when $m\geq 1$, the single model (1.1) one of the initial boundary value problems of the system

$$\begin{cases} u_{tt} + (-\Delta)^m u = |u|^{p-2} u \ \text{ in } \Omega\times(0,+\infty), \\ u(x,0)=u_0(x), \ u_t(x,0)=u_1(x), \quad \text{in } \Omega, \\ \frac{\partial^i u}{\partial\nu^i}=0, \ i=0,1,...,m-1, \ \text{on } \partial\Omega\times(0,+\infty), \end{cases} \tag{1.2}$$

which many authors have examined. In particular, P. Brenner and W. Von Wahl [16] established the existence and uniqueness of classical solutions to the problem presented in (1.2) within the framework of Hilbert spaces. H. Pecher [18] investigated the existence and uniqueness of the Cauchy problem associated with the equation in (1.2), using the potential well method developed by L. Payne and D.H. Sattinger [17] as well as D.H. Sattinger [19]. B.X. Wang [20] demonstrated that the scattering operators map a band in H^s to H^s when the nonlinearities have critical or subcritical powers in H^^Furthermore, C.X. Miao [21] derived scattering theory at low energy levels through time-space estimates and nonlinear estimates. He also established the global existence and uniqueness of solutions under low energy conditions. Moreover, M. Nakao [22] used the Galerkin method to show the existence and uniqueness of bounded solutions, periodic solutions, and almost periodic solutions for the problem (1.2), as the equation in (1.2) includes a linear dissipative term μu_t. By applying a difference inequality, M. Nakao and H. Kuwahara [23] explored the decay estimates of global solutions for the problem (1.2) with a degenerate dissipative term $a(x)u_t$. Given that the equation in (1.2) features a nonlinear dissipative term $|u_t|^{q-2} u_t$, Y.J. Ye [24] investigates the existence and asymptotic behavior of global solutions for the problem (1.2). A.B. Aliev and B.H. Lichaei [25] simultaneously address the Cauchy problem for the equation referenced in (1.2). They uncover criteria for the existence and nonexistence of global solutions by applying the $L^p - L^q$ estimate for the corresponding linear problem. Additionally, they establish the asymptotic behavior of solutions and their derivatives as $t\to+\infty$. Considerable effort has been devoted to studying the problem referenced in (1.1), which involves both constant and variable-exponent nonlinearities. Messaoudi et al. [2] examined the existence and blow-up of solutions for the nonlinear damped wave equation, considering the exponents $m(.)$ and $p(.)$ as measurable functions.

$$u_t - \Delta u + a|u_t|^{m(.)-2} u_t = b|u|^{p(.)-2}u, \quad \text{in} \quad \Omega\times(0,T)$$

where a and b are positive constants. They established that a solution with negative initial energy blows up in finite time for the following quasilinear wave equation with variable exponent nonlinearities:

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{r(.)-2}\nabla u\right) + a|u_t|^{m(.)-2} u_t = b|u|^{p(.)-2}u, \quad \text{in} \quad \Omega\times(0,T)$$

Messaoudi and Talahmeh [3] investigated the blow-up behavior in solutions of viscoelastic hyperbolic equations. They established blow-up results for solutions with negative initial energy, as well as certain solutions with positive energy. To our knowledge, there are currently no established blow-up results for solutions of viscoelastic hyperbolic equations that involve variable exponents. In the case where the exponent $p(.)$ is a constant and the dissipation term is absent, Yaojun [1] focused on a higher-order viscoelastic wave equation with nonlinear damping.

$$u_{tt} + (-\Delta)^m u - \int_0^t g(t-s)(-\Delta)^m u(x,s)\,\mathrm{d}s = |u|^{p-2} u \quad \text{in } \Omega \times (0, +\infty). \tag{1.3}$$

The Faedo-Galerkin method is employed to demonstrate the existence of global weak solutions. Furthermore, the author indicates that under suitable conditions for the relaxation function $g(.)$, the solution can exhibit blow-up properties within a finite time, regardless of whether the initial energy is positive or non-positive. The author also provides estimates regarding the lifespan of these solutions. For plate problems involving memory terms with variable exponents, there is a great deal of results in the literature regarding existence, uniqueness and asymptotic behavior for these plate-type equations. For example, Al-Mahdi et al. [32] studied the semi-linear dissipative plate equation with variable-exponent nonlinearity, where the linear part is given by

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta u(s)ds + |u_t|^{p(x)-2}u_t - \Delta u_t = 0,$$

and established the global existence of solutions and the polynomial decay of the energy. Park and Kang in [33] and Piskin in [34] considered a similar problem involving a variable exponent in the second term, using the following equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a|u_t|^{m(x)-2}u_t = b|u|^{p(x)-2}u,$$

proved the global existence and established blow-up of solutions. For stability, we refer to Park [35] who treated the following viscoelastic von Karman system

$$\begin{cases} u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)\mathrm{d}s + |u_t|^{m(.)-2}u_t = [u, \chi(u)] + |u|^{p(.)-2}u \\ \Delta^2 \chi(u) = -[u, u] \end{cases}$$

and established energy decay estimates under the general assumption on the function g. Recently, Messaoudi et.al. in [5] investigate the general decay rate of the solutions for a class of fractional Laplacian equations given by

$$u_{tt} + (-\Delta)^\alpha u - \int_0^t g(t-s)(-\Delta)^\beta u(s)\,ds = 0, \quad \text{in } \mathbb{R}^n, \ n \geq 1.$$

The authors utilized the energy method in Fourier space to construct an appropriate Lyapunov functional, allowing them to obtain the necessary estimates for the solution's image in Fourier space. They also provided a two-dimensional numerical example to illustrate their findings. Building on previous research, this paper demonstrates the local existence of weak solutions to problem (1.1) by employing the Faedo-Galerkin method in conjunction with the fixed point theorem. Additionally, under suitable conditions on the relaxation function $g(.)$ and for both positive and non-positive initial energy, we derive results regarding blow-up phenomena and establish both lower and upper bounds for the blow-up time. The upper bound supports the expansion of the solution, while the lower bound may offer a reliable time interval for implementation when using (1.1) to model a physical phenomenon. To our knowledge, similar results in the context of variable exponents for such problems are quite rare.

2. Preliminaries

Let $H^m(\Omega$ denote the Sobolev space equipped with the usual scalar products and norms. $H_0^m(\Omega)$ represents the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$. For simplicity, we will denote the Lebesgue space $L^q(\Omega)$ norm by $\|.\|_q$ for $1 \leq q \leq \infty$. We will use the equivalent norm $\|D^m.\|$ as a substitute for the $H_0^m(\Omega)$ norm $\|.\|_{H_0^m(\Omega)}$, where D denotes the gradient operator, specifically $D. = \nabla. = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)$.

Furthermore, if $m=2j$, we have $D^m. = \Delta^j$, while if $m=2j+1$, it follows that $D^m. = D\Delta^j$. Additionally, $C_i (i = 1, 2, \ldots)$ denotes various positive constants that depend on known constants and may vary with each occurrence. Here, $m \geq 1$ is a natural number, and the measurable functions $p(.)$ and $r(.)$ satisfy the bounded exponent condition.

(H1) The function $p(.)$ satisfying

$$\begin{cases} 2 < p_{1,2} < \infty, \ n \leq 2m, \\ 2 < p_1 \leq p(x) \leq p_2 < \frac{2m}{n-2m}, \ n \geq 2m+1, \end{cases}$$

with

$$p_2 = ess \sup_{x \in \Omega} p(x), \ p_1 = ess \inf_{x \in \Omega} p(x). \tag{2.1}$$

(H2) The function $r(.)$ satisfying $2 \leq r_1 \leq r(x) \leq r_2 < \infty$, with

$$r_2 = ess \sup_{x \in \Omega} r(x), \ r_1 = ess \inf_{x \in \Omega} r(x).$$

Assume except that $r(.)$, and $p(.)$ verify the log-Hölder continuity condition:

$$|q(x) - q(y)| \leq M(|x-y|), \tag{2.2}$$

where $M(r)$ satisfies

$$\lim_{r \to 0^+} \sup M(r) \ln \left(\frac{1}{r} \right) = c < \infty.$$

The relaxation function g fulfills

(H3) $g : [0, +\infty) \to [0, +\infty)$ is a C^1 non-increasing function satisfying

$$1 - \int_0^\infty g(s) \mathrm{d}s = \beta > 0, \tag{2.3}$$

and

$$\int_0^\infty g(s) \mathrm{d}s < \frac{p_1(p_1 - 2)}{(p_1 - 1)^2}. \tag{2.4}$$

Now, we define the following functionals.

The total energy associated with the problem (1.1) by $\mathrm{E} : H_0^m(\Omega) \cap L^{p(\cdot)}(\Omega) \to \mathbb{R}$, which is defined by

$$\mathrm{E(t)} = \frac{1}{2} \|u_t\|_2^2 + \frac{1}{2} \left(1 - \int_0^t g(s) \mathrm{d}s\right) \|D^m u(t)\|_2^2 + \frac{1}{2} (g \circ D^m u)(t) - \int_\Omega \frac{1}{p(x)} |u(t)|^{p(x)} \mathrm{d}x. \tag{2.5}$$

where

$$(g \circ D^m u)(t) = \int_0^t g(t-s) \|D^m u(t) - D^m u(s)\|_2^2 \, ds.$$

and $\mathrm{E}(0) = \frac{1}{2} \|u_1\|_2^2 + \frac{1}{2} \|D^m u_0\|_2^2 - \int_\Omega \frac{1}{p(x)} |u_0|^{p(x)} \mathrm{d}x$ is the initial total energy.

By simple calculations, we have.

$$\begin{aligned} -\int_0^t g(t-s) \int_\Omega D^m u_t(t) D^m u(s) \mathrm{d}x \mathrm{d}s &= \tfrac{1}{2} g(t) \int_\Omega |D^m u(t)|^2 \mathrm{d}x - \tfrac{1}{2} (g' \circ D^m u)(t) \\ &+ \tfrac{1}{2} \tfrac{\mathrm{d}}{\mathrm{d}t} \left((g \circ D^m u)(t) - \left(\int_0^t g(s) \mathrm{d}s \int_\Omega |D^m u(t)|^2 \mathrm{d}x \right) \right). \end{aligned} \tag{2.6}$$

In the following, we prepare some lemmas needed in the proof of the main results.

Lemma 2.1 (Sobolev-Poincaré inequality) *Let $p(.)$ satisfy condition (H1). For all $u \in H_0^m(\Omega)$, the following embedding*

$$H_0^m(\Omega) \hookrightarrow L^{p_2}(\Omega) \hookrightarrow L^{p(.)}(\Omega) \hookrightarrow L^{p_1}(\Omega) \hookrightarrow L^2(\Omega),$$

is continuous. We obtain the following inequalities:

$$\|u\|_{p(.)} \leq B\|D^m u\|_2,\ \|u\|_{\frac{2n}{n-2m}} \leq \bar{B}\|D^m u\|_2 \quad (n > 2m),\ \|u\|_{p_2} \leq \hat{B}\|D^m u\|_2,$$

where B, $\bar{B}$, $\hat{B}$ are the optimal constants of the Sobolev embedding, and $\|.\|_{p(.)}$ denotes the norm in $L^{p(.)}(\Omega)$. This norm has the property that:

$$\min\left(\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right) \leq \varrho(u) = \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x \leq \max\left(\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right),$$

for any $u \in L^{p(.)}(\Omega)$.

We denote $\|.\|_q$ and $\|.\|_{H^m(\Omega)}$ as the usual $L^q(\Omega)$ norm and $H^m(\Omega)$, respectively.

