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Stability analysis for a class of fuzzy fractional differential equations with time delay involving generalized Atangana-Baleanu derivative

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ABSTRACT: This research delves into a comprehensive investigation of a specific category of fuzzy fractional differential equations, focusing on issues of existence, uniqueness, and Ulam-Hyers stability of solutions. The considered equation incorporate the generalized Atangana-Baleanu derivative in the Caputo sense and encompass time-delays. Integral to the derivation of substantial results are functional analysis techniques, notably the Schaefer fixed point theorem for establishing existence and the Banach fixed point theorem for ensuring uniqueness. The study further extends its contributions by examining Ulam-Hyers stability concerning variations in parameters, encompassing both initial conditions or parameters of the equation. These insights are grounded in the application of generalized forms of Gronwall's inequality. To illustrate and reinforce the obtained results, the research includes a demonstrative example.

Key Words: Atangana-Baleanu fractional derivative, Schaefer fixed point theorem, generalized Gronwall's inequality, Ulam-Hyers stability, time-delays.

Contents

1. Introduction

In recent years, the study of fractional calculus has gained significant attention due to its ability to provide a more accurate description of complex systems exhibiting memory and hereditary properties. This mathematical framework, characterized by derivatives and integrals of non-integer order, has found applications in various scientific disciplines, including physics, engineering, biology, and bimolecular dynamics [17,18,21,19]. One of the key advancements in fractional calculus is the introduction of the Atangana-Baleanu fractional derivative in the Caputo sense (ABC) [20]. Named after the mathematicians Abdon Atangana and Dumitru Baleanu, this fractional derivative offers a versatile and robust tool for modeling real-world phenomena with fractional dynamics. Its broad applicability has led to its adoption in diverse fields, making it an attractive choice for researchers exploring the dynamics of complex systems (see [1,2,22,23,5]). Recently, Vu et al. [30] have introduced a new definition of fractional derivative called generalized ABC derivative, by replacing the Mittag-Leffler function kernel with the generalized Mittag-Leffler function. In this context, we delve into the realm of fuzzy fractional differential equations (FFDEs) with time-delay, a subject that adds an extra layer of complexity to the modeling process. The inclusion of fuzziness allows for a more realistic representation of uncertainty and imprecision inherent in many natural systems. Time-delay, on the other hand, captures the delayed response observed in various dynamic processes, introducing an additional temporal dimension to the mathematical models. The amalgamation of fuzzy logic [33] and fractional calculus in the presence of

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time-delay presents a formidable challenge and promises to enhance our understanding of intricate systems. This article aims to explore the dynamics of such systems by formulating and analyzing fuzzy fractional differential equations under the influence of time-delay, employing the powerful framework of the generalized Atangana-Baleanu fractional derivative. By addressing the intricate interplay between fuzzy logic, fractional calculus, and time-delay, this work contributes to the growing body of knowledge in the broader field of applied mathematics, paving the way for more accurate and nuanced models of complex systems. Recently, the theory of fractional differential equations have been the subject of important studies, then many scientists extended these equations into new forms and presented the solvability aspect of those problems both numerically and theoretically, see [27,25,6,28,3,4,29,9,10,11,12,13,14,15,16]. We can observe that many authors have achieved some outstanding results in fuzzy fractional differential equations involving Atangana-Baleanu fractional derivatives, we refer the reader to the interesting papers [7,8,26,24] and the references therein. Motivated by the results mentioned above, we are concerned with a novel class of fuzzy fractional differential equations with generalized Atangana-Baleanu fractional derivatives given as follow:

$$\begin{cases} {}^{ABC}\mathcal{D}_{0^+}^{\gamma;\psi}\mathbf{y}(u) = \mathfrak{f}\big(u, \mathbf{y}(u), \mathbf{y}(u-\xi)\big), & u \in [0, D], \\ \mathbf{y}(u) = \varphi(u), & u \in [-\xi, 0], \end{cases} \tag{1.1}$$

where ${}^{ABC}\mathcal{D}_{0^+}^{\gamma;\psi}$ denotes the generalized ABC derivative of order $0 < \gamma < 1$ and $\mathfrak{f} : [0, D] \times \mathbf{E^n} \times \mathbf{E^n} \longrightarrow \mathbf{E^n}$ is continuous function, $\xi \in \mathbb{R}^+$ represents the delay, $\varphi(u)$ is history function.
The remainder of this article is organized as follows: Section 2 outlines the mathematical preliminaries, introducing the necessary concepts and definitions. In Section 3, we present the existence and uniqueness results of our study, followed by the Ulam-Hyers stability result in Section 4. Section 5 includes an example to demonstrate the usefulness of our findings. Finally, conclusions is discussed in Section 6.

2. Preliminaries

In this part, we will review some essential ideas of fuzzy fractional integrals that will be used in the next sections.

Definition 2.1 *[32] The set of fuzzy subsets of $\mathbb{R}^n$ is denoted by $\mathbf{E^n} := \big\{\Upsilon : \mathbb{R}^n \longrightarrow [0, 1]\big\}$ which satisfies:*

(i) Υ is upper semicontinous on $\mathbb{R}^n$,

(ii) Υ is fuzzy convex, i.e, for $0 \le \lambda \le 1$

$$\Upsilon\big(\lambda z_1 + (1-\lambda)z_2\big) \ge min\left\{\Upsilon(z_1), \Upsilon(z_2)\right\}, \ \forall z_1, z_2 \in \mathbb{R}^n,$$

(iii) $[\Upsilon]^0 = \overline{\{z \in \mathbb{R}^n : \Upsilon(z) > 0\}}$ is compact,

(iv) Υ is normal, i.e, $\exists z_0 \in \mathbb{R}^n$ such that $\Upsilon(z_0) = 1$.

Definition 2.2 *[32] The p-level set of $\Upsilon \in \mathbf{E^n}$ is defined by:*
For $p \in (0, 1]$, we have $[\Upsilon]^p = \big\{z \in \mathbb{R}^n | \Upsilon(z) \ge p\big\}$ and for $p = 0$ we have $[\Upsilon]^0 = \overline{\big\{z \in \mathbb{R}^n | \Upsilon(z) > 0\big\}}$.

Remark 2.1 From Definition 2.1, it follows that the p-level set $[\Upsilon]^p$ of Υ, is a nonempty compact interval and $[\Upsilon]^p = [\underline{\Upsilon}(p), \overline{\Upsilon}(p)]$. Moreover, $len([\Upsilon]^p) = l([\Upsilon]^p) := \overline{\Upsilon}(p) - \underline{\Upsilon}(p)$.

Definition 2.3 *[32] For addition and scalar multiplication in fuzzy set space $\mathbf{E^n}$, we have*

$$[\Upsilon_1 + \Upsilon_2]^p = [\Upsilon_1]^p + [\Upsilon_2]^p = \big\{z_1 + z_2 \mid z_1 \in [\Upsilon_1]^p, z_2 \in [\Upsilon_2]^p\big\},$$

and

$$[\alpha\Upsilon]^p = \alpha[\Upsilon]^p = \big\{\alpha z \mid z \in [\Upsilon]^p\big\},$$

for all $p \in [0, 1]$.