Proof: By the Sobolev embedding theorem, we obtain $H_0^m(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ and $\|u(t)\|_{p(.)} \leq C\|u(t)\|_{H_0^m(\Omega)}$. From the Poincaré inequality, we know that $\|u(t)\|_{H_0^m(\Omega)}$ and $\|D^m u(t)\|_2$ are equivalent, leading to the conclusion. □

Lemma 2.2 *Assuming that condition (H3) holds and that $u(t)$ is a solution to problem (3.1), then $E(t)$ is a non-increasing function for $t \geq 0$ and*

$$\mathrm{E}'(t) = \frac{1}{2}\left(g' \circ D^m u\right)(t) - \frac{1}{2}g(t)\|D^m u(t)\|_2^2 - \int_\Omega |u_t(t)|^{m(x)}\mathrm{d}x \leq 0, \tag{2.7}$$

where

$$\left(g' \circ D^m u\right)(t) = \int_0^t g'(t-s)\left\|D^m v(t) - D^m v(s)\right\|_2^2 ds.$$

Proof: Multiplying the equation (1.1) by u_t and integrating over $\Omega \times [0, t]$, we apply integration by parts to obtain that

$$\mathrm{E}(t) = \mathrm{E}(0) + \frac{1}{2}\int_0^t \left[\left(g' \circ D^m u\right)(s) - g(s)\int_\Omega |D^m u(s)|^2\mathrm{d}x - 2\int_\Omega |u_t(s)|^{m(x)}\mathrm{d}x\right]\mathrm{d}s,$$

for $t \geq 0$. As the primitive of an integrable function, $\mathrm{E}(t)$ is continuous. Furthermore, based on assumptions (H3), equality (2.7) holds true. □

3. Local solutions

We are now ready to present the result of local existence. To do this, we first examine a related, simpler problem. Then, we demonstrate the local existence of solutions to problem (1.1) by employing the Faedo-Galerkin arguments in conjunction with the fixed-point theorem. We outline the key steps of the proof as follows. Given v, we consider the following auxiliary initial-boundary value problem

$$\begin{cases} u_{tt} + (-\Delta)^m u - \int_0^t g(t-s)(-\Delta)^m u(x,s)\,\mathrm{d}s + |u_t|^{r(.)-2}u_t = v(x,t), & \text{in } \Omega \times (0,T), \\ \frac{\partial^i u}{\partial \nu^i} = 0,\ i = 0, 1, ..., m-1, \text{ on } \partial\Omega \times (0,T), & \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \quad \text{in } \Omega, & \end{cases} \tag{3.1}$$

where $v \in L^2(\Omega \times (0,T))$, $(u_0, u_1) \in H_0^m(\Omega) \times L^2(\Omega)$, Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, and $r(.)$ is a given measurable function satisfying conditions (2.1) and (2.2). We now need to state the following well-posedness result for problem (3.1).

Theorem 3.1 (Local existence) *Supposed that (H1)-(H3) hold, for any given* $(u_0, u_1) \in H_0^m(\Omega) \times L^2(\Omega)$*, the problem* (1.1) *has a unique local solution such that* $u \in C\left([0,T]; H_0^m(\Omega)\right)$, $u_t \in C\left([0,T]; L^2(\Omega)\right) \cap L^{r(\cdot)}(\Omega \times (0,T))$*, and* $u_{tt} \in L^\infty\left((0,T); L^2(\Omega)\right)$ *for some* $T > 0$.

Lemma 3.1 *Under the conditions of Theorem* (3.1)*, problem* (3.1) *possesses a unique local solution in* W*, where*

$$W = \left\{ w \in L^\infty\left((0,T); H_0^m(\Omega)\right),\ w_t \in L^\infty\left((0,T); L^2(\Omega)\right)\right) \cap L^{r(\cdot)}\left(\Omega \times (0,T)\right)\right\}.$$

Proof: Uniqueness: Let u and v be the solutions of (3.1), then, $w = u - v$ satisfies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega w_t^2 \mathrm{d}x + \left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\|D^m w\|_2^2\right) + \left(|u_t|^{r(\cdot)-2} u_t - |v_t|^{r(\cdot)-2} v_t, u_t - v_t\right) = 0.$$

From (H3), we observe that

$$\int_\Omega w_t^2 \mathrm{d}x + \left(1 - \int_0^\infty g(s)\,\mathrm{d}s\right)\|D^m w\|_2^2 \le 0.$$

which implies that $w = 0$. Hence, the uniqueness of the solution.

Existence:

Let $\{\varphi_i\}_{i=1}^\infty$ be an ortho-normal basis of $H_0^m(\Omega)$, with

$$(-\Delta)^m \varphi_i = \lambda_i \varphi_i \text{ in } \Omega,\ \frac{\partial \varphi_i}{\partial \nu^i} = 0,\ i = 0, 1, ..., m-1, \text{ on } \partial\Omega,$$

and define the finite-dimensional subspace $\Phi_k = \operatorname{span}\{\varphi_1, \dots, \varphi_k\}$. By normalizing we obtain $\|\varphi_i\| = 1$. Let define

$$u^k(x,t) = \sum_{i=1}^k c_i(t)\varphi_i,$$

where $u^k(x,t)$ represents the solution of the following approximate problem

$$\begin{array}{c} \int_\Omega u_{tt}^k(x,t)\varphi_i(x)\mathrm{d}x + \int_\Omega D^m u^k(x,t) D^m \varphi_i(x)\mathrm{d}x \\ - \int_\Omega \int_0^t g(t-s) D^m u^k(x,s) D^m \varphi_i(x)\mathrm{d}s\mathrm{d}x \\ + \int_\Omega \left|u_t^k(x,t)\right|^{r(x)-2} u_t^k(x,t)\varphi_i(x)\mathrm{d}x = \int_\Omega v\varphi_i(x)\mathrm{d}x \\ u^k(x,0) = u_0^k, \quad u_t^k(x,0) = u_1^k, \quad \forall i = 1, \dots, k, \end{array} \tag{3.2}$$

where $u_0^k = \sum_{i=1}^k (u_0, \varphi_i)\varphi_i \to u_0$ in $H_0^m(\Omega)$ and $u_1^k = \sum_{i=1}^k (u_1, \varphi_i)\varphi_i \to u_1$ in $L^2(\Omega)$, respectively. System (3.2) has a local solution in $[0, t_k)$, where $0 < t_k < T$ for any$T > 0$. Next, we must to prove that $t_k = T$, $\forall k \ge 1$, multiplying $(3.2)_1$ by $c_i'(t)$ and summing the resulting products result over i, we get integrating over $(0,t)$ using the hypotheses on g

$$\begin{array}{c} \frac{1}{2}\left\{\int_\Omega \left|u_t^k(t)\right|^2 \mathrm{d}x + \left(1 - \int_0^t g(s)\mathrm{d}s\right)\int_\Omega \left|D^m u^k(t)\right|^2 \mathrm{d}x + \left(g \circ D^m u^k\right)(t)\right\} \\ + \int_0^t \int_\Omega \left|u_t^k(x,s)\right|^{r(x)} \mathrm{d}x\mathrm{d}s \\ \le \frac{1}{2}\int_\Omega \left|u_1^k\right|^2 \mathrm{d}x + \frac{1}{2}\int_\Omega \left|D^m u_0^k\right|^2 \mathrm{d}x + \int_0^t \int_\Omega v(s) u_t^k(x,s)\mathrm{d}x\mathrm{d}s \\ \le \frac{1}{2}\int_\Omega \left|u_1^k\right|^2 \mathrm{d}x + \frac{1}{2}\int_\Omega \left|D^m u_0^k\right|^2 \mathrm{d}x + \frac{1}{4}\int_0^t \int_\Omega \left|u_t^k(x,s)\right|^2 \mathrm{d}x\mathrm{d}s \\ + \int_0^t \int_\Omega |v(s)|^2 \mathrm{d}x\mathrm{d}s \\ \le C + \frac{1}{4}\int_\Omega \left|u_t^k(x,t)\right|^2 \mathrm{d}x, \quad \forall t \in [0, t_k). \end{array}$$

So, on $(0, t_k)$, we have

$$\int_\Omega \left|u_t^k(t)\right|^2 \mathrm{d}x + \beta \int_\Omega \left|D^m u^k(t)\right|^2 \mathrm{d}x + \int_0^{t_k} \int_\Omega \left|u_t^k(x,s)\right|^{r(x)} \mathrm{d}x\mathrm{d}s \le C.$$

Then the solution can be extended to the interval $[0, T\ )$ and we obtain

$$\begin{array}{l} \left(u^k\right) \text{ is a bounded sequence in } L^\infty\left((0,T);H_0^m(\Omega)\right) \\ \left(u_t^k\right) \text{ is a bounded sequence in } L^\infty\left((0,T);L^2(\Omega)\right)\cap L^{r(.)}(\Omega\times(0,T)). \end{array}$$

Hence, there exists a subsequence (u^μ) of $\left(u^k\right)$ such that

$$\begin{array}{l} u^\mu \to u \text{ weak star in } L^\infty\left((0,T);H_0^m(\Omega)\right) \\ u_t^\mu \to u_t \text{ weak star in } L^\infty\left((0,T);L^2(\Omega)\right) \text{ and weakly in } L^{r(.)}(\Omega\times(0,T)). \end{array}$$

On the other hand, from Lion's lemma [31, Lemme 1.2.], we conclude that $u\in C\left([0,T];L^2(\Omega)\right)$. Since (u_t^μ) is bounded in $L^{r(.)}(\Omega\times(0,T))$, then $|u_t^\mu|^{r(x)-2}\,u_t^\mu$ is bounded in $L^{\frac{r(.)}{r(.)-1}}(\Omega\times(0,T))$, and consequently, by making use of Lion's Lemma [31, Lemme 1.3.], we can infer

$$|u_t^\mu|^{r(.)-2}\,u_t^\mu \to |u_t|^{r(.)-2}\,u_t \text{ weakly in } \ L^{\frac{r(-)}{r(.)-1}}(\Omega\times(0,T)).$$

Hence, we get, for all $\varphi\in L^{r(.)}\left((0,T)\times H_0^m(\Omega)\right)$

$$\int_\Omega\left(u_{tt}\varphi + D^m u D^m\varphi + \int_0^t g(t-s)D^m u(s)D^m\varphi \mathrm{d}s + |u_t|^{r(x)-2}\,u_t\varphi\right)\mathrm{d}x = \int_\Omega v\varphi \mathrm{d}x,$$

which gives

$$u_{tt} + (-\Delta)^m\,u - \int_0^t g(t-s)\,(-\Delta)^m\,u(s)\mathrm{d}s + |u_t|^{m(x)-2}\,u_t = v, \text{ in } D'(\Omega\times(0,T)).$$

□

Proof: [Proof of Theorem 3.1] **Existence:** For $v\in L^\infty\left((0,T)\,;H_0^m(\Omega)\right)$, we have

$$\left\|\,|v|^{p(.)-2}\,v\right\|_2^2 \le \int_\Omega |v|^{2p_1-2}\,\mathrm{d}x + \int_\Omega |v|^{2p_2-2}\,\mathrm{d}x < \infty,$$

since

$$1 < p_1 \le p\,(x) \le p_2 < \frac{2n}{n-2m}. \tag{3.3}$$

So, in this case,

$$|v|^{p(.)-2}\,v\in L^\infty\left((0,T),L^2(\Omega)\right)\subset L^2(\Omega\times(0,T).$$

Therefore, for each $v\in L^\infty\left((0,T),H_0^m(\Omega)\right)$, there exists a unique u, such that

$$u\in L^\infty\left((0,T)\,;H_0^m(\Omega)\right),\ u_t\in L^2\left((0,T)\,;L^2(\Omega)\right),$$

satisfying

$$\left\{\begin{array}{c} u_{tt} + (-\Delta)^m\,u - \int_0^t g\,(t-s)\,(-\Delta)^m\,u\,(x,s)\,\mathrm{d}s + |u_t|^{r(.)-2}\,u_t \\ = |v|^{p(.)-2}\,v, \text{ in } \Omega\times(0,T), \\ \frac{\partial^i u}{\partial\nu^i} = 0,\ i=0,1,...,m-1 \text{ on } \partial\Omega\times(0,T), \\ u(x,0)=u_0(x),\ \ u_t(x,0)=u_1(x) \text{ in } \Omega. \end{array}\right. \tag{3.4}$$

Let define a class of functions $X_{R_0,T}$ which consists of all functions v in W satisfying the initial conditions in (1.1). Namely,

$$X_{R_0,T} = \left\{\begin{array}{c} v\,(t)\in W, \\ \left\|v'\,(t)\right\|_2^2 + \beta\left\|D^m v\,(t)\right\|_2^2 \le R_0^2 \text{ on } [0,T]\,, \\ v\,(0)=v_0,\ v'\,(0)=u_1. \end{array}\right\}$$

where R_0 be a positive number such that

$$R_0 = \sqrt{2\left(\|u_1\|_2^2 + \beta\|D^m u_0\|_2^2\right)}.$$

We define the metric d on the space $B_T(R_0)$ as

$$\mathrm{d}(u,v) = \sup_{0\le t\le T}\left(\|u_t(t) - v_t(t)\|_2^2 + \|D^m u(t) - D^m v(t)\|_2^2\right) \text{ for } u,\ v \in X_{R_0,T}.$$

Then the space $X_{R_0,T}$ is the complete metric space with d. Let $v \in X_{R_0,T}$, then $\|v_t(t)\|_2 \le R_0$, $\|D^m v(t)\|_2 \le \frac{1}{\sqrt{\beta}}R_0$ for all $t \in [0,T]$. We define the nonlinear mapping Φ by $u = \Phi(v)$ is the unique solution of problem (3.4). Then we have that

$$\Phi(v) = u \in X_{R_0,T} \text{ for } v \in X_{R_0,T}, \tag{3.5}$$
$$\Phi : X_{R_0,T} \to X_{R_0,T} \text{ is a contractive mapping.} \tag{3.6}$$

To show (3.5) we multiply (3.4) by u_t and integrate over $\Omega \times (0,t)$, yielding

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\left\{\int_\Omega |u_t(t)|^2\,\mathrm{d}x + \left(1 - \int_0^t g(s)\mathrm{d}s\right)\int_\Omega |D^m u(t)|^2\,\mathrm{d}x + (g\circ D^m u)(t)\right\}\\ &+ \int_\Omega |u_t(x,t)|^{r(x)}\,\mathrm{d}x\\ &= -\frac{1}{2}g(t)\int_\Omega |D^m u(t)|^2\,\mathrm{d}x + \frac{1}{2}(g'\circ D^m u)(t) + \int_\Omega |v|^{p(x)-2}v(x,t)u_t(x,t)\mathrm{d}x.\end{aligned} \tag{3.7}$$

Applying Young's inequality, we have

$$\begin{aligned}\left|\int_\Omega |v|^{p(x)-2}vu_t\mathrm{d}x\right| &\le \frac{1}{2}\int_\Omega u_t^2\mathrm{d}x + \frac{1}{2}\int_\Omega |v|^{2p(x)-2}\mathrm{d}x\\ &\le \frac{1}{2}\int_\Omega u_t^2\mathrm{d}x + \frac{1}{2}\left[\int_\Omega |v|^{2p_2-2}\mathrm{d}x + \int_\Omega |v|^{2p_1-2}\mathrm{d}x\right]\\ &\le \frac{1}{2}\int_\Omega u_t^2\mathrm{d}x + \frac{\mathrm{c_e}}{2}\left[\|D^m v\|_2^{2p_2-2} + \|D^m v\|_2^{2p_1-2}\right].\end{aligned}$$

So (3.7) becomes

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\left\{\int_\Omega |u_t(t)|^2\,\mathrm{d}x + \left(1 - \int_0^t g(s)\mathrm{d}s\right)\int_\Omega |D^m u(t)|^2\,\mathrm{d}x\right\}\\ &\le \frac{1}{2}\int_\Omega u_t^2\mathrm{d}x + \frac{\mathrm{c_e}}{2}\left[\left(\frac{1}{\sqrt{\beta}}R_0\right)^{2p_2-2} + \left(\frac{1}{\sqrt{\beta}}R_0\right)^{2p_1-2}\right],\end{aligned}$$

in which

$$\begin{aligned}\psi_v(u)(t) &\le \psi_v(u)(0) + \int_0^t\left(c_e\left[\left(\frac{1}{\sqrt{\beta}}R_0\right)^{2p_2-2} + \left(\frac{1}{\sqrt{\beta}}R_0\right)^{2p_1-2}\right] + \psi_v(u)(t)\right)\mathrm{d}s\\ &\le \frac{1}{2}R_0^2 + \lambda_0\int_0^t(1 + \psi_v(u)(t))\,\mathrm{d}s,\end{aligned}$$

where $\lambda_0 = \max\left(\mathrm{c_e}\left[\left(\frac{1}{\sqrt{\beta}}R_0\right)^{2p_2-2} + \left(\frac{1}{\sqrt{\beta}}R_0\right)^{2p_1-2}\right], 1\right)$, $\mathrm{c_e}$ is the Sobolev embedding constant, and

$$\psi_v(u)(t) = \|u_t\|_2^2 + \beta\|D^m u\|_2^2.$$

By the Gronwall inequality and straightforward calculations, we have that

$$\|u_t\|_2^2 + \beta\|D^m u\|_2^2 \le \left(\frac{1}{2}R_0^2 + \lambda_0 T_0\right)e^{\lambda_0 T_0} < R_0^2, \quad 0 \le t \le T_0,$$

for sufficiently small $0 < T_0 \le T$. Thus (3.5) is satisfied.

Next, we present (3.6), let $v_1, v_2 \in X_{R_0,T}$ and $u_1 = \Phi(v_1)$, $u_2 = \Phi(v_2)$ be the corresponding solutions to problem (3.4). By setting $w = u_1 - u_2$, we find that w satisfies the following system:

$$\begin{aligned}(w_{tt},v) + (D^m w, D^m v) &- \int_\Omega\int_0^t g(t-s)D^m w(x,s)D^m v(x)\mathrm{d}s\mathrm{d}x\\ &+ \left(|u_{1t}(t)|^{r(x)-1}u_{1t}(t) - |u_{2t}(t)|^{r(x)-1}u_{2t}(t), v\right)\\ &= \left(|v_1|^{p(x)-2}v_1 - |v_2|^{p(x)-2}v_2, v\right), \text{ in } L^2\left(0,T_1;L^2(\Omega)\right).\end{aligned} \tag{3.8}$$

Set

$$\varphi(w)(t) = \|w_t\|_2^2 + \beta\|D^m w\|_2^2 .$$

By multiplying w_t to (3.8), and considering that

$$\left(|u_{1t}|^{r(.)-2}u_{1t} - |u_{2t}|^{r(.)-2}u_{2t}, u_{1t}-u_{2t}\right) \geq \int_\Omega |w_t|^{r(x)}\mathrm{d}x \geq 0,\ a.e.\ x\in\Omega,$$

we obtain

$$\frac{1}{2}\int_\Omega w_t^2\mathrm{d}x + \frac{1}{2}\beta\|D^m w\|_2^2 \leq \int_0^t\int_\Omega \left(|v_1|^{p(.)-2}v_1 - |v_2|^{p(.)-2}v_2\right)w_t\mathrm{d}x\mathrm{d}s. \tag{3.9}$$

Using the fact that

$$|v_1|^{p(.)-2}v_1 - |v_2|^{p(.)-2}v_2 = (p(x)-1)\,\zeta^{p(x)-2}v,\ \text{for any } x\in\Omega \text{ fixed},$$

with $v = v_1 - v_2$, and $\zeta = sv_1 + (1-s)v_2$, $s\in(0,1)$. Thus Young's inequality implies

$$\begin{aligned} I &= \left|\int_\Omega \left(|v_1(s)|^{p(.)-2}v_1(s) - |v_2(s)|^{p(.)-2}v_2(s)\right)w_t\mathrm{d}x\right| \\ &\leq \tfrac{1}{2}\int_\Omega w_t^2\mathrm{d}x + \tfrac{1}{2}\int_\Omega \left|(p(x)-1)\,\zeta^{p(x)-2}\right|^2|v|^2\mathrm{d}x \\ &\leq \tfrac{1}{2}\int_\Omega w_t^2\mathrm{d}x + \tfrac{p_2^2}{2}\int_\Omega |sv_1+(1-s)v_2|^{2(p(x)-2)}|v|^2\mathrm{d}x \\ &\leq \tfrac{1}{2}\int_\Omega w_t^2\mathrm{d}x + \tfrac{p_2^2}{2}\left(\int_\Omega |v|^{\frac{2n}{n-2}}\mathrm{d}x\right)^{\frac{n-2}{n}}\left[\left(\int_\Omega |sv_1+(1-s)v_2|^{n(p_2-2)}\right)^{\frac{2}{n}}\mathrm{d}x\right. \\ &\quad \left. + \left(\int_\Omega |sv_1+(1-s)v_2|^{n(p_1-2)}\right)^{\frac{2}{n}}\mathrm{d}x\right]. \end{aligned}$$

By recalling (3.3), we estimate I in the following way

$$\begin{aligned} I &\leq \tfrac{1}{2}\int_\Omega w_t^2\mathrm{d}x \\ &+ \mathrm{c_e}\tfrac{p_2^2}{2}\|D^m v\|_2^2\left[\|D^m v_1\|_2^{2(p_2-2)} + \|D^m v_1\|_2^{2(p_1-2)} + \|D^m v_2\|_2^{2(p_2-2)} + \|D^m v_2\|_2^{2(p_1-2)}\right] \\ &\leq \tfrac{1}{2}\beta(w)(t) + 2p_2^2 c_e R_0^{2(p_2-2)}\mathrm{d}(v_1,v_2). \end{aligned}$$

Therefore, (3.9) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(w)(t) \leq \beta(w)(t) + 4\mathrm{c_e}p_2^2 R_0^{2(p_2-2)}\mathrm{d}(v_1,v_2).$$

Thus we have from Gronwall's inequality and $\beta(w)(0) = 0$ that

$$\mathrm{d}(u_1,u_2) \leq 4\mathrm{c_e}p_2^2 R_0^{2(p_2-2)}Te^T\mathrm{d}(v_1,v_2).$$

By choosing $0 < T_1 \leq T$ so small that

$$4\mathrm{c_e}R_0^{2(p_2-2)}T_1e^{T_1} < 1.$$

then Φ is a contraction mapping. According to Banach fixed point theorem, we obtain a local existence result. This completes the proof of Theorem 3.1.

Uniqueness: Suppose that we have two solutions u and v. Then $U = u - v$ satisfies

$$\frac{1}{2}\int_\Omega U_t^2\mathrm{d}x + \frac{1}{2}\beta\int_\Omega |D^m U|^2\mathrm{d}x \leq \int_0^t\int_\Omega \left(|u|^{p(.)-2}u - |v|^{p(.)-2}v\right)U_t\mathrm{d}x.$$

By repeating the same estimates as above, we arrive at

$$\int_\Omega \left(U^2 + \beta|D^m U|^2\right)\mathrm{d}x \leq C\int_0^t\int_\Omega \left(U^2(x,s) + |D^m U(x,s)|^2\right)\mathrm{d}x\mathrm{d}s.$$

Gronwall's inequality and (H3) yields

$$\int_\Omega \left(U^2 + \beta|D^m U|^2\right)\mathrm{d}x = 0.$$

Thus, $U \equiv 0$. This shows the uniqueness. The proof of Theorem (3.1) is completed. □

4. Blow-up results

4.1. First Theorem of blow-up result

Let consider

$$R_1=-2\frac{\left(1+\tau-\int_0^t g(s)\mathrm{d}s\right)}{1-\int_0^t g(s)\mathrm{d}s},\ R_2=\frac{\left(1+\tau-\int_0^t g(s)\mathrm{d}s\right)}{1-\int_0^t g(s)\mathrm{d}s}+1,\ R_0=\min\left(R_2,\frac{1}{2}\right),\tag{4.1}$$

where

$$\tau\in\left[\frac{1}{2}\left(\sqrt{1-\int_0^t g(s)\mathrm{d}s}+\int_0^t g(s)\mathrm{d}s-1\right),\ \frac{1}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right)(p_1-2)\right].$$

To examine the phenomenon of finite-time blow-up, the following lemmas are necessary.

Lemma 4.1 *Suppose that a positive, twice-differentiable function $\alpha(t)$ satisfies on $t\geq 0$ the inequality*

$$\alpha''\alpha-(1+a)(\alpha')^2\geq 0,\ \ a>0.$$

If

$$\alpha(0)>0,\ \text{and}\ \alpha'(0)>0,$$

then, there exists $t_1\in\left(0,\frac{\alpha(0)}{a\alpha'(0)}\right)$ such that

$$\alpha(t)\to\infty\ \text{as}\ t\to t_1.$$

Our blow-up result reads as follows

Theorem 4.1 *If $p(.)$, and $r(.)$ satisfy (2.1) and (H1)-(H2) hold, there is no solution can exist for all time to the problem (1.1), whose initial data satisfies $\int_\Omega u_0u_1\mathrm{d}x\geq\frac{1}{R_0}[R_1\mathrm{E}(0)+|\Omega|]>0$, and*

$$\mathrm{E}(0)>\frac{2(p_1+1)R_0}{|\Omega|\mathrm{C_e}}.\tag{4.2}$$

In other words, there exists a $T_1\leq T_{\max}$ such that $\lim\limits_{t\to T_1}\int_0^t\|u(s)\|_2^2\mathrm{d}s=+\infty$, i.e. the solution u blows up in finite time in $L^2(\Omega)$-norm. R_0 and R_1 as in (4.1), and $\mathrm{C_e}$ is the Sobolev constant of the embedding $L^{r(.)}(\Omega)\hookrightarrow L^2(\Omega)$.

Lemma 4.2 *Under the assumptions of Theorem (3.1), for $p_1>r_2$ the following inequality*

$$\int_\Omega uu_t\ \mathrm{d}x\geq \mathrm{e}^{R_0t}\left[\int_\Omega u_0u_1\mathrm{d}x-\frac{R_1}{R_0}\mathrm{E}(0)-\frac{|\Omega|}{R_0}\right]+\frac{R_1}{R_0}\mathrm{E}(t)+\frac{|\Omega|}{R_0},\ \ t>0,\tag{4.3}$$

holds for any solution u of problem (1.1), where R_0 and R_1 are as in (4.1).