Definition 2.4 *[32] The Hausdorff distance is given by*

$$\begin{aligned}\mathbf{D}_\infty(\Upsilon_1,\Upsilon_2) &= \sup_{0\le p\le 1}\left\{|\underline{\Upsilon}_1(p)-\underline{\Upsilon}_2(p)|,|\overline{\Upsilon}_1(p)-\overline{\Upsilon}_2(p)|\right\},\\ &= \sup_{0\le p\le 1}\mathcal{D}_H\big([\Upsilon_1]^p,[\Upsilon_2]^p\big).\end{aligned}$$

Remark 2.2 $(\mathbf{E^n},\mathbf{D}_\infty)$ is complet metric space with the above definition (see [32]) and we have the following properties of $\mathbf{D}_\infty$:
(i) $\mathbf{D}_\infty(\Upsilon_1+\Upsilon_3,\Upsilon_2+\Upsilon_3)=\mathbf{D}_\infty(\Upsilon_1,\Upsilon_2)$,
(ii) $\mathbf{D}_\infty(\lambda\Upsilon_1,\lambda\Upsilon_2)=|\lambda|\mathbf{D}_\infty(\Upsilon_1,\Upsilon_2)$ and $\mathbf{D}_\infty(\lambda_1\Upsilon,\lambda_2\Upsilon)=|\lambda_1-\lambda_2|\mathbf{D}_\infty(\Upsilon,\hat{0})$,
(iii) $\mathbf{D}_\infty(\Upsilon_1,\Upsilon_3)\le\mathbf{D}_\infty(\Upsilon_1,\Upsilon_2)+\mathbf{D}_\infty(\Upsilon_2,\Upsilon_3)$,
(iv) $\mathbf{D}_\infty(\Upsilon_1+\Upsilon_2,\hat{0})\le\mathbf{D}_\infty(\Upsilon_1,\hat{0})+\mathbf{D}_\infty(\Upsilon_2,\hat{0})$,
for all $\Upsilon_1,\Upsilon_2,\Upsilon_3\in\mathbf{E^n}$ and $\lambda,\lambda_1,\lambda_2\in\mathbb{R}^n$.

Definition 2.5 *[32] Let Υ_1, $\Upsilon_2\in\mathbf{E^n}$, if there exists $\Upsilon_3\in\mathbf{E^n}$ such that $\Upsilon_1=\Upsilon_2+\Upsilon_3$, then Υ_3 is called the Hukuhara difference of Υ_1 and Υ_2 noted by $\Upsilon_1\ominus\Upsilon_2$.*

Definition 2.6 *[25] The generalized Hukuhara difference (gH-difference) of $\Upsilon_1,\Upsilon_2\in\mathbf{E^n}$ is defined as follows:*

$$\Upsilon_1\ominus_{gH}\Upsilon_2=\Upsilon_3\Leftrightarrow\begin{cases}(i)\quad\Upsilon_1=\Upsilon_2+\Upsilon_3,\ \text{if}\quad len([\Upsilon_1]^p)\ge len([\Upsilon_2]^p).\\ (ii)\quad\Upsilon_2=\Upsilon_1+(-1)\Upsilon_3,\ \text{if}\quad len([\Upsilon_2]^p)\ge len([\Upsilon_1]^p).\end{cases}$$

Definition 2.7 *[32] Let a fuzzy function $\Upsilon:[a,b]\longrightarrow\mathbf{E^n}$. If for every $p\in[0,1]$, the function $u\longmapsto len[\Upsilon(u)]^p$ is increasing (decreasing) on $[a,b]$, then Υ is called d-increasing (d-decreasing) on $[a,b]$.*

Remark 2.3 If Υ is d-increasing or d-decreasing, then we say that Υ is d-monotone on $[a,b]$.

Definition 2.8 *[30] A fuzzy function $\Upsilon:(a,b)\longrightarrow\mathbf{E^n}$ is called the gH-differentiable at $u\in(a,b)$ if there is $\Upsilon'(u)\in\mathbf{E^n}$ such that*

$$\Upsilon'(u)=\lim_{k\longrightarrow0}\frac{\Upsilon(u+k)\ominus_{gH}\Upsilon(u)}{k}.$$

Notation:
- $C^{\mathbf{E^n}}([c,d]):=C\big([c,d],\mathbf{E^n}\big)$ denote the set of all continuous fuzzy functions.
- $C^1\big([c,d],\mathbb{R}^+\big)$ denote the space of real-valued continuously differentiable functions on $[c,d]$.
- $\mathbb{S}$ denote the set of real-valued functions $\psi\in C^1\big([c,d],\mathbb{R}^+\big)$ satisfying: ψ is increasing, $\psi'(u)$ is positive and for all $u\in(c,d)$ we have $\psi'(u)\neq0$.
- $\mathcal{L}^{\mathbf{E^n}}([c,d]):=\mathcal{L}\big([c,d],\mathbf{E^n}\big)$ denote the space of all fuzzy integrable functions on $[c,d]$.

Definition 2.9 *[30] Let $0<\gamma<1$ and $\psi\in\mathbb{S}$. The generalized fuzzy fractional integral concerning the kernel ψ-function of $\mathbf{z}:[c,d]\longrightarrow\mathbf{E^n}$ is defined by*

$$\mathcal{I}_{c^+}^{\gamma;\psi}\mathbf{z}(u)=\frac{1}{\Gamma(\gamma)}\int_c^u\psi'(v)\big(\psi(u)-\psi(v)\big)^{\gamma-1}\mathbf{z}(v)dv,$$

Remark 2.4 [30] Let $\gamma_1,\gamma_2>0$, and $\psi(u)\in\mathbb{S}$
(i) $\mathcal{I}_{c^+}^{\gamma_1;\psi}\mathcal{I}_{c^+}^{\gamma_2;\psi}\mathbf{z}(u)=\mathcal{I}_{c^+}^{\gamma_1+\gamma_2;\psi}\mathbf{z}(u)$,
(ii) $\mathcal{I}_{c^+}^{\gamma;\psi}\big(\mathbf{z}(u)+\mathbf{w}(u)\big)=\mathcal{I}_{c^+}^{\gamma;\psi}\mathbf{z}(u)+\mathcal{I}_{c^+}^{\gamma;\psi}\mathbf{w}(u)$.