Proof: [proof of Lemma 4.2] Let

$$L(t)=\int_\Omega uu_t\mathrm{d}x.$$

Then

$$L'(t)=\int_\Omega u_t^2\mathrm{d}x+\int_\Omega uu_{tt}\mathrm{d}x.$$

We get integrating over $[0,t]$ for all $t\in[0,T]$, using Eq. (1.1) we obtain

$$\begin{aligned}L'(t)=\|u_t\|_2^2-\|D^mu(t)\|_2^2+\int_0^t g(t-s)\int_\Omega D^mu(t)(D^mu(s)-D^mu(t))\mathrm{d}x\mathrm{d}s\\+\int_0^t g(s)\mathrm{d}s\|D^mu(t)\|_2^2+\varrho(u)-\int_\Omega|u_t|^{r(.)-2}u_tu\mathrm{d}x.\end{aligned}\tag{4.4}$$

From Young's inequality it follows

$$\int_\Omega |u_t|^{r(x)-1}|u|\mathrm{d}x \le \frac{1}{r_1}\int_\Omega \zeta^{r(x)}|u|^{r(x)}\mathrm{d}x + \frac{r_2-1}{r_2}\int_\Omega \zeta^{-\frac{r(x)}{r(x)-1}}|u_t|^{r(x)}\,\mathrm{d}x,\ \forall\zeta>0.$$

Taking ζ so that

$$\zeta^{-\frac{r(x)}{r(x)-1}} = k,\ \ k>0,$$

For a suitably large constant k, which will be determined later, we find that by substituting it into equation (4.4)

$$\begin{aligned} L'(t) = \|u_t\|_2^2 - \|D^m u(t)\|_2^2 + \textstyle\int_0^t g(t-s)\int_\Omega D^m u(t)(D^m u(s)-D^m u(t))\mathrm{d}x\mathrm{d}s \\ + \textstyle\int_0^t g(s)\mathrm{d}s\|D^m u(t)\|_2^2 + \varrho(u) - \frac{k^{1-r_1}}{r_1}\int_\Omega |u|^{r(x)}\mathrm{d}x - \frac{r_2-1}{r_2}k\int_\Omega |u_t|^{r(x)}\,\mathrm{d}x. \end{aligned} \tag{4.5}$$

Using Hölder's inequality and Young's inequality, for any $\tau>0$ we have

$$\begin{aligned} &\left|\textstyle\int_0^t g(t-s)\int_\Omega D^m u(t)(D^m u(s)-D^m u(t))\mathrm{d}x\mathrm{d}s\right| \\ &= \left|\textstyle\int_\Omega D^m u(t)\int_0^t g(t-s)(D^m u(s)-D^m u(t))\mathrm{d}s\mathrm{d}x\right| \\ &\le \tau\textstyle\int_\Omega |D^m u(t)|^2\mathrm{d}x + \frac{1}{4\tau}\int_\Omega\left(\int_0^t g(t-s)(D^m u(s)-D^m u(t))\mathrm{d}s\right)^2\mathrm{d}x \\ &\le \tau\|D^m u(t)\|_2^2 \\ &+\tfrac{1}{4\tau}\textstyle\int_\Omega\left(\int_0^t g(t-s)\mathrm{d}s\right)\left(\int_0^t g(t-s)(D^m u(s)-D^m u(t))^2\mathrm{d}s\right)\mathrm{d}x \\ &= \tau\|D^m u(t)\|_2^2 + \tfrac{1}{4\tau}\textstyle\int_0^t g(s)\mathrm{d}s(g\circ D^m u)(t), \end{aligned} \tag{4.6}$$

Using (4.5) and (4.6), we deduce

$$\begin{aligned} L'(t) \ge \|u_t\|_2^2 + \left(-1-\tau+\textstyle\int_0^t g(s)\mathrm{d}s\right)\|D^m u(t)\|_2^2 - \frac{1}{4\tau}\int_0^t g(s)\mathrm{d}s(g\circ D^m u)(t) \\ +\tfrac12\varrho(u) - \tfrac{k^{1-r_1}}{r_1}\textstyle\int_\Omega |u|^{r(x)}\mathrm{d}x + \frac12\varrho(u) - \frac{r_2-1}{r_2}k\int_\Omega |u_t|^{r(x)}\,\mathrm{d}x. \end{aligned} \tag{4.7}$$

Furthermore, since $p_1>2$, it is evident that

$$\begin{aligned} \textstyle\int_\Omega |u|^{p(x)}\mathrm{d}x &= \textstyle\int_{\{x\in\Omega:|u|\le1\}}|u|^{p(x)}\mathrm{d}x + \int_{\{x\in\Omega:|u|\ge1\}}|u|^{p(x)}\mathrm{d}x \\ &\ge \textstyle\int_{\{x\in\Omega:|u|\ge1\}}|u|^2\mathrm{d}x \ge \int_\Omega|u|^2\mathrm{d}x - \int_{\{x\in\Omega:|u|\le1\}}|u|^2\mathrm{d}x \\ &\ge \textstyle\int_\Omega |u|^2\mathrm{d}x - |\Omega|, \end{aligned}$$

and also, since $p_1>r_2$;

$$\begin{aligned} \textstyle\int_\Omega |u|^{p(x)}\mathrm{d}x &= \textstyle\int_{\{x\in\Omega:|u|\le1\}}|u|^{p(x)}\mathrm{d}x + \int_{\{x\in\Omega:|u|\ge1\}}|u|^{p(x)}\mathrm{d}x \\ &\ge \textstyle\int_{\{x\in\Omega:|u|\ge1\}}|u|^{r(x)}\mathrm{d}x \ge \int_\Omega|u|^{r(x)}\mathrm{d}x - \int_{\{x\in\Omega:|u|\le1\}}|u|^{r(x)}\mathrm{d}x \\ &\ge \textstyle\int_\Omega |u|^{r(x)}\mathrm{d}x - |\Omega|, \end{aligned}$$

joining it with (4.7) gives

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\left(L(t) - \tfrac{r_2-1}{r_2}k\mathrm{E}(t)\right) &\ge \left(-1-\tau+\textstyle\int_0^t g(s)\mathrm{d}s\right)\frac{2\mathrm{E}(t)-\|u_t\|_2^2-(g\circ D^m u)(t)+\frac{2}{p_1}\varrho(u)}{1-\int_0^t g(s)\mathrm{d}s} \\ &+\|u_t\|_2^2 + \varrho(u) - \tfrac{1}{4\tau}\textstyle\int_0^t g(s)\mathrm{d}s(g\circ D^m u)(t)| - \frac{k^{1-m_1}}{m_1}\int_\Omega |u|^{r(x)}\mathrm{d}x \\ &\ge -2\frac{\left(1+\tau-\int_0^t g(s)\mathrm{d}s\right)}{1-\int_0^t g(s)\mathrm{d}s}\mathrm{E}(t) + \left[\frac{\left(1+\tau-\int_0^t g(s)\mathrm{d}s\right)}{1-\int_0^t g(s)\mathrm{d}s}+1\right]\|u_t\|_2^2 \\ &+\left[\frac{\left(1+\tau-\int_0^t g(s)\mathrm{d}s\right)}{1-\int_0^t g(s)\mathrm{d}s} - \tfrac{1}{4\tau}\textstyle\int_0^t g(s)\mathrm{d}s\right](g\circ D^m u)(t) \\ &+\left(\tfrac12\left[1-\frac{\left(1+\tau-\int_0^t g(s)\mathrm{d}s\right)}{1-\int_0^t g(s)\mathrm{d}s}\frac{2}{p_1}\right] - \frac{k^{1-r_1}}{r_1}\right)\textstyle\int_\Omega |u|^{r(x)}\mathrm{d}x \\ &+\tfrac12\left[1-\frac{\left(1+\tau-\int_0^t g(s)\mathrm{d}s\right)}{1-\int_0^t g(s)\mathrm{d}s}\frac{2}{p_1}\right]\textstyle\int_\Omega |u|^2\mathrm{d}x - |\Omega|. \end{aligned} \tag{4.8}$$

Choosing k large enough so that

$$k \geq \max\left(\left(\frac{r_1}{2}\right)^{\frac{1}{r_1-1}}, \frac{r_2}{r_2-1}\frac{R_1}{R_0}, \left(\frac{1}{2} - \frac{\left(1+\tau-\int_0^t g(s)\mathrm{d}s\right)}{\left(1-\int_0^t g(s)\mathrm{d}s\right)}\frac{1}{p_1}\right)^{\frac{1}{r_1-1}}\right),$$

From $\mathrm{E}(t) \leq \mathrm{E}(0) \leq 0$ and (4.8), we deduce the existence of a constant $\gamma > 0$ such that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(L(t) - \frac{R_1}{R_0}\mathrm{E}(t) - \frac{|\Omega|}{R_0}\right) \\ &\geq R_2\left\|u_t\right\|_2^2 + \left(\frac{1}{2} - \frac{\left(1+\tau-\int_0^t g(s)\mathrm{d}s\right)}{\left(1-\int_0^t g(s)\mathrm{d}s\right)}\frac{1}{p_1}\right)\int_\Omega |u|^2\mathrm{d}x - R_1\mathrm{E}(t) - |\Omega| \\ &\geq R_0\left(L\left(t\right) - \frac{R_1}{R_0}\mathrm{E}(t) - \frac{|\Omega|}{R_0}\right). \end{aligned} \tag{4.9}$$

We conclude from (4.9) that

$$L(t) \geq \mathrm{e}^{R_0 t}\left[L(0) - \frac{R_1}{R_0}\mathrm{E}(0) - \frac{|\Omega|}{M_0}\right] + \frac{R_1}{R_0}\mathrm{E}(t) + \frac{|\Omega|}{R_0}.$$

□

The proof of the main result is presented as follows.

We observe that the primary method used in this proof is derived from Levin's concavity technique [29], which integrates ideas employed in [30, Theorem 2.2.].

Proof: [Proof of Theorem 4.1] We first assume that $T_{\max} = +\infty$, i.e., u is defined in the whole interval $(0, +\infty)$, which leads to a contradiction. Let

$$\varphi\left(t\right) = \left\|u\right\|_2^2, \text{ for } 0 < t < \infty,$$

then

$$\varphi'\left(t\right) = 2\left(u, u_t\right). \tag{4.10}$$

For $r\left(.\right)$ satisfy (H2) we have

$$H'(t) = -\mathrm{E}'(t) \geq \int_\Omega |u_t(t)|^{r(x)}\mathrm{d}x \geq \mathrm{C_e}\int_\Omega |u_t(t)|^2\mathrm{d}x, \tag{4.11}$$

from which we have used the following embedding

$$L^{r_2}(\Omega) \hookrightarrow L^{r(.)}(\Omega) \hookrightarrow L^{r_1}(\Omega) \hookrightarrow L^2(\Omega),$$

Now, we distinguish two cases:

1. **Case.1** $\mathrm{E}(u(t)) \geq 0$, for all $t > 0$. Using (4.2) we can choose β that satisfies

$$1 < \beta\mathrm{C_e} < \frac{|\Omega|\mathrm{E}(0)}{2\left(p_1+1\right)R_0}. \tag{4.12}$$

By adding $4\left(p_1+1\right)\beta\mathrm{E}\left(0\right) - 4\left(p_1+1\right)\beta\mathrm{E}\left(0\right)$, and driving us (4.3), (4.10), and (4.11) it yields

$$\begin{aligned} \varphi'(t) &\geq 2\mathrm{e}^{R_0 t}\left[\varphi'(0) - \frac{R_1}{R_0}\mathrm{E}(0) - \frac{|\Omega|}{R_0}\right] + 2\left(\frac{R_1}{R_0} + 2\left(p_1+1\right)\beta\right)\mathrm{E}(t) \\ &\quad -4\left(p_1+1\right)\beta\mathrm{E}\left(t\right) + 2\frac{|\Omega|}{R_0} \\ &\geq 2\mathrm{e}^{R_0 t}\left[\varphi'(0) - \frac{R_1}{R_0}\mathrm{E}(0) - \frac{|\Omega|}{R_0}\right] - 4\left(p_1+1\right)\beta\mathrm{E}\left(0\right) + 2\frac{|\Omega|}{R_0} \\ &\quad +2\left(p_1+1\right)\beta\mathrm{C_e}\int_0^t \left\|u_t\left(., s\right)\right\|_2^2\mathrm{d}s, \end{aligned} \tag{4.13}$$

for all $t>0$, let define $\phi(t)=\int_0^t \|u(.,s)\|_2^2 ds$, then we have,

$$\phi'(t)=\|u(t)\|_2^2=\varphi(t)\,,\ \phi''(t)=\varphi'(t)\,.$$

Using (4.13) to find

$$\begin{array}{c}\phi''(t)\geq 2(p_1+1)\beta\int_0^t \|u_t(.,s)\|_{2,\Omega_2}^2\,\mathrm{d}s+2\left[\varphi'(0)-\frac{R_1}{R_0}\mathrm{E}(0)-\frac{|\Omega|}{R_0}\right]\mathrm{e}^{M_0 t}\\ +2\frac{|\Omega|}{R_0}-4(p_1+1)\beta\mathrm{E}(0).\end{array}\tag{4.14}$$

Let defined ψ be an auxiliary function as

$$\psi(t)=\phi^2(t)+\varepsilon^{-1}\varphi(0)\phi(t)+\gamma,$$

in which we take $\varepsilon>0$ small enough such that

$$0<\varepsilon\leq\frac{\left[\varphi'(0)-\frac{R_1}{R_0}\mathrm{E}(0)-\frac{|\Omega|}{R_0}\right]+\frac{|\Omega|}{R_0}-2(p_1+1)\beta\mathrm{E}(0)}{(p_1+1)\beta\mathrm{C_e}\varphi(0)},$$

and $\gamma>0$ taken large enough (if needed), so that

$$4\varepsilon^2\gamma>\varphi^2(0)\,.\tag{4.15}$$

Therefore,

$$\psi'(t)=\left(2\phi(t)+\varepsilon^{-1}\varphi(0)\right)\phi'(t),\tag{4.16}$$

$$\psi''(t)=\left(2\phi(t)+\varepsilon^{-1}\varphi(0)\right)\phi''(t)+2(\phi'(t))^2.\tag{4.17}$$

From (4.16), we obtain

$$\begin{array}{c}(\psi'(t))^2=\left(2\phi(t)+\varepsilon^{-1}\varphi(0)\right)^2(\phi'(t))^2\\ =\left(4\phi^2(t)+\varepsilon^{-2}\varphi^2(0)+4\varepsilon^{-1}\phi(t)\varphi(0)\right)(\phi'(t))^2\\ =\left(4\phi^2(t)+4\varepsilon^{-1}\phi(t)\varphi(0)+4\gamma-\delta\right)(\phi'(t))^2\\ =(4\psi(t)-\delta)(\phi'(t))^2,\end{array}\tag{4.18}$$

where $\delta=4\gamma-\varepsilon^{-2}\varphi^2(0)>0$, then

$$(\psi'(t))^2+\delta(\phi'(t))^2=4\psi(t)(\phi'(t))^2\,.\tag{4.19}$$

Observing that

$$\int_0^t (u_t(.,s),u)\,\mathrm{d}s=\frac{1}{2}\int_0^t\left(\frac{\mathrm{d}}{\mathrm{d}s}\|u\|_2^2\right)\mathrm{d}s=\frac{1}{2}\|u(t)\|_2^2-\frac{1}{2}\|u_0\|_2^2\,.$$

Therefore,

$$\|u(t)\|_2^2=\|u_0\|_2^2+2\int_0^t\int_\Omega u_t(.,s)\,u(s)\,\mathrm{d}x\mathrm{d}s.$$

From Hölder's and Young's inequalities we have

$$\begin{array}{c}(\phi'(t))^2=\|u(t)\|_2^4\\ =\left(\|u_0\|_2^2+2\int_0^t\int_\Omega u_t(.,s)\,u(s)\,\mathrm{d}x\mathrm{d}s\right)^2\\ \leq\left(\|u_0\|_2^2+2\left(\int_0^t\|u\|_2^2\,\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_0^t\|u_t(.,s)\|_2^2\,\mathrm{d}s\right)^{\frac{1}{2}}\right)^2\\ \leq\|u_0\|_2^4+2\|u_0\|_2^2\left(\int_0^t\|u\|_2^2\,\mathrm{d}s+\int_0^t\|u_t(.,s)\|_2^2\,\mathrm{d}s\right)\\ +4\left(\int_0^t\|u\|_2^2\,\mathrm{d}s\right)\left(\int_0^t\|u_t(.,s)\|_2^2\,\mathrm{d}s\right)\\ =\|u_0\|_2^4+2\varepsilon^{-1}\|u_0\|_2^2\int_0^t\|u_t(.,s)\|_2^2\,\mathrm{d}s+2\varepsilon\|u_0\|_2^2\,\phi(t)\\ +4\phi(t)\int_0^t\|u_t(.,s)\|_2^2\,\mathrm{d}s.\end{array}\tag{4.20}$$

From (4.17) and (4.19), we get

$$\begin{aligned} 2\psi''(t)\psi(t) &= 2\left(2\phi(t)+\varepsilon^{-1}\varphi(0)\right)\phi''(t)\psi(t)+4(\phi'(t))^2\psi(t) \\ &= 2\left(2\phi(t)+\varepsilon^{-1}\varphi(0)\right)\phi''(t)\psi(t)+(\psi'(t))^2+\delta(\phi'(t))^2. \end{aligned} \tag{4.21}$$

The following estimates are provided

$$\begin{aligned} &2\psi''(t)\psi(t)-(1+\beta\mathrm{C_e})(\psi'(t))^2 \\ &= 2\left(2\phi(t)+\varepsilon^{-1}\varphi(0)\right)\phi''(t)\psi(t)+\delta(\phi'(t))^2-\beta\mathrm{C_e}(\psi'(t))^2 \\ &= 2\left(2\phi(t)+\varepsilon^{-1}\varphi(0)\right)\phi''(t)\psi(t)+\delta(\phi'(t))^2-\beta\mathrm{C_e}(4\psi(t)-\delta)(\phi'(t))^2 \\ &= 2\left(2\phi(t)+\varepsilon^{-1}\varphi(0)\right)\phi''(t)\psi(t)-4\beta\mathrm{c_e}\psi(t)(\phi'(t))^2+\delta(1+\beta\mathrm{C_e})(\phi'(t))^2 \\ &\geq 2\psi(t)\left(2\phi(t)+\varepsilon^{-1}\varphi(0)\right)\begin{pmatrix} 2\mathrm{e}^{R_0 t}\left[\varphi'(0)-\frac{R_1}{R_0}\mathrm{E}(0)-\frac{|\Omega|}{R_0}\right]+2\frac{|\Omega|}{R_0} \\ -4(p_1+1)\beta\mathrm{E}(0)+2(p_1+1)\beta\mathrm{C_e}\int_0^t\|u_t(.,s)\|_2^2\,\mathrm{d}s \end{pmatrix} \\ &\quad -4\beta\mathrm{C_e}\psi(t)\begin{pmatrix} \|u_0\|_2^4+2\varepsilon^{-1}\|u_0\|_2^2\int_0^t\|u_t(.,s)\|_{2,\Omega_2}^2\,\mathrm{d}s+2\varepsilon\|u_0\|_2^2\phi(t) \\ +4\phi(t)\int_0^t\|u_t(.,s)\|_2^2\,\mathrm{d}s \end{pmatrix}. \end{aligned}$$

In which we use (4.21), (4.18), (4.14) and (4.20). According the values of β and ε, taking into account that $\mathrm{e}^{M_0 t}>1$, $p_1+1>2$, $\psi>0$, it reaches

$$\begin{aligned} &2\psi''(t)\psi(t)-(1+\beta\mathrm{C_e})(\psi'(t))^2 \\ &\geq 4\mathrm{C_e}\beta\psi(t)\left(2\phi(t)+\varepsilon^{-1}\varphi(0)\right)\left((p_1+1)\int_0^t\|u_t(.,s)\|_2^2\,\mathrm{d}s+(p_1+1)\varepsilon\varphi(0)\right) \\ &\quad -4\mathrm{C_e}\beta\psi(t)\begin{pmatrix} \|u_0\|_2^4+2\varepsilon^{-1}\|u_0\|_2^2\int_0^t\|u_t(.,s)\|_2^2\,\mathrm{d}s+2\varepsilon\|u_0\|_2^2\phi(t) \\ +4\phi(t)\int_0^t\|u_t(.,s)\|_2^2\,\mathrm{d}s, \end{pmatrix} \geq 0. \end{aligned}$$

In this case, we demonstrate that T cannot be infinite, and therefore a weak solution does not exist at all times. From Lemma (4.1), it follows that there exists a finite time $0<t_1<+\infty$ such that $\psi(t)\to\infty$ as $t\to t_1$, where

$$0<t_1<\frac{2\psi(0)}{(\beta\mathrm{c_e}-1)\psi'(0)}=\frac{2\gamma\varepsilon}{(\beta\mathrm{c_e}-1)\|u_0\|_2^4}<+\infty.$$

Since ψ is continuous with respect to ϕ, we conclude that there exists a time $T_1\leq t_1$ such that

$$\lim_{t\to T_1}\int_0^t\|u(s)\|_2^2\,\mathrm{d}s=+\infty\Rightarrow\lim_{t\to T_1}\sup\|u(t)\|_2^2=+\infty.$$

Thus, $u(x,t)$ becomes discontinuous at some finite time T_1, indicating that $u(x,t)$ does not exist for all time; in other words, $u(x,t)$ blows up at time T_1. This leads to the nonexistence result stated in the theorem, which implies that ϕ blows up at time T_1 in $L^2(\Omega)$-norm, resulting in a contradiction.

Hence, for the data satisfying (4.2), any solution has a finite explosion time.

2. **Case 2.** Assume that there exists $t_0>0$ such that $\mathrm{E}(u(t_0))<0$, $(u(t_0)\neq 0)$. Noticing that $\mathrm{E}(0)>0$ and considering the continuity of $\mathrm{E}(t)$, we can conclude that there exists some $t_1\in(0,t_0)$ such that $\mathrm{E}(t_1)=0$. Furthermore, by applying the monotonicity of $\mathrm{E}(t)$, we can deduce that $\mathrm{E}(t)\geq 0$, $0<t\leq t_1$. Repeating the proof as in **Case 1** allows us to conclude that the solution to problem (1.1) is exploding before the time t_0.

This concludes the proof.

□

4.2. Second Theorem of blow-up result

For our result, we need to consider the following functions:

$$\alpha(t)=\left[\beta\|D^m u(t)\|_2^2+(g\circ D^m u)(t)\right]^{\frac{1}{2}}, \tag{4.22}$$

and for ε (a small positive value) and N, precise positive constants to be determined later,

$$\mathrm{A}(t):=H^{1-\alpha}(t)+\varepsilon\int_\Omega u(x,t)u_t(x,t)\mathrm{d}x+\varepsilon N\mathrm{E}_1 t,\quad t\in[0,T), \tag{4.23}$$

and

$$\varphi(t)=\int_\Omega |u|^{p_2}\mathrm{d}x. \tag{4.24}$$

Let B, α_1, α_0, c_* and E_1 be positive auxiliary constants that satisfy

$$\begin{gathered} c_*=\max\left((2B)^{p_1},(2B)^{p_2}\right),\ B=\sqrt{\beta}c_*^{\frac{-1}{p_2}}B_1,\ \alpha_1=\left(\tfrac{p_1}{p_2}B_1^{-P_2}\right)^{\frac{1}{P_2-2}} \\ \alpha(0)=\alpha_0=\beta^{\frac{1}{2}}\left\|D^m u_0\right\|_2,\quad \mathrm{E}_1=\left(\tfrac{1}{2}-\tfrac{1}{p_2}\right)\alpha_1^2. \end{gathered} \tag{4.25}$$

The second Theorem result of the blow-up is as follows

Theorem 4.2 *Considering that g, $r(.)$, and $p(.)$ meet conditions (H1)-(H2) with $p_1>r_2$, the local solution of problem (1.1) under boundary conditions that satisfy $\mathrm{E}(0)<\mathrm{E}_1$, $\beta^{\frac{1}{2}}\|D^m u_0\|>\alpha_1$ will blow up in finite time T. This shows to the following estimates:*

$$\int_{\varphi(0)}^{+\infty}\frac{\mathrm{d}z}{C_7\left(z^\delta+z^{\delta\frac{p_1}{p_2}}+z+z^{\frac{p_1}{p_2}}+1\right)}\le T^*\le\frac{1-\alpha}{\alpha\frac{\delta_1}{\delta_2}\mathrm{A}^{\frac{\alpha}{1-\alpha}}(0)},$$

where

$$0<\alpha\le\min\left\{\frac{p_1-2}{2p_1},\frac{p_1-r_2}{p_1\left(r_2-1\right)}\right\}, \tag{4.26}$$

and δ, δ_1, δ_2 are given in (4.59), (4.50), (4.55), respectively.