Definition 2.10 *[30] The generalized Atangana-Baleanu fractional integral of order $\gamma\in(0,1)$ is given by*

$${}^{AB}\mathcal{I}_{c^+}^{\gamma;\psi}\mathbf{z}(u)=\frac{1-\gamma}{\mathbb{A}(\gamma)}\mathbf{z}(u)+\frac{\gamma}{\mathbb{A}(\gamma)}\mathcal{I}_{c^+}^{\gamma;\psi}\mathbf{z}(u),$$

where $\mathbf{z}:[c,d]\longrightarrow\mathbf{E^n}$ and $\mathbf{z}\in C^{\mathbf{E^n}}([c,d])\cap\mathcal{L}^{\mathbf{E^n}}([c,d])$ and $\mathbb{A}(\gamma)=(1-\gamma)+\dfrac{\gamma}{\Gamma(\gamma)}$ is known as the normalization function which satisfies $\mathbb{A}(0)=\mathbb{A}(1)=1$.

Definition 2.11 *[30] The generalized Atangana-Baleanu fractional fuzzy derivative in Caputo sense is defined by*

$$^{ABC}\mathcal{D}_{c^+}^{\gamma;\psi}\mathbf{z}(u)=\frac{\mathbb{A}(\gamma)}{1-\gamma}\int_c^u \mathbb{E}_\gamma\Big(\frac{-\gamma}{1-\gamma}\big(\psi(u)-\psi(v)\big)^\gamma\Big)\mathbf{z}'(v)dv,$$

where $\mathbf{z}\in C^{\mathbf{E^n}}([c,d])\cap\mathcal{L}^{\mathbf{E^n}}([c,d])$ *and* $\mathbb{E}_\gamma$ *is the Mittag-Leffler function.*

Corollary 2.1 *[30] Let* $\gamma\in(0,1)$, $\psi\in\mathbb{R}$ *and* $\mathbf{z}:[c,d]\longrightarrow\mathbf{E^n}$ *be d-monotone fuzzy function. If* $\mathbf{z}'(u)\in\mathcal{L}^{\mathbf{E^n}}([c,d])$, *we get*

$$^{AB}\mathcal{I}_{c^+}^{\gamma;\psi}\big(^{ABC}\mathcal{D}_{c^+}^{\gamma;\psi}\big)\mathbf{z}(u)=\mathbf{z}(u)\ominus_{gH}\mathbf{z}(c).$$

Lemma 2.1 *[31] Let* $\mathbf{x},\mathbf{y}$ *be two positive integrable functions and* $\mathbf{z}$ *continuous on* $[c,d]$. *Let* $\psi\in C^1([c,d],\mathbb{R}^+)$ *an increasing function such that* $\psi'(u)\neq 0$ *for all* $u\in[c,d]$. *Assume that*

$$\mathbf{x}(u)\leq\mathbf{y}(u)+\mathbf{z}(u)\int_c^u\psi'(v)\big(\psi(u)-\psi(v)\big)^{\gamma-1}\mathbf{x}(v)dv,\quad u\in[c,d],$$

then

$$\mathbf{x}(u)\leq\mathbf{y}(u)+\int_c^u\sum_{i=1}^{\infty}\frac{(\mathbf{z}(v)\Gamma(q))^i}{\Gamma(i\gamma)}\psi'(v)\big(\psi(u)-\psi(v)\big)^{\gamma-1}\mathbf{y}(v)dv,\quad u\in[c,d].$$

Corollary 2.2 *[31] Under the hypotheses of Lemma 2.1, let* $\mathbf{y}$ *be a nondecreasing function on* $[c,d]$. *Then, we have*

$$\mathbf{x}(u)\leq\mathbf{y}(u)\mathbb{E}_\gamma\Big(\Gamma(\gamma)\mathbf{z}(u)\big(\psi(u)-\psi(c)\big)^\gamma\Big),\tag{2.1}$$

where $\mathbb{E}_\gamma(u)=\sum_{i=0}^{\infty}\frac{u^i}{\Gamma(i\gamma+1)}$.

Lemma 2.2 *[30] A d-monotone fuzzy fonction* $\mathbf{z}\in C^{\mathbf{E^n}}([c,d])$ *is a solution of the following IVP:*

$$\begin{cases}^{ABC}\mathcal{D}_{0^+}^{\gamma;\psi}\mathbf{z}(u)=\mathfrak{g}(u), & u\in[c,d],\\ \mathbf{z}(0)=\mathbf{z}_0,\end{cases}\tag{2.2}$$

if and only if $\mathbf{z}\in C^{\mathbf{E^n}}([c,d])$ *satisfies the integral equation provided as follows*

$$\begin{aligned}\mathbf{z}(u)\ominus_{gH}\mathbf{z}_0&=\frac{1-\gamma}{\mathbb{A}(\gamma)}\mathfrak{g}(u)+\frac{\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma)}\int_c^u\psi'(v)\big(\psi(u)-\psi(v)\big)^{\gamma-1}\mathfrak{g}(v)dv,\\&=\frac{1-\gamma}{\mathbb{A}(\gamma)}\mathfrak{g}(u)+\frac{\gamma}{\mathbb{A}(\gamma)}\mathcal{I}_{c^+}^{\gamma;\psi}\mathfrak{g}(u),\end{aligned}\tag{2.3}$$

and $u\mapsto^{AB}\mathcal{I}_{c^+}^{\gamma;\psi}\mathfrak{g}(u)$ *is d-increasing on* $(c,d]$.

Remark 2.5 • If $\mathbf{z}\in C^{\mathbf{E^n}}([c,d])$ such that $len([\mathbf{z}(u)]^p)\geq len([\mathbf{z}_0]^p)$, then (2.3) becomes

$$\mathbf{z}(u)=\mathbf{z}_0+\frac{1-\gamma}{\mathbb{A}(\gamma)}\mathfrak{g}(u)+\frac{\gamma}{\mathbb{A}(\gamma)}\mathcal{I}_{c^+}^{\gamma;\psi}\mathfrak{g}(u).\tag{2.4}$$

• If $\mathbf{z}\in C^{\mathbf{E^n}}([c,d])$ such that $len([\mathbf{z}(u)]^p)\leq len([\mathbf{z}_0]^p)$, then (2.3) becomes

$$\mathbf{z}(u)=\mathbf{z}_0\ominus(-1)\Big(\frac{1-\gamma}{\mathbb{A}(\gamma)}\mathfrak{g}(u)+\frac{\gamma}{\mathbb{A}(\gamma)}\mathcal{I}_{c^+}^{\gamma;\psi}\mathfrak{g}(u)\Big).\tag{2.5}$$

Remark 2.6 • Let $\psi(u) = u$, then the equation (2.3) becomes the following AB-Riemann–Liouville fuzzy fractional integral equations

$$\mathbf{z}(u) \ominus_{gH} \mathbf{z}_0 = \frac{1-\gamma}{\mathbb{A}(\gamma)}\mathfrak{g}(u) + \frac{\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma)} \int_c^u \big(u-v\big)^{\gamma-1}\mathfrak{g}(v)dv.$$