Lemma 4.3 *Let $h:[0,+\infty)\to\mathbb{R}$ be a function defined by*

$$h\left(t\right):=h\left(\alpha\right)=\frac{1}{2}\alpha^2-\frac{B_1^{p_2}}{p_1}\alpha^{p_2}, \tag{4.27}$$

then h possesses the following properties:

(i) *h is increasing for $0<\alpha\le\alpha_1$ and decreasing for $\alpha\ge\alpha_1$,*

(ii) $\lim\limits_{\alpha\to+\infty}h\left(\alpha\right)=-\infty$ *and* $h\left(\alpha_1\right)=\mathrm{E}_1$,

iii $\mathrm{E}(t)\ge h(\alpha(t))$,

where $\alpha(t)$ is given in (4.22), α_1 and E_1 are given in (4.25).

Proof: $h(\alpha)$ is continuous and differentiable in $[0,+\infty)$,

$$h'(\alpha)=\alpha\left(1-B_1^{p_2}\alpha^{p_2-2}(t)\right)\begin{cases} >0, & \alpha\in(0,\alpha_1)\\ <0, & \alpha\in(\alpha_1,+\infty),\end{cases}$$

which implies that

$$\begin{gathered} h(\alpha)\text{ is strictly increasing in }\left(0,\alpha_1\right), \\ h(\alpha)\text{ is strictly decreasing in }\left(\alpha_1,+\infty\right). \end{gathered} \tag{4.28}$$

Then **(i)** comes. Since $p_2-2>0$, we have $\lim_{\alpha\to+\infty} h(\alpha)=-\infty$. A straightforward calculation results to $h(\alpha_1)=\mathrm{E}_1$. Then **(ii)** holds.

By Lemma (2.1)

$$\begin{aligned}\int_\Omega |u(.)|^{p(x)}\mathrm{d}x &\le \max\left\{\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right\}\\ \le c_*\max&\left(\left(\int_{\{\|D^m u\|_2\ge 1\}}|D^m u(t)|^2\mathrm{d}x\right)^{p_1}, \left(\int_{\{\|D^m u\|_2\ge 1\}}|D^m u(t)|^2\mathrm{d}x\right)^{p_2}\right)\\ &= c_*\left(\int_{\{\|D^m u\|_2\ge 1\}}|D^m u(t)|^2\mathrm{d}x\right)^{p_2}\le c_*\left(\int_\Omega |D^m u(t)|^2\mathrm{d}x\right)^{p_2}.\end{aligned}$$

From (H_1), (2.5) and Lemma (2.1), we have

$$\begin{aligned}\mathrm{E}(t) &\ge \tfrac{1}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right)\|D^m u(t)\|_2^2+\tfrac{1}{2}(g\circ D^m u)(t)-\tfrac{1}{p_1}\int_\Omega |u(t)|^{p(x)}\mathrm{d}x\\ &\ge \tfrac{1}{2}\beta\|D^m u(t)\|_2^2+\tfrac{1}{2}(g\circ D^m u)(t)-\tfrac{1}{p_1}B^{p_2}c_*\left(\int_\Omega |D^m u(t)|^2\mathrm{d}x\right)^{p_2}\\ &\ge \tfrac{1}{2}\left[\beta\|D^m u(t)\|_2^2+(g\circ D^m u)(t)\right]-\tfrac{B_1^{p_2}}{p_1}\left[\beta\|D^m u(t)\|_2^2+(g\circ D^m u)(t)\right]^{\frac{p_2}{2}}\\ &= \tfrac{1}{2}\alpha^2(t)-\tfrac{B_1^{p_2}}{p_1}\alpha^{p_2}(t)=h(\alpha(t)),\end{aligned}$$

in which **(iii)** is verified. □

Lemma 4.4 *Assuming the conditions in Theorem (4.2) are satisfied, there exists a positive constant $\alpha_2>\alpha_1$ such that*

$$\alpha(t)\ge\alpha_2>\alpha_1,\ t\ge 0; \tag{4.29}$$

$$\varrho(u)\ge B_1^{p_2}\alpha_2^{p_2}, \tag{4.30}$$

where α_1, B_1 and E_1 are indicated in (4.25).

Proof: Since $\mathrm{E}(0)<\mathrm{E}_1$ and $h(\alpha)$ is a continuous function, there exist α_2' and α_2 with $\alpha_2'<\alpha_1<\alpha_2$ such that $h(\alpha_2')=h(\alpha_2)=\mathrm{E}(0)$ This, combined with Lemma (4.3) yields

$$h(\alpha_0)\le \mathrm{E}(0)=h(\alpha_2). \tag{4.31}$$

From Lemma (4.3)**(i)**, we deduce that

$$\alpha_0\ge\alpha_2, \tag{4.32}$$

so for $t=0$, (4.29) holds.

Now, we establish (4.29), we proceed by contradiction and suppose there exists $t^*>0$ such that $\alpha(t^*)<\alpha_2$, then we distinguish the following two cases,

Case 1. If $\alpha_2'<\alpha(t^*)<\alpha_2$, using Lemma (4.3) and (4.28), we know that

$$h(\alpha(t^*))>\mathrm{E}(0)\ge\mathrm{E}(t^*),$$

this contradicts Lemma (4.3)**(iii)**.

Case 2. If $\alpha(t^*)\le\alpha_2'$, then $\alpha(t^*)\le\alpha_2'<\alpha_2$. Putting $\lambda(t)=\alpha(t)-\frac{\alpha_2+\alpha_2'}{2}$, then $\lambda(t)$ is a continuous function, $\lambda(t^*)<0$ and by joining (4.32), $\lambda(0)>0$. Hence, there exists $t_0\in(0,t^*)$ such that $\lambda(t_0)=0$, that means $\alpha(t_0)=\frac{\alpha_2+\alpha_2'}{2}$, which shows

$$h(\alpha(t_0))>\mathrm{E}(0)\ge\mathrm{E}(t_0).$$

This contradicts Lemma (4.3)**(iii)**, consequently (4.29) follows.

From (2.5), we have

$$\frac{1}{2}\left[\left(1-\int_0^t g(s)\mathrm{d}s\right)\|D^m u(t)\|_2^2+(g\circ D^m u)(t)\right]\le\mathrm{E}(t)+\frac{1}{p_1}\int_\Omega |u(t)|^{p(x)}\mathrm{d}x,$$

which implies

$$\begin{aligned}\tfrac{1}{p_1}\int_\Omega |u(t)|^{p(x)}\mathrm{d}x &\geq \tfrac{1}{2}\left[\left(1-\int_0^t g(s)\mathrm{d}s\right)\|D^m u(t)\|_2^2+(g\circ D^m u)(t)\right]-\mathrm{E}(t)\\ &\geq \tfrac{1}{2}\left[\beta\|D^m u(t)\|_2^2+(g\circ D^m u)(t)\right]-\mathrm{E}(0)\\ &\geq \tfrac{1}{2}\alpha_2^2-h(\alpha_2)=\tfrac{B_1^{p_2}}{P_1}\alpha_2^{p_2},\end{aligned}$$

then, the second inequality in (4.30) remains valid. □

Let

$$H(t)=\mathrm{E}_1-\mathrm{E}(t) \text{ for } t\geq 0. \tag{4.33}$$

The following lemma hold

Lemma 4.5 *Under the assumptions of Theorem (4.2) and $0\leq \mathrm{E}(0)<\mathrm{E}_1$, the functional $H(t)$ defined in (4.33) satisfies the solutions of (1.1),*

$$0<H(0)\leq H(t)\leq \int_\Omega \frac{1}{p(x)}|u(t)|^{p(x)}\mathrm{d}x\leq \frac{1}{p_1}\varrho(u),\ \ t\geq 0. \tag{4.34}$$

Proof: By using Lemma (2.1), we easily see that

$$H(t)\geq H(0)=\mathrm{E}_1-\mathrm{E}(0)>0,\quad t\geq 0. \tag{4.35}$$

it follows from (4.25) and Lemma (4.4)

$$\begin{aligned}&\mathrm{E}_1-\left[\tfrac{1}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right)\|D^m u(t)\|_2^2+\tfrac{1}{2}\|u_t\|_2^2+\tfrac{1}{2}(g\circ D^m u)(t)\right]\\ &\leq \mathrm{E}_1-\left[\tfrac{1}{2}\|u_t\|_2^2+\tfrac{1}{2}\left(\beta\|D^m u(t)\|_2^2+(g\circ D^m u)(t)\right)\right]\\ &=\mathrm{E}_1-\left[\tfrac{1}{2}\|u_t\|_2^2+\tfrac{1}{2}\alpha^2(t)\right]\\ &\leq \mathrm{E}_1-\tfrac{1}{2}\alpha^2(t)\leq \mathrm{E}_1-\tfrac{1}{2}\alpha_1^2=-\tfrac{1}{p_2}\alpha_1^2<0,\ \text{for all } t\in[0,T),\end{aligned}$$

which gives

$$\begin{aligned}H(t)=\mathrm{E}_1&-\left[\tfrac{1}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right)\|D^m u(t)\|_2^2+\tfrac{1}{2}\|u_t\|_2^2+\tfrac{1}{2}(g\circ D^m u)(t)\right]\\ &+\int_\Omega \tfrac{1}{p(x)}|u(t)|^{p(x)}\mathrm{d}x\leq \int_\Omega \tfrac{1}{p(x)}|u(t)|^{p(x)}\mathrm{d}x\leq \tfrac{1}{p_1}\varrho(u).\end{aligned} \tag{4.36}$$

(4.34) follows from (4.35) and (4.36). □

Lemma 4.6 *For the conditions in Theorem (4.2) there exists a positive constant C such that*

$$\|D^m u(t)\|_2^2\leq C\varrho(u)\ \ \textit{for all } t\in[0,T). \tag{4.37}$$

Proof: From Lemma (4.4) and $\alpha_2>\alpha_1$, we have

$$\varrho(u)\geq B_1\alpha_2^{p_2}>B_1\alpha_1^{p_2-2}\alpha_1^2=\frac{p_1}{p_2}B_1^{1-P_2}\alpha_1^2,$$

which combining with (4.25) gives

$$\mathrm{E}_1\leq B_1^{1-P_2}\frac{p_2}{p_1}\left(\frac{1}{2}-\frac{1}{p_2}\right)\varrho(u). \tag{4.38}$$

joining (4.33), (4.38) and the definition of $H(t)$, we have

$$\begin{array}{c} \frac{1}{2}\beta\|D^m u(t)\|_2^2 \leq \frac{1}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right)\|D^m u(t)\|_2^2 \\ = \mathrm{E}(t) - \frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\left(g\circ D^m u\right)(t) + \int_\Omega \frac{1}{p(x)}|u(t)|^{p(x)}\mathrm{d}x \\ \leq B_1^{1-P_2}\frac{p_2}{p_1}\left(\frac{1}{2}-\frac{1}{p_2}\right)\varrho(u) - H(t) - \frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}(g\circ D^m u)(t) + \frac{1}{p_1}\varrho(u) \\ = \left(B_1^{1-P_2}\frac{p_2}{p_1}\left(\frac{1}{2}-\frac{1}{p_2}\right)+\frac{1}{p_1}\right)\varrho(u) - H(t) - \frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}(g\circ D^m u)(t) \\ \leq \left(B_1^{1-P_2}\frac{p_2}{p_1}\left(\frac{1}{2}-\frac{1}{p_2}\right)+\frac{1}{p_1}\right)\varrho(u). \end{array} \tag{4.39}$$

So the expected result with $C = \frac{\left(B_1^{1-P_2}\frac{p_2}{p_1}\left(1-\frac{2}{p_2}\right)+\frac{2}{p_1}\right)}{\beta}$. □

Based on the lemmas as mentioned above, the proof of Theorem (4.2) is presented as follows

Proof: [Proof of Theorem 4.2] **Case 1.** If $0 \leq \mathrm{E}(0) < \mathrm{E}_1$, differentiating (4.23) gives

$$\mathrm{A}'(t) = (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\int_\Omega\left(u_t^2 + uu_{tt}\right)\mathrm{d}x + N\mathrm{E}_1.$$

Recalling Eq (1.1), we get integrating by parts on Ω that

$$\begin{array}{c} \mathrm{A}'(t) = (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\|u_t\|_2^2 - \varepsilon\|D^m u(t)\|_2^2 \\ +\varepsilon\int_0^t g(t-s)\int_\Omega D^m u(t)D^m u(s)\mathrm{d}x\mathrm{d}s - \varepsilon\int_\Omega |u_t|^{r(x)-2}u_t u\mathrm{d}x \\ +\varepsilon\int_\Omega |u(t)|^{p(x)}\mathrm{d}x + \varepsilon N\mathrm{E}_1 \\ = (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\|u_t\|_2^2 - \varepsilon\|D^m u(t)\|_2^2 \\ +\varepsilon\int_0^t g(t-s)\int_\Omega D^m u(t)(D^m u(s) - D^m u(t))\mathrm{d}x\mathrm{d}s \\ +\varepsilon\int_0^t g(t-s)\int_\Omega |D^m u(t)|^2\mathrm{d}x\mathrm{d}s - \varepsilon\int_\Omega |u_t|^{r(x)-2}u_t u\mathrm{d}x + \varepsilon\int_\Omega |u(t)|^{p(x)}\mathrm{d}x + \varepsilon N\mathrm{E}_1. \end{array} \tag{4.40}$$

We get from Young's inequality that

$$\begin{array}{c} \left|\int_0^t g(t-s)\int_\Omega D^m u(t)(D^m u(s) - D^m u(t))\mathrm{d}x\mathrm{d}s\right| \\ \leq \tau\int_0^t g(t-s)\|D^m u(s) - D^m u(t)\|_2^2\mathrm{d}s + \frac{1}{4\tau}\int_0^t g(s)\mathrm{d}s\|D^m u(t)\|_2^2 \\ = \tau(g\circ D^m u)(t) + \frac{1}{4\tau}\int_0^t g(s)\mathrm{d}s\|D^m u(t)\|_2^2 \\ \text{for any } \tau > 0. \end{array} \tag{4.41}$$

Replacing (4.41) in (4.40), using (2.5), and picking $\tau > 0$ such that $0 < \tau < \frac{p_1}{2}$, we deduce

$$\begin{array}{c} \mathrm{A}'(t) \geq (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\|u_t\|_2^2 - \varepsilon\|D^m u(t)\|_2^2 + \int_0^t g(s)\mathrm{d}s\|D^m u(t)\|_2^2 \\ -\int_\Omega |u_t|^{r(x)-2}u_t u\mathrm{d}x - \tau\varepsilon(g\circ D^m u)(t) \\ -\frac{1}{4\tau}\varepsilon\int_0^t g(s)\mathrm{d}s\|D^m u(t)\|_2^2 + \varepsilon p_1\left(H(t) - \mathrm{E}_1\right) + \frac{p_1}{2}\varepsilon(g\circ D^m u)(t) \\ +\frac{p_1}{2}\varepsilon\|u_t\|_2^2 + \frac{p_1}{2}\varepsilon\left(1-\int_0^t g(s)\mathrm{d}s\right)\|D^m u(t)\|_2^2 \\ \geq (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\left(\frac{p}{2}+1\right)\|u_t\|_2^2 + \varepsilon\left(\frac{p_1}{2}-\tau\right)\left(g\circ D^m u\right)(t) \\ +\varepsilon(N-p_1)\mathrm{E}_1 + \varepsilon p_1 H(t) - \varepsilon\int_\Omega |u_t|^{r(x)-2}u_t u\mathrm{d}x \\ +\varepsilon\left[\left(\frac{p_1}{2}-1\right) - \left(\frac{p_1}{2}-1+\frac{1}{4\tau}\right)\int_0^\infty g(s)\mathrm{d}s\right]\|D^m u(t)\|_2^2. \end{array} \tag{4.42}$$

Joining (2.4) and (4.42), we obtain

$$\begin{array}{c} \mathrm{A}'(t) \geq (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\left(\frac{p_1}{2}+1\right)\|u_t\|_2^2 + a_1\varepsilon\left(g\circ D^m u\right)(t) \\ +a_2\varepsilon\|D^m u(t)\|_2^2 + \varepsilon(N-p_1)\mathrm{E}_1 + \varepsilon p_1 H(t) - \varepsilon\int_\Omega |u_t|^{m(x)-2}u_t u\mathrm{d}x, \end{array} \tag{4.43}$$

where

$$a_1 = \left(\frac{p_1}{2}-\tau\right) > 0,\ a_2 = \left(\frac{p_1}{2}-1\right) - \left(\frac{p_1}{2}-1+\frac{1}{4\tau}\right)\int_0^\infty g(s)\mathrm{d}s > 0.$$

For a large enough constant $\alpha > 1$ where to be determined later, we have

$$\int_\Omega |u_t|^{r(x)-1} |u| \mathrm{d}x \leq \frac{1}{\lambda^{r_1}} \int_\Omega H^{\alpha(r(x)-1)}(t) |u|^{r(x)} \mathrm{d}x + \lambda^{\frac{r_1}{r_1-1}} H^{-\alpha}(t) \int_\Omega |u_t|^{r(x)} \mathrm{d}x. \tag{4.44}$$

where the Hölder inequality is used.

Joining (4.43) with (4.44) gives

$$\begin{aligned} \mathrm{A}'(t) \geq & \left[(1-\alpha) - \varepsilon \lambda^{\frac{r_1}{r_1-1}}\right] H^{-\alpha}(t) H'(t) + \varepsilon \left(\tfrac{p_1}{2}+1\right) \|u_t\|_2^2 + \varepsilon a_1 \left(g \circ D^m u\right)(t) \\ & + \varepsilon a_2 \|D^m u(t)\|_2^2 + \varepsilon (N-p_1) \mathrm{E}_1 + \varepsilon p_1 H(t) - \varepsilon \lambda^{-r_1} H^{\alpha(r_2-1)}(t) \textstyle\int_\Omega |u|^{r(x)} \mathrm{d}x. \end{aligned} \tag{4.45}$$

If $0 < H(t) \leq 1$, from (4.36), we have

$$\begin{aligned} \textstyle\int_\Omega |u|^{r(x)} H^{\alpha(r(x)-1)}(t) \mathrm{d}x & \leq \textstyle\int_\Omega |u|^{r(x)} \mathrm{d}x \leq \max\left(\|u\|_{r(.)}^{r_1}, \|u\|_{r(.)}^{r_2}\right) \\ & \leq c_1 \max\left(\|u\|_{p(.)}^{r_1}, \|u\|_{p(.)}^{r_2}\right) \\ & \leq c_1 \max\left(\left(\textstyle\int_\Omega |u|^{p(x)} \mathrm{d}x\right)^{\frac{r_1}{p_2}}, \left(\textstyle\int_\Omega |u|^{p(x)} \mathrm{d}x\right)^{\frac{r_2}{p_1}}\right) \\ & \leq c_1 \max\left(\left(\textstyle\int_\Omega |u|^{p(x)} \mathrm{d}x\right)^{\frac{r_1-p_2}{p_2}}, \left(\textstyle\int_\Omega |u|^{p(x)} \mathrm{d}x\right)^{\frac{r_2-p_1}{p_1}}\right) \textstyle\int_\Omega |u|^{p(x)} \mathrm{d}x \\ & \leq c_1 \max\left(\left(p_1 H(0)\right)^{\frac{r_1-p_2}{p_2}}, \left(p_1 H(0)\right)^{\frac{r_2-p_1}{p_1}}\right) \textstyle\int_\Omega |u|^{p(x)} \mathrm{d}x = c_2 \textstyle\int_\Omega |u|^{p(x)} \mathrm{d}x. \end{aligned}$$

If $H(t) > 1$, we have

$$\begin{aligned} & \textstyle\int_\Omega |u|^{r(x)} H^{\alpha(r(x)-1)}(t) \mathrm{d}x \leq H^{\alpha(r_2-1)}(t) \textstyle\int_\Omega |u|^{r(x)} \mathrm{d}x \\ & \leq c_1 H^{\alpha(r_2-1)}(t) \max\left(\left(\textstyle\int_\Omega |u|^{p(x)} \mathrm{d}x\right)^{\frac{m_1}{p_1}}, \left(\textstyle\int_\Omega |u|^{p(x)} \mathrm{d}x\right)^{\frac{r_2}{p_1}}\right) \\ & \leq c_1 \left(\tfrac{1}{p_1}\right)^{\alpha(r_2-1)} \left(\textstyle\int_\Omega |u|^{p(x)} \mathrm{d}x\right)^{\alpha(r_2-1)} \max\left(\left(\textstyle\int_\Omega |u|^{p(x)} \mathrm{d}x\right)^{\frac{r_1}{p_2}}, \right. \\ & \qquad \left. \left(\textstyle\int_\Omega |u|^{p(x)} \mathrm{d}x\right)^{\frac{r_2}{p_1}}\right) \\ & \leq c_1 \left(\tfrac{1}{p_1}\right)^{\alpha(r_2-1)} \max\left(\left(\textstyle\int_\Omega |u|^{p(x)} \mathrm{d}x\right)^{\frac{r_1-p_2}{p_2}+\alpha(r_2-1)}, \right. \\ & \qquad \left. \left(\textstyle\int_\Omega |u|^{p(x)} \mathrm{d}x\right)^{\frac{r_2-p_1}{p_1}+\alpha(r_2-1)}\right) \textstyle\int_\Omega |u|^{p(x)} \mathrm{d}x \\ & \leq c_2 H^{\alpha(r_2-1)}(0) \textstyle\int_\Omega |u|^{p(x)} \mathrm{d}x. \end{aligned}$$

Combining the two cases, we have

$$H^{\alpha(r(x)-1)}(t) \int_\Omega |u|^{r(x)} \mathrm{d}x \leq c_3 \int_\Omega |u|^{p(x)} \mathrm{d}x, \tag{4.46}$$

where

$$c_1 = \left(1 + |\Omega|^{\frac{p_2-r_1}{p_2} \frac{r_2}{r_1}}\right),$$

$$c_2 = c_1 \max\left(\left(p_1 H(0)\right)^{\frac{r_1-p_2}{p_2}}, \left(p_1 H(0)\right)^{\frac{r_2-p_1}{p_1}}\right),$$

$$c_3 = c_2 \left(1 + H^{\alpha(r_2-1)}(0)\right).$$

Joining (4.45) and (4.46) yields

$$\begin{aligned} \mathrm{A}'(t) \geq & \left[(1-\alpha) - \varepsilon \lambda^{\frac{r_1}{r_1-1}}\right] H^{-\alpha}(t) H'(t) + \varepsilon \left(\tfrac{p_1}{2}+1\right) \|u_t\|_2^2 + \varepsilon a_1 \left(g \circ D^m u\right)(t) \\ & + \varepsilon a_2 \|D^m u(t)\|_2^2 + \varepsilon (N-p_1) \mathrm{E}_1 + \varepsilon p_1 H(t) - \varepsilon \lambda^{-r_1} c_3 \textstyle\int_\Omega |u|^{r(x)} \mathrm{d}x, \end{aligned} \tag{4.47}$$

clearly

$$H(t) \geq \mathrm{E}_1 - \frac{1}{2}\|D^m u(t)\|_2^2 - \frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}(g \circ D^m u)(t) + \frac{1}{p_2}\varrho(u), \tag{4.48}$$

Rewriting p_1 as $p_1 = p_1 - 2a_3 + 2a_3$, with $0 < a_3 < \min\left(a_1, a_2, \frac{p_1}{2}\right)$, (4.48) in (4.47) implies

$$\begin{aligned}\mathrm{A}'(t) \geq \left[(1-\alpha) - \varepsilon\lambda^{\frac{r_1}{r_1-1}}\right] H^{-\alpha}(t)H'(t) + \varepsilon\left(\tfrac{p_1}{2} + 1 - a_3\right)\|u_t\|_2^2 \\ +\varepsilon\left(a_1 - a_3\right)\left(g \circ D^m u\right)(t) + \varepsilon\left(a_2 - a_3\right)\|D^m u(t)\|_2^2 \\ +\varepsilon(N - (p_1 - 2a_3))\mathrm{E}_1 + \varepsilon\left(p_1 - 2a_3\right)H(t) \\ +\varepsilon\left(\tfrac{2}{p_2}a_3 - \lambda^{-r_1}c_3\right)\varrho(u).\end{aligned}$$

At this step, we select λ and N to be sufficiently large so that

$$\begin{aligned}\gamma_1 = \frac{2}{p_2}a_3 - \lambda^{-r_1}c_3 > 0, \\ N - (p_1 - 2a_3) > 0.\end{aligned}$$

When N and λ are fixed (thus γ_1), we choose ε small sufficiently so that

$$(1-\alpha) - \varepsilon\lambda^{\frac{r_1}{r_1-1}} > 0, \text{ and } \mathrm{A}(0) = H^{1-\alpha}(0) + \varepsilon\int_\Omega u_0u_1\mathrm{d}x > 0, \text{ since } H(0) > 0. \tag{4.49}$$