• Let $\psi(u) = u^\rho$, then the equation (2.3) becomes the following AB-Katugampola fuzzy fractional integral equations

$$\mathbf{z}(u) \ominus_{gH} \mathbf{z}_0 = \frac{1-\gamma}{\mathbb{A}(\gamma)}\mathfrak{g}(u) + \frac{\gamma\rho^{1-\gamma}}{\mathbb{A}(\gamma)\Gamma(\gamma)} \int_c^u \big(u^\rho - v^\rho\big)^{\gamma-1}\mathfrak{g}(v)\frac{dv}{v}.$$

• Let $\psi(u) = ln(u)$, then the equation (2.3) becomes the following AB-Hadamard fuzzy fractional integral equations

$$\mathbf{z}(u) \ominus_{gH} \mathbf{z}_0 = \frac{1-\gamma}{\mathbb{A}(\gamma)}\mathfrak{g}(u) + \frac{\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma)} \int_c^u \big(\ln(u) - \ln(v)\big)^{\gamma-1}\mathfrak{g}(v)\frac{dv}{v}.$$

Lemma 2.3 *A d-monotone fuzzy fonction $\mathbf{y}(\cdot) \in C^{\mathbf{E^n}}([-\xi, D])$ is a solution of (1.1) if and only if $\mathbf{y}(\cdot)$ satisfies one of the following fuzzy fractional integral equations:*

(**C1**) *If $\mathbf{y}(\cdot) \in C^{\mathbf{E^n}}([-\xi, D])$ is d-increasing, then*

$$\mathbf{y}(u) = \begin{cases} \varphi(t), \qquad u \in [-\xi, 0], \\ \varphi(0) + \dfrac{\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma)} \displaystyle\int_0^u \psi^{'}(v)\big(\psi(u)-\psi(v)\big)^{\gamma-1}\mathfrak{f}\big(v, \mathbf{y}(v), \mathbf{y}(v-\xi)\big)dv \\ \qquad + \dfrac{1-\gamma}{\mathbb{A}(\gamma)}\mathfrak{f}\big(u, \mathbf{y}(u), \mathbf{y}(u-\xi)\big), \quad u \in (0, D], \end{cases} \tag{2.6}$$

(**C2**) *If $\mathbf{y}(\cdot) \in C^{\mathbf{E^n}}([-\xi, D])$ is d-decreasing, then*

$$\mathbf{y}(u) = \begin{cases} \varphi(t), \qquad u \in [-\xi, 0], \\ \varphi(0) \ominus (-1)\Big(\dfrac{\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma)} \displaystyle\int_0^u \psi^{'}(v)\big(\psi(u)-\psi(v)\big)^{\gamma-1}\mathfrak{f}\big(v, \mathbf{y}(v), \mathbf{y}(v-\xi)\big)dv \\ \qquad + \dfrac{1-\gamma}{\mathbb{A}(\gamma)}\mathfrak{f}\big(u, \mathbf{y}(u), \mathbf{y}(u-\xi)\big)\Big), \quad u \in (0, D], \end{cases} \tag{2.7}$$

Proof: The proof is similar to the proof of Lemma 3.1 in [30]. □

3. Existence and uniqueness results

The purpose of this section is to investigate the characteristics of system (1.1) under the conditions that the fuzzy solution $\mathbf{y}(\cdot) \in C^{\mathbf{E^n}}([-\xi, D])$ is d-increasing and the fuzzy function $u \longmapsto {}^{AB}\mathcal{I}_{0+}^{\gamma;\psi}\mathfrak{f}\big(u, \mathbf{y}(u), \mathbf{y}(u-\xi)\big)$ is d-increasing. According to Lemma 2.3, we define a mapping $\mathbf{L} : C^{\mathbf{E^n}}([-\xi, D]) \longrightarrow C^{\mathbf{E^n}}([-\xi, D])$ as follow

$$(\mathbf{Ly})(u) := \begin{cases} \varphi(t), \qquad u \in [-\xi, 0], \\ \varphi(0) + \dfrac{\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma)} \displaystyle\int_0^u \psi^{'}(v)\big(\psi(u)-\psi(v)\big)^{\gamma-1}\mathfrak{f}\big(v, \mathbf{y}(v), \mathbf{y}(v-\xi)\big)dv \\ \qquad + \dfrac{1-\gamma}{\mathbb{A}(\gamma)}\mathfrak{f}\big(u, \mathbf{y}(u), \mathbf{y}(u-\xi)\big), \quad u \in (0, D], \end{cases} \tag{3.1}$$

Theorem 3.1 *Suppose that the following hypothesis:*
($\mathbf{H}1$) *For all* $\phi_1, \phi_2,\ \nu_1, \nu_2 \in \mathbf{E^n}$ *and* $u \in [-\xi, D]$, *there exist* $\beta > 0$ *such that*

$$\mathbf{D}_\infty\big[\mathfrak{f}(u,\phi_1,\nu_1),\mathfrak{f}(u,\phi_2,\nu_2)\big] \leq \beta\Big(\mathbf{D}_\infty[\phi_1,\phi_2]+\mathbf{D}_\infty[\nu_1,\nu_2]\Big),$$

hold. Then, system (1.1) has at least one solution on $[-\xi, D]$.

Proof: We will utilize the Schaefer fixed point theorem to demonstrate that $\mathbf{L}$ defined by (3.1) has a fixed point. We divide the subsequent proof into three steps.
Step 1: Let prove that $\mathbf{L}$ is continuous. For any integer $m \geq 1$, define $\mathbf{z}_m(u) = \varphi(u)$ for all $u \in [-\xi, 0]$. For all $u \in [0, D]$, using the properties of metric $\mathbf{D}_\infty$ and hypothesis ($\mathbf{H1}$), one has

$$\begin{aligned}
&\mathbf{D}_\infty\big[\mathbf{L}(\mathbf{y}_m(u)),\mathbf{L}(\mathbf{y}(u))\big]\\
&\leq \frac{\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma)}\int_0^u \psi'(v)\big(\psi(u)-\psi(v)\big)^{\gamma-1}\mathbf{D}_\infty\Big[\mathfrak{f}\big(v,\mathbf{y}_m(v),\mathbf{y}_m(v-\xi)\big),\mathfrak{f}\big(v,\mathbf{y}(v),\mathbf{y}(v-\xi)\big)\Big]dv\\
&+\frac{1-\gamma}{\mathbb{A}(\gamma)}\mathbf{D}_\infty\Big[\mathfrak{f}\big(u,\mathbf{y}_m(u),\mathbf{y}_m(u-\xi)\big),\mathfrak{f}\big(u,\mathbf{y}(u),\mathbf{y}(u-\xi)\big)\Big]\\
&\leq \beta\left(\frac{\gamma\big(\psi(D)-\psi(0)\big)^\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma+1)}+\frac{1-\gamma}{\mathbb{A}(\gamma)}\right)\sup_{0\leq u\leq D}\Bigg(\mathbf{D}_\infty\big[\mathbf{y}_m(u),\mathbf{y}(u)\big]+\mathbf{D}_\infty\big[\mathbf{y}_m(u-\xi),\mathbf{y}(u-\xi)\big]\Bigg),
\end{aligned}$$