Then, there exists a constant δ_1 that satisfies

$$0 < \delta_1 \leq \min\left\{\frac{p_1}{2} + 1 - a_3, a_1 - a_3, \gamma_1, p_1 - 2a_3\right\}, \tag{4.50}$$

and

$$\mathrm{A}'(t) \geq \delta_1\varepsilon\left[\|u_t\|_2^2 + (g \circ D^m u)(t) + \|D^m u(t)\|_2^2 + H(t) + \varrho(u)\right], \tag{4.51}$$

which combined with (4.49), infer

$$\mathrm{A}(t) \geq \mathrm{A}(0) > 0, \ \forall t \in [0, T).$$

Choosing $\varepsilon > 0$ such that $\varepsilon < \frac{1}{T}\left(\frac{\alpha_2}{\alpha_1}\right)^{p_2}$, recalls Lemma (4.4), and we obtain

$$|\varepsilon N\mathrm{E}_1 T|^{\frac{1}{1-\alpha}} \leq \left(\frac{\alpha_2}{\alpha_1}\right)^{p_2} N\mathrm{E}_1 \leq \frac{N\mathrm{E}_1}{B_1\alpha_1^{p_2}}\varrho(u). \tag{4.52}$$

Recalling $L^{p(\cdot)}(\Omega) \hookrightarrow L^2(\Omega)$, we see that

$$\begin{aligned}\left|\int_\Omega uu_t\mathrm{d}x\right|^{\frac{1}{1-\alpha}} \leq \|u\|_2^{\frac{1}{1-\alpha}}\|u_t\|_2^{\frac{1}{1-\alpha}} \\ \leq (1+|\Omega|)^{\frac{p_1-2}{p_1(1-\alpha)}}\|u\|_{p(x)}^{\frac{1}{1-\alpha}}\|u_t\|_2^{\frac{1}{1-\alpha}} \\ \leq c_4\left(\|u_t\|_2^2 + \|u\|_{p(x)}^{\frac{2}{1-2\alpha}}\right) \\ \leq c_4\|u_t\|_2^2 + c_4\max\left\{\left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{2}{(1-2\alpha)p_1}}, \left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{2}{(1-2\alpha)p_2}}\right\} \\ \leq c_4\|u_t\|_2^2 + c_5\int_\Omega |u|^{p(x)}\mathrm{d}x,\end{aligned} \tag{4.53}$$

in which we applied Holder's and Young's inequalities, where

$$\begin{aligned}c_4 = (1+|\Omega|)^{\frac{p_1-2}{p_1(1-\alpha)}} \\ c_5 = c_4\max\left\{(p_1H(0))^{\frac{2}{(1-2\alpha)p_1}-1}, (p_1H(0))^{\frac{2}{(1-2\alpha)p_2}-1}\right\}.\end{aligned}$$

We have from (4.23), (4.52), (4.53) and Cauchy-Schwarz's inequality that

$$\begin{aligned}\mathrm{A}^{\frac{1}{1-\alpha}}(t) \leq 2^{1/(1-\alpha)+1}\left(H(t) + \varepsilon^{\frac{1}{1-\alpha}}\left|\int_\Omega uu_t(x,t)\mathrm{d}x\right|^{\frac{1}{1-\alpha}} + \varepsilon^{\frac{1}{1-\alpha}}(N\mathrm{E}_1T)^{\frac{1}{1-\alpha}}\right) \\ \leq \delta_2\left[H(t) + \|u_t\|_2^2 + \varrho(u)\right],\end{aligned} \tag{4.54}$$

where δ_2 is a positive constant chosen as

$$\delta_2 = 2^{1/(1-\alpha)+1} \max\left(1, \varepsilon^{\frac{1}{1-\alpha}}, c_4, c_5 + \frac{N\mathrm{E}_1}{B_1\lambda_1^{p_2}}\right). \tag{4.55}$$

combining (4.52), (4.53), (4.51), yields

$$\mathrm{A}'(t) \geq \frac{\delta_1}{\delta_2}\mathrm{A}^{\frac{1}{1-\alpha}}(t), \text{ for all } t \geq 0, \tag{4.56}$$

A simple integration of (4.41) over $(0,t)$ implies that

$$\mathrm{A}^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{\mathrm{A}^{\frac{\alpha}{1-\alpha}}(0) - \frac{\alpha}{1-\alpha}\frac{\delta_1}{\delta_2}t}. \tag{4.57}$$

Therefore, $\mathrm{A}(t)$ blows up in a finite time $\widehat{T}$,

$$\widehat{T} \leq \frac{1-\alpha}{\alpha\frac{\delta_1}{\delta_2}\mathrm{A}^{\frac{\alpha}{1-\alpha}}(0)}.$$

As $\mathrm{A}(0) > 0$, (4.57) shows that $\lim_{t\to T^*}\mathrm{A}(t) = \infty$, where $T^* = \frac{1-\alpha}{\alpha\frac{\delta_1}{\delta_2}\mathrm{A}^{\frac{\alpha}{1-\alpha}}(0)}$. This achieve the proof.

Case 2. In the scenario where $E(0) < 0$, substituting $H(t) = -\mathrm{E}(t)$ in Lemma (4.6) produces a result akin to that of Lemma (4.6). Consequently, we have $0 < -\mathrm{E}(0) = H(0) \leq H(t)$ and $H(t) \leq \frac{1}{p_1}\varrho(u)$. By setting $N = 0$ in (4.23) and applying the same reasoning as in **Case 1**, we can arrive at our conclusion.

Lower bound of the blowing-up time:

we have from (4.24)

$$\varphi'(t) = p_2\int_\Omega |u|^{p_2-2}uu_t\mathrm{d}x \leq p_2\int_\Omega |u|^{2p_2-2}\mathrm{d}x + p_2\int_\Omega |u_t|^2\,\mathrm{d}x. \tag{4.58}$$

To evaluate the initial term on the right side of inequality (4.58), we discuss the following three scenarios.

Case.1. $n < 2m$. By the embedding inequality, we have

$$\int_\Omega |u|^{2p_2-2}\mathrm{d}x \leq \hat{B}^{2p_2-2}\|D^m u\|_2^{2(p_2-1)} \leq \hat{B}^{2p_2-2}\left(\|D^m u\|_2^2 + \int_\Omega |u|^{p_2}\mathrm{d}x\right)^{p_2-1}.$$

Case.2. $2 < p_2 < \frac{n}{n-2m+1}$, $n \geq 2m+1$. Applying Hölder's and embedding inequalities

$$\begin{aligned}
\textstyle\int_\Omega |u|^{2p_2-2}\mathrm{d}x &= \textstyle\int_\Omega |u|^{2p_2-4}u^2\mathrm{d}x \\
&\leq \left(\textstyle\int_\Omega |u|^{\frac{(p_2-2)n}{m}}\mathrm{d}x\right)^{\frac{2m}{n}}\left(\textstyle\int_\Omega |u|^{\frac{2n}{n-2m}}\mathrm{d}x\right)^{1-\frac{2m}{n}} \\
&\leq |\Omega|^{\frac{2m}{n}-\frac{2(p_2-2)}{p_2}}\|u\|^2_{\frac{2n}{n-2m}}\left(\textstyle\int_\Omega |u|^{p_2}\mathrm{d}x\right)^{\frac{2(p_2-2)}{p_2}} \\
&\leq B_1^2|\Omega|^{\frac{2m}{n}-\frac{2(p_2-2)}{p_2}}\|D^m u\|_2^2\left(\textstyle\int_\Omega |u|^{p_2}\mathrm{d}x\right)^{\frac{2(p_2-2)}{p_2}} \\
&\leq B_1^2|\Omega|^{\frac{2m}{n}-\frac{2(p_2-2)}{p_2}}\left(\|D^m u\|_2^2 + \textstyle\int_\Omega |u|^{p_2}\mathrm{d}x\right)^{\frac{3p_2-4}{p_2}}.
\end{aligned}$$

Case.3. $\frac{n+1}{n-2m+1} \leq p_2 < \frac{n}{n-2m}$, $n \geq 2m+1$.

Let $\gamma = 2p-2$, $\mu = \frac{n}{m}(p-2)$, $2^* = \frac{2n}{n-2m}$ and using Hölder's inequality, we have

$$\int_\Omega |u|^\gamma\mathrm{d}x = \int_\Omega |u|^{\gamma a}\mathrm{d}x + \int_\Omega |u|^{\gamma(1-a)}\mathrm{d}x \leq \left(\int_\Omega |u|^\mu\mathrm{d}x\right)^{\frac{\gamma a}{\mu}}\left(\int_\Omega |u|^{2^*}\mathrm{d}x\right)^{\frac{\gamma(1-a)}{2^*}},$$

where a satisfies that $\frac{\gamma a}{\mu} + \frac{\gamma(1-a)}{2^*} = 1$. A direct calculation shows

$$\begin{aligned}
a &= \frac{1-\frac{\gamma}{2^*}}{\frac{\gamma}{\mu}-\frac{\gamma}{2^*}} = \frac{\mu\left(2^*-\gamma\right)}{\gamma\left(2^*-\mu\right)}, \\
\frac{a\gamma}{\mu} &= \frac{2^*-\gamma}{2^*-\mu} = \frac{2m}{n}, \quad \frac{\gamma-a\gamma}{2^*} = \frac{n-2m}{n},
\end{aligned}$$

and then we have

$$\begin{aligned}\|u\|_\gamma^\gamma &\leq \|u\|_\mu^{\gamma a}\|u\|_{2^*}^{\gamma(1-a)} = \|u\|_\mu^{\frac{2m\mu}{n}}\|u\|_{2^*}^2\\ &\leq C_1^2\left(1+|\Omega|^{\frac{2m(p_2-\mu)}{np_2}}\right)\|u\|_{p_2}^{\frac{2m\mu}{n}}\|D^m u\|_2^2\\ &\leq C_2\left(\|u\|_{p_2}^{p_2}+\|D^m u\|_2^2\right)^{q_1},\end{aligned}$$

with $q_1 = \frac{3p_2-4}{p_2}$. Therefore, we obtain

$$\int_\Omega |u|^{2p_2}\mathrm{d}x \leq c^*\left(\int_\Omega |u|^{p_2}\mathrm{d}x + \int_\Omega |D^m u|^2\mathrm{d}x\right)^\delta. \tag{4.59}$$

where $\delta = C\left(p_2-1, \frac{3p_2-4}{p_2}\right)$, for above three Cases. From the definition of $\mathrm{E}(t)$, we see that

$$\begin{aligned}\tfrac{1}{2}\int_\Omega |u_t|^2\,\mathrm{d}x + \tfrac{1}{2}\beta\int_\Omega |D^m u|^2\mathrm{d}x &\leq \mathrm{E}(0) + \tfrac{1}{p_1}\int_\Omega |u|^{p(x)}\mathrm{d}x\\ &\leq \mathrm{E}(0) + \tfrac{c_6}{p_1}\left(\varphi(t)+\varphi^{\frac{p_1}{p_2}}(t)\right)\end{aligned} \tag{4.60}$$

where $c_6 = (1+|\Omega|)^{\frac{p_2-p_1}{p_1}}$. Combining (4.58)-(4.60), we obtain

$$\begin{aligned}\varphi'(t) &\leq c^* p_2\left[2\mathrm{E}(0)+\left(1+\tfrac{2c_6}{p_1}\right)\varphi(t)+\tfrac{2c_6}{p_1}\varphi^{\frac{p_1}{p_2}}(t)\right]^\delta\\ &\quad + p_2\left(2\mathrm{E}(0)+\tfrac{2c_6}{p_1}\varphi(t)+\tfrac{2c_6}{p_1}\varphi^{\frac{p_1}{p_2}}(t)\right)\\ &\leq c_7\left(\varphi^\delta(t)+\varphi^{\frac{p_1}{p_2}\delta}(t)+\varphi(t)+\varphi^{\frac{p_1}{p_2}}(t)+1\right),\end{aligned}$$

where

$$c_7 = \max\left(4^{\delta-1}c^* p_2\left(1+\frac{2c_6}{p_1}\right)^\delta, \frac{2^{3\delta-2}cc_6^\delta p_2}{p_1^\delta}, \frac{2p_2c_6}{p_1}, 2^{3\delta-2}cp_2\mathrm{E}^\delta(0)+2p_2\mathrm{E}(0)\right). \tag{4.61}$$

The definition of T^* allows us

$$\lim_{t\to T^*}\int_\Omega |u|^{p_2}\mathrm{d}x = +\infty,$$

in which we obtain

$$\int_{\varphi(0)}^{+\infty}\frac{\mathrm{d}z}{c_7\left(z^\delta+z^{\delta\frac{p_1}{p_2}}+z+z^{\frac{p_1}{p_2}}+1\right)} \leq T^*.$$

The proof is complete. □

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