or, by using the definition of $\mathbf{D}_\infty$, we get

$$\begin{aligned}
&\mathbf{D}_\infty\big[\mathbf{y}_m(v-\xi),\mathbf{y}(v-\xi)\big]\\
&\quad=\sup_{0\leq p\leq 1}\max_{0\leq v\leq u}\big\{\,|\,\underline{\mathbf{y}_m}(v-\xi,p)-\underline{\mathbf{y}}(v-\xi,p)\,|,|\,\overline{\mathbf{y}_m}(v-\xi,p)-\overline{\mathbf{y}}(v-\xi,p)\,|\,\big\}\\
&\quad=\sup_{0\leq p\leq 1}\max_{-\xi\leq \mu\leq u-\xi}\big\{\,|\,\underline{\mathbf{y}_m}(\mu,p)-\underline{\mathbf{y}}(\mu,p)\,|,|\,\overline{\mathbf{y}_m}(\mu,p)-\overline{\mathbf{y}}(\mu,p)\,|\,\big\}\\
&\quad\leq\sup_{0\leq p\leq 1}\max_{-\xi\leq \mu\leq 0}\big\{\,|\,\underline{\mathbf{y}_m}(\mu,p)-\underline{\mathbf{y}}(\mu,p)\,|,|\,\overline{\mathbf{y}_m}(\mu,p)-\overline{\mathbf{y}}(\mu,p)\,|\,\big\}\\
&\qquad\qquad+\sup_{0\leq p\leq 1}\max_{0\leq \mu\leq u-\xi}\big\{\,|\,\underline{\mathbf{y}_m}(\mu,p)-\underline{\mathbf{y}}(\mu,p)\,|,|\,\overline{\mathbf{y}_m}(\mu,p)-\overline{\mathbf{y}}(\mu,p)\,|\,\big\}\\
&\quad\leq\sup_{0\leq p\leq 1}\max_{0\leq v\leq u}\big\{\,|\,\underline{\mathbf{y}_m}(v,p)-\underline{\mathbf{y}}(s,r)\,|,|\,\overline{\mathbf{y}_m}(v,p)-\overline{\mathbf{y}}(v,p)\,|\,\big\}=\mathbf{D}_\infty\big[\mathbf{y}_m(v),\mathbf{y}(v)\big].
\end{aligned}$$

Therefore, we obtain

$$\mathbf{D}_\infty\big[\mathbf{L}(\mathbf{y}_m(u)),\mathbf{L}(\mathbf{y}(u))\big]\leq 2\beta\left(\frac{\gamma\big(\psi(D)-\psi(0)\big)^\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma+1)}+\frac{1-\gamma}{\mathbb{A}(\gamma)}\right)\sup_{0\leq u\leq D}\mathbf{D}_\infty\big[\mathbf{y}_m(u),\mathbf{y}(u)\big], \tag{3.2}$$

this shows that $\mathbf{D}_\infty\big[\mathbf{L}(\mathbf{y}_m(u)),\mathbf{L}(\mathbf{y}(u))\big]\longrightarrow 0$ as $m\longrightarrow\infty$. As a result of this, $\mathbf{L}$ is continuous.
Step 2:
ⓐ- Let us show that $\mathbf{B}_{\varsigma_1} := \Big\{\mathbf{y}(u)\in C^{\mathbf{E^n}}([-\xi,D])|\mathbf{D}_\infty[\mathbf{y}(u),\hat{0}]\leq\varsigma_1\Big\}$ is bounded. Then, let prove that there exists a positive constant ξ_1 and for all $\varsigma_1>0$ satisfying for all $\mathbf{z}(u)\in\mathbf{B}_{\varsigma_1}$ one has $\mathbf{D}_\infty\big[\mathbf{L}\big(\mathbf{y}(u)\big),\hat{0}\big]\leq\xi_1$. In fact, for all $u\in[0,D]$ and $\mathbf{y}(u)\in\mathbf{B}_{\varsigma_1}$, we have

$$\begin{aligned}
&\mathbf{D}_\infty\big[\mathbf{L}(\mathbf{y}(u)),\hat{0}\big]\\
&\leq\mathbf{D}_\infty\big[\varphi(0),\hat{0}\big]+\frac{1-\gamma}{\mathbb{A}(\gamma)}\mathbf{D}_\infty\Big[\mathfrak{f}\big(u,\mathbf{y}(u),\mathbf{y}(u-\xi)\big),\hat{0}\Big]\\
&+\frac{\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma)}\int_0^u\psi'(v)\big(\psi(u)-\psi(v)\big)^{\gamma-1}\mathbf{D}_\infty\Big[\mathfrak{f}\big(v,\mathbf{y}(v),\mathbf{y}(v-\xi)\big),\hat{0}\Big]dv\\
&\leq\mathbf{D}_\infty\big[\varphi(0),\hat{0}\big]+\frac{1-\gamma}{\mathbb{A}(\gamma)}\mathbf{D}_\infty\Big[\mathfrak{f}\big(u,\mathbf{y}(u),\mathbf{y}(u-\xi)\big),\hat{0}\Big]+\frac{\gamma\big(\psi(D)-\psi(0)\big)^\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma+1)}\sup_{0\leq u\leq D}\mathbf{D}_\infty\Big[\mathfrak{f}\big(u,\mathbf{y}(u),\mathbf{y}(u-\xi)\big),\hat{0}\Big].
\end{aligned}$$

Since the function $\mathfrak{f}$ is continuous, there is exist a constant $M_{\mathfrak{f}} > 0$ such that $\mathbf{D}_\infty\big[\mathfrak{f}(u,\phi,\mu),\hat{0}\big] \leq M_{\mathfrak{f}}$. Therefore

$$\mathbf{D}_\infty\big[\mathbf{L}\big(\mathbf{y}(u)\big),\hat{0}\big] \leq \mathbf{D}_\infty\big[\varphi(0),\hat{0}\big] + M_{\mathfrak{f}}\left(\frac{\gamma\big(\psi(D)-\psi(0)\big)^\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma+1)} + \frac{1-\gamma}{\mathbb{A}(\gamma)}\right) := \xi_1. \tag{3.3}$$

It yields, for every $\mathbf{y}(u) \in \mathbf{B}_{\varsigma_1}$, that $\mathbf{D}_\infty\big[\mathbf{L}\big(\mathbf{y}(u)\big),\hat{0}\big] \leq \xi_1$, this implies that $\mathbf{L}(\mathbf{B}_{\varsigma_1}) \subseteq \mathbf{B}_{\xi_1}$.
ⓑ- Let prove that $\mathbf{L}$ maps bounded set into equi-continuous set. For each $\mathbf{y}(u) \in \mathbf{B}_{\varsigma_2}$ and $s_1, s_2 \in [0,D]$ such that $s_1 < s_2$, we have

$$\begin{aligned}
&\mathbf{D}_\infty\Big[\mathbf{L}\big(\mathbf{y}(s_1)\big),\mathbf{L}\big(\mathbf{y}(s_2)\big)\Big]\\
&\leq \frac{\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma)}\mathbf{D}_\infty\Bigg[\int_0^{s_1}\frac{\psi'(v)\mathfrak{f}\big(v,\mathbf{y}(v),\mathbf{y}(v-\xi)\big)}{\big(\psi(s_1)-\psi(v)\big)^{1-\gamma}}dv, \int_0^{s_1}\frac{\psi'(v)\mathfrak{f}\big(v,\mathbf{y}(v),\mathbf{y}(v-\xi)\big)}{\big(\psi(s_2)-\psi(v)\big)^{1-\gamma}}dv\\
&\qquad + \int_{s_1}^{s_2}\frac{\psi'(v)\mathfrak{f}\big(v,\mathbf{y}(v),\mathbf{y}(v-\xi)\big)}{\big(\psi(s_2)-\psi(v)\big)^{1-\gamma}}dv\Bigg]\\
&\leq \frac{\gamma\Big(\big(\psi(s_1)-\psi(0)\big)^\gamma + \big(\psi(s_2)-\psi(s_1)\big)^\gamma - \big(\psi(s_2)-\psi(0)\big)^\gamma\Big)}{\mathbb{A}(\gamma)\Gamma(\gamma+1)}\sup_{0\leq u\leq D}\mathbf{D}_\infty\Big[\mathfrak{f}\big(v,\mathbf{y}(v),\mathbf{y}(v-\xi)\big),\hat{0}\Big]\\
&\qquad + \frac{\gamma\big(\psi(s_2)-\psi(s_1)\big)^\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma+1)}\sup_{0\leq u\leq D}\mathbf{D}_\infty\Big[\mathfrak{f}\big(v,\mathbf{y}(v),\mathbf{y}(v-\xi)\big),\hat{0}\Big]\\
&\leq \frac{\gamma\Big(\big(\psi(s_1)-\psi(0)\big)^\gamma + 2\big(\psi(s_2)-\psi(s_1)\big)^\gamma - \big(\psi(s_2)-\psi(0)\big)^\gamma\Big)}{\mathbb{A}(\gamma)\Gamma(\gamma+1)}\sup_{0\leq u\leq D}\mathbf{D}_\infty\Big[\mathfrak{f}\big(v,\mathbf{y}(v),\mathbf{y}(v-\xi)\big),\hat{0}\Big]\\
&\leq \frac{\gamma\Big(\big(\psi(s_1)-\psi(0)\big)^\gamma + 2\big(\psi(s_2)-\psi(s_1)\big)^\gamma - \big(\psi(s_2)-\psi(0)\big)^\gamma\Big)}{\mathbb{A}(\gamma)\Gamma(\gamma+1)}M_{\mathfrak{f}} := \Psi.
\end{aligned}$$

We have Ψ is independent of $\mathbf{y}(u)$ and $\Psi \longrightarrow 0$ as $s_2 \longrightarrow s_1$. Then, we obtain

$$\mathbf{D}_\infty\big[\mathbf{L}\big(\mathbf{y}(s_1)\big),\mathbf{L}\big(\mathbf{y}(s_2)\big)\big] \longrightarrow 0.$$

It means that $\mathbf{L}(\mathbf{B}_{\varsigma_2})$ is equi-continuous. Then, according to Arzela-Ascoli theorem, $\mathbf{L}$ is relatively compact. As a consequence of steps 1 and 2, $\mathbf{L}$ is completely continuous.
Step 3: Let prove that $\mathbf{B}_\delta = \Big\{\mathbf{y}(u) \in C^{\mathbf{E^n}}([-\xi,D])|\mathbf{y} = \delta(\mathbf{L}\mathbf{y}), \delta \in (0,1)\Big\}$ is bounded set. Let $\mathbf{y} \in \mathbf{B}_\delta$, then $\mathbf{y} = \delta(\mathbf{L}\mathbf{y})$ for $\delta \in (0,1)$. So for each $u \in [0,D]$, one has

$$\begin{aligned}
\mathbf{y}(u) = \delta\Bigg(&\varphi(0) + \frac{\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma)}\int_0^u \psi'(v)\big(\psi(u)-\psi(v)\big)^{\gamma-1}\mathfrak{f}\big(v,\mathbf{y}(v),\mathbf{y}(v-\xi)\big)dv\\
&+ \frac{1-\gamma}{\mathbb{A}(\gamma)}\mathfrak{f}\big(u,\mathbf{y}(u),\mathbf{y}(u-\xi)\big)\Bigg).
\end{aligned}$$

It follows from (3.3) that

$$\mathbf{D}_\infty\big[\mathbf{y}(u),\hat{0}\big] \leq \delta\mathbf{D}_\infty\big[\varphi(0),\hat{0}\big] + \delta M_{\mathfrak{f}}\left(\frac{\gamma\big(\psi(D)-\psi(0)\big)^\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma+1)} + \frac{1-\gamma}{\mathbb{A}(\gamma)}\right). \tag{3.4}$$

According to (3.4), we conclude that $\mathbf{B}_\delta$ is bounded set. As a consequence of Schaefer's fixed point theorem, $\mathbf{L}$ has a fixed point which is a solution of system (1.1). □

For the uniqueness result, we have the following theorem:

Theorem 3.2 *Assume that the hypothesis* (**H1**) *hold. If*

$$\Upsilon := 2\beta\left(\frac{\gamma\big(\psi(D)-\psi(0)\big)^\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma+1)} + \frac{1-\gamma}{\mathbb{A}(\gamma)}\right) < 1,$$

then the solution of system (1.1) *is unique.*

Proof: Let $\mathbf{y}_1,\ \mathbf{y}_2 \in C^{\mathbf{E^n}}([-\xi, D])$ and for $u \in [-\xi, 0]$, $\mathbf{y}_1(u) = \mathbf{y}_2(u) = \varphi(u)$. For all $u \in [0, D]$, using the properties of metric $\mathbf{D}_\infty$ and hypothesis (**H1**), we get

$$\begin{aligned}
&\mathbf{D}_\infty\big[\mathbf{L}(\mathbf{y}_1(u)), \mathbf{L}(\mathbf{y}_2(u))\big] \\
&\leq \frac{\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma)} \int_0^u \psi'(v)\big(\psi(u)-\psi(v)\big)^{\gamma-1} \mathbf{D}_\infty\Big[\mathfrak{f}\big(v, \mathbf{y}_1(v), \mathbf{y}_1(v-\xi)\big), \mathfrak{f}\big(v, \mathbf{y}_2(v), \mathbf{y}_2(v-\xi)\big)\Big] dv \\
&+ \frac{1-\gamma}{\mathbb{A}(\gamma)} \mathbf{D}_\infty\Big[\mathfrak{f}\big(u, \mathbf{y}_1(u), \mathbf{y}_1(u-\xi)\big), \mathfrak{f}\big(u, \mathbf{y}_2(u), \mathbf{y}_2(u-\xi)\big)\Big] \\
&\leq \beta\left(\frac{\gamma\big(\psi(D)-\psi(0)\big)^\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma+1)} + \frac{1-\gamma}{\mathbb{A}(\gamma)}\right) \sup_{0\leq u\leq D}\Big(\mathbf{D}_\infty\big[\mathbf{y}_1(u), \mathbf{y}_2(u)\big] + \mathbf{D}_\infty\big[\mathbf{y}_1(u-\xi), \mathbf{y}_2(u-\xi)\big]\Big) \\
&\leq 2\beta\left(\frac{\gamma\big(\psi(D)-\psi(0)\big)^\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma+1)} + \frac{1-\gamma}{\mathbb{A}(\gamma)}\right) \sup_{0\leq u\leq D} \mathbf{D}_\infty\big[\mathbf{y}_1(u), \mathbf{y}_2(u)\big].
\end{aligned}$$

Hence

$$\sup_{0\leq u\leq D} \mathbf{D}_\infty\big[\mathbf{L}(\mathbf{y}_1(u)), \mathbf{L}(\mathbf{y}_2(u))\big] \leq 2\beta\left(\frac{\gamma\big(\psi(D)-\psi(0)\big)^\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma+1)} + \frac{1-\gamma}{\mathbb{A}(\gamma)}\right) \sup_{0\leq u\leq D} \mathbf{D}_\infty\big[\mathbf{y}_1(u), \mathbf{y}_2(u)\big].$$

Since $\Upsilon < 1$, it follows that $\mathbf{L}$ is a contractive mapping. Based on the Banach contraction principle, $\mathbf{L}$ has an unique fixed point $\mathbf{y}(u)$.

□

4. Stability analysis result

Definition 4.1 *[33] The solution of problem (1.1) is said to be Ulam-Hyers stable if there is a constant $\mathbf{C}_\mathfrak{f} > 0$ such that for each $\varepsilon > 0$ and each solution $\mathbf{z} \in C^{\mathbf{E^n}}([0, D])$ of the inequality*

$$\mathbf{D}_\infty\left[{}^{ABC}\mathcal{D}_{0^+}^{\gamma;\psi}\mathbf{z}(u), \mathfrak{f}\big(u, \mathbf{z}(u), \mathbf{z}(u-\xi)\big)\right] \leq \varepsilon, \qquad u \in [0, D], \tag{4.1}$$

there is a solution $\mathbf{y} \in C^{\mathbf{E^n}}([0, D])$ of (1.1) such that

$$\mathbf{D}_\infty\Big[\mathbf{z}(u), \mathbf{y}(u)\Big] \leq \mathbf{C}_\mathfrak{f}\varepsilon, \qquad u \in [0, D].$$

Definition 4.2 *[33] The solution of problem (1.1) is said to be generalized Ulam-Hyers stable if there exists $\Phi \in C^1\big([0, D], \mathbb{R}^+\big)$, $\Phi(0) = 0$ such that for each solution $\mathbf{z} \in C^{\mathbf{E^n}}([0, D])$ of (4.1), there exists a solution $\mathbf{y} \in C^{\mathbf{E^n}}([0, D])$ of (1.1) such that*

$$\mathbf{D}_\infty\Big[\mathbf{z}(u), \mathbf{y}(u)\Big] \leq \mathbf{C}_\mathfrak{f}\Phi(\varepsilon), \qquad u \in [0, D].$$

Remark 4.1 A fuzzy function $\mathbf{z} \in C^{\mathbf{E^n}}([0, D])$ is the solution of inequality (4.1) if and only if $\exists\Theta \in C^{\mathbf{E^n}}([0, D])$ such that

(*i*) $\mathbf{D}_\infty\big[\Theta(u), \hat{0}\big] \leq \varepsilon$, for all $u \in [0, D]$.

(*ii*) ${}^{ABC}\mathcal{D}_{0^+}^{\gamma;\psi}\mathbf{z}(u) = \mathfrak{f}\big(u, \mathbf{z}(u), \mathbf{z}(u-\xi)\big) + \Theta(u)$, for all $u \in [0, D]$.

Theorem 4.1 *Under the hypothesis (**H**1), the system (1.1) is Ulam-Hyers stable under inequality (4.1).*

Proof: Let $\mathbf{z}(u)$ be the solution of inequality (4.1) and $\mathbf{y}(u)$ be the solution of the proposed system (1.1). In light of Remark 4.1, Lemma 2.3 and Remark 2.4, one has

$$\begin{aligned}\mathbf{z}(u) &= \varphi(0) + \frac{1-\gamma}{\mathbb{A}(\gamma)}\Big(\mathfrak{f}\big(u, \mathbf{y}(u), \mathbf{y}(u-\xi)\big) + \Theta(u)\Big) \\ &+ \frac{\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma)}\int_0^u \psi^{'}(v)\big(\psi(u)-\psi(v)\big)^{\gamma-1}\Big(\mathfrak{f}\big(v, \mathbf{y}(v), \mathbf{y}(v-\xi)\big) + \Theta(v)\Big)dv.\end{aligned}$$

Note that for $u \in [-\xi, 0]$, we have $\mathbf{D}_\infty\big[\mathbf{z}(u), \mathbf{y}(u)\big] = 0$. For $u \in [0, D]$, using hypothesis ($\mathbf{H}1$) one has

$$\begin{aligned}&\mathbf{D}_\infty\big[\mathbf{z}(u), \mathbf{y}(u)\big] \\ &\leq \frac{1-\gamma}{\mathbb{A}(\gamma)}\Big(\mathbf{D}_\infty\Big[\mathfrak{f}\big(u, \mathbf{z}(u), \mathbf{z}(u-\xi)\big), \mathfrak{f}\big(u, \mathbf{y}(u), \mathbf{y}(u-\xi)\big)\Big] + \mathbf{D}_\infty\Big[\Theta(u), \hat{0}\Big]\Big) \\ &+ \frac{\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma)}\int_0^u \psi'(v)\big(\psi(u)-\psi(v)\big)^{\gamma-1}\mathbf{D}_\infty\Big[\mathfrak{f}\big(v, \mathbf{z}(v), \mathbf{z}(v-\xi)\big), \mathfrak{f}\big(v, \mathbf{y}(v), \mathbf{y}(v-\xi)\big)\Big]dv \\ &\qquad + \frac{\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma)}\int_0^u \psi'(v)\big(\psi(u)-\psi(s)\big)^{\gamma-1}\mathbf{D}_\infty\Big[\Theta(v), \hat{0}\Big]dv \\ &\leq \frac{2\beta(1-\gamma)}{\mathbb{A}(\gamma)}\mathbf{D}_\infty\big[\mathbf{z}(u), \mathbf{y}(u)\big] + \frac{\varepsilon(1-\gamma)}{\mathbb{A}(\gamma)} + \frac{\varepsilon\big(\psi(u)-\psi(0)\big)^\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma)} \\ &\qquad + \frac{2\beta\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma)}\int_0^u \psi'(v)\big(\psi(u)-\psi(v)\big)^{\gamma-1}\mathbf{D}_\infty\big[\mathbf{z}(v), \mathbf{y}(v)\big]dv \\ &\leq \left(\frac{1-\gamma}{\mathbb{A}(\gamma)-2\beta(1-\gamma)} + \frac{\big(\psi(u)-\psi(0)\big)^\gamma}{\Gamma(\gamma)\big(\mathbb{A}(\gamma)-2\beta(1-\gamma)\big)}\right)\varepsilon \\ &\qquad + \frac{2\beta\gamma}{\Gamma(\gamma)\big(\mathbb{A}(\gamma)-2\beta(1-\gamma)\big)}\int_0^u \psi'(v)\big(\psi(u)-\psi(v)\big)^{\gamma-1}\mathbf{D}_\infty\big[\mathbf{z}(v), \mathbf{y}(v)\big]dv.\end{aligned}$$

Let $\alpha(u) = \dfrac{1-\gamma}{\mathbb{A}(\gamma)-2\beta(1-\gamma)} + \dfrac{\big(\psi(u)-\psi(0)\big)^\gamma}{\Gamma(\gamma)\big(\mathbb{A}(\gamma)-2\beta(1-\gamma)\big)}$ and $\mu = \dfrac{2\beta\gamma}{\Gamma(\gamma)\big(\mathbb{A}(\gamma)-2\beta(1-\gamma)\big)}$, we obtain

$$\mathbf{D}_\infty\big[\mathbf{z}(u), \mathbf{y}(u)\big] \leq \alpha(u)\varepsilon + \mu\int_0^u \psi'(v)\big(\psi(u)-\psi(v)\big)^{\gamma-1}\mathbf{D}_\infty\big[\mathbf{z}(v), \mathbf{y}(v)\big]dv.$$

It can be seen from Lemma 2.1 that

$$\mathbf{D}_\infty\big[\mathbf{z}(u), \mathbf{y}(u)\big] \leq \alpha(u)\varepsilon + \int_0^u \sum_{m=1}^{\infty}(\mu\Gamma(\gamma))^m\frac{\psi'(v)\big(\psi(u)-\psi(v)\big)^{m\gamma-1}}{\Gamma(m\gamma)}\alpha(v)\varepsilon dv.$$

We have that $\alpha(u)\varepsilon$ is a real-valued nondecreasing function on $[0, D]$, then from Corollary 2.2, we get

$$\mathbf{D}_\infty\big[\mathbf{z}(u), \mathbf{y}(u)\big] \leq \alpha(u)\varepsilon\mathbb{E}_q\Big(\mu\Gamma(\gamma)\big(\psi(u)-\psi(0)\big)^\gamma\Big).$$

Letting $\mathbf{C}_\mathfrak{f} = \alpha(u)\mathbb{E}_q\Big(\mu\Gamma(\gamma)\big(\psi(u)-\psi(0)\big)^\gamma\Big)$, we get

$$\mathbf{D}_\infty\big[\mathbf{z}(u), \mathbf{y}(u)\big] \leq \mathbf{C}_\mathfrak{f}\varepsilon.$$

Therefore, from Definition 4.1, the system (1.1) is Ulam–Hyers stable. Also, if $\Phi(\varepsilon) = \varepsilon$, then our system is also generalized Hyers-Ulam stable. □

5. Example

In this section, we will provide the following example to illustrate the prior important theoretical conclusions.

$$\begin{cases} \mathcal{D}_{0^+}^{\gamma;\psi}\mathbf{z}(u) = \dfrac{u}{1+u^2} + \dfrac{\cos(\mathbf{z}(u))}{3+u} + \dfrac{\mathbf{z}(u-1)}{u^2+3}, & u \in [0,3]. \\ \mathbf{z}(u) = (-u-1, 1.6, u+1), & u \in [-1,0], \end{cases} \tag{5.1}$$

Here $\mathfrak{f}(u,\mathbf{z}(u),\mathbf{z}(u-1)) = \dfrac{u}{1+u^2} + \dfrac{\cos(\mathbf{z}(u))}{3+u} + \dfrac{\mathbf{z}(u-1)}{u^2+3}$. Letting $\gamma = 0.7$, $\psi(u) = e^{-u}$.

$$\begin{aligned} &\mathbf{D}_\infty\big[\mathfrak{f}(u,\mathbf{z}(u),\mathbf{z}(u-1)), \mathfrak{f}(u,\mathbf{w}(u),\mathbf{w}(u-1))\big] \\ &\leq \frac{1}{3+u}\mathbf{D}_\infty\big[\cos(\mathbf{z}(u)), \cos(\mathbf{w}(u))\big] + \frac{1}{u^2+3}\mathbf{D}_\infty\big[\mathbf{z}(u-1),\mathbf{w}(u-1)\big], \\ &\leq \frac{1}{3}\mathbf{D}_\infty\big[\mathbf{z}(u),\mathbf{w}(u)\big] + \frac{1}{3}\mathbf{D}_\infty\big[\mathbf{z}(u-1),\mathbf{w}(u-1)\big]. \end{aligned}$$

Hence $\beta = \frac{1}{3}$, according to Theorem 3.1, system (5.1) has at least one solution on $[-1,3]$. Moreover, by the Mathematical software, we can calculate the following:

$$2\beta\left(\frac{\gamma\big(\psi(D)-\psi(0)\big)^\gamma}{\mathbb{A}(\gamma)\Gamma(\gamma+1)} + \frac{1-\gamma}{\mathbb{A}(\gamma)}\right) = 2\cdot\frac{1}{3}\left(\frac{0.7\cdot\big(e^{-3}-e^0\big)^{0.7}}{\mathbb{A}(0.7)\Gamma(0.7+1)} + \frac{1-0.7}{\mathbb{A}(0.7)}\right) \simeq 0.28 < 1.$$

According to Theorem 3.2, the solution of system (5.1) is unique. On the other hand, the system (5.1) is also satisfy the conditions of Theorem 4.1, thus system (5.1) is Ulam-Hyers stable and then generalized Hyers-Ulam stable on $[0,3]$.

6. Conclusion

This research has effectively investigated a specific class of fuzzy fractional differential equations, incorporating the generalized Atangana-Baleanu derivative and time-delays. Through the application of functional analysis techniques, particularly the Schaefer fixed point theorem for existence and the Banach fixed point theorem for uniqueness, the study has provided meaningful insights into the existence, uniqueness, and Ulam-Hyers stability of solutions. The exploration of Ulam-Hyers stability across variations in parameters, such as initial conditions and other factors, adds depth to the contributions of this research. The derivation of results relies on generalized forms of Gronwall's inequality, and the inclusion of a demonstrative example further reinforces the significance of the obtained findings.

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