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Bifurcation Structures and Persistence Conditions in a Nonlinear Predator-Prey Model with Allee and Switching Effects

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ABSTRACT: This paper studies the dynamics of a nonlinear ecological model with one predator and two prey species governed by several biologically relevant factors such as prey-switching, refuges, and Allee effects. The first prey species receives a regular refuge, while the second prey species is affected by a strong Allee effect on its population growth at low density. Furthermore, the predator has switching dynamics of prey availability, which improves the survival strategy as well. The equilibrium points of the model are investigated to study the existence and biological relevance of these points. The local dynamics of the interior and boundary equilibria are examined through a combination of analytical methods and bifurcation theory. Using Sotomayor's theorem, we give necessary and sufficient conditions for transcritical and Hopf bifurcations. The analysis is based on the Dulac–Bendixson criterion and Lyapunov functions to ensure that all species have uniform persistence. Finally, persistence is discussed as a global property of the switching functional response to a single predator's consumption of two prey. This has happened with the presence of the refuge effect on the first prey and the Allee effect on the second prey only, regardless of mortality and saturation rates concerning the predator. Assumed beneficial virtual dealing with parameters for the study of the numerical results emphasizes the right of the analysis of the analytical results. Visualizing the two main phenomena, the first is local bifurcation, and the second is the firmness of all the creatures that exist in this model. The results show that the system's stability is affected by the prey switching and refuge provision, depending on the strength of the Allee effect and harvesting pressure.

Key Words: Switching, Allee effect, Refuge, local bifurcation, saddle-node, transcritical, Hopf-bifurcation, persistence.

Contents

1. Introduction

The interactions between species in ecological systems are a long-standing mathematical biology problem that is considered challenging. The population dynamics and species coexistence are investigated through the predator–prey models, which are the fundamental basis for such purpose. These interactions are classically handled using the Lotka-Volterra system, which captures basic predator-prey interaction. In reality, interactions through the predator and the selection of one of the two prey are more complex and require supplementary mechanisms, like prey switching, refuge strategy, and the Allee effect, which have a deep impact on the system's conduct. Prey switching is a combined stabilizing factor in multi-prey systems. It is the dynamic adjustment of a predator's feeding preference in response to the relative abundance of its prey. It may be authorized to preclude overprofiteering of prey, which leads to a higher bio

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veriety [1]. Numerous authors, for suggestion, [2,3,4,5,6], have studied the phenomenon of a switching influence on the dynamics of predator-prey systems, comprehensive biostability, and oscillations.

The benefits of using refuge strategy in ecological systems are that it enables some members in the prey population to stay away from predators via hiding in locations beyond reach or by adopting a safeguarding attitude against predation. The biological models, including a factor of refuge in the prey populations, play a crucial role in leading to increased prey endurance and conceivable stabilization of predator–prey dynamics [7,8]. The dynamics of constant or density-dependent refuges have been investigated in a few other studies; for example, in [9], Ruan and Wei have proved the existence of refuge effects that cause alternative stable states and can change the stability criteria. The Allee effect adds an additional layer of complexity, a phenomenon where population growth is suppressed or becomes negative at low densities. It adds threshold dynamics to population models, which can cause species extinction even in the presence of favorable conditions. Recently, by including the Allee effect in prey and/or predator populations, many studies [10,11,12] have determined the threshold parameter values that distinguish persistence from extinction. While discussion of the climate change in [13,14]. Furthermore studies focus on the diseases spread as [15,16,17].

Bifurcation theory scientifically explains the changes that occur over time and how the structure of a moving system transforms. The study of changes includes: quality, vector fields, and solutions to differential equations for hypothetical systems, which simulate the competition and interaction of organisms in nature. Bifurcation can generally be attributed to a smooth change in the values of the parameters of the model under study (bifurcation coefficients), leading to a sudden, differential transformation in its "topological" behavior. It is worth noting that this term is widely used when many researchers study various systems that change over time [18,19,20]. Bifurcations are classified into two major categories: local bifurcations occur as the specific parameter crosses the critical threshold when varying values. For example, the saddle-node, transcritical, pitchfork, period-doubling flip, Hopf [21,22], and Neimark secondary Hopf bifurcations are analyzed and studied to determine how their local stability properties change. The second category is global bifurcation, which occurs when bigger invariant sets and stability, like periodic orbits, collide. Conducting bifurcation analysis for dynamic models is important because it allows one to clearly see what the stability limits of the parameters of the proposed nonlinear system will lead to.

In dynamical systems, ordinary differential equations have often been used in many systems, whether linear or nonlinear, independent or dependent. Many researchers have studied bifurcation in [23] for fractional-order predator-prey systems. One more type is a predator-prey model in the occupancy of a stage-structured model with overload hunting and poisoning, which is studied. Kadhim and Majeed have added a study of [24] they aim to study concepts, the bifurcation as well as the persistence of the proposed biological model. Al-Jubouri and Naji [25] further studied predator–prey with delay plus additional behaviour responses (i.e., fear plus cooperative hunting) and the Allee effect, revealing essential insights on the bifurcation structure and long-term behaviour.

Some recent studies have used generalized predators to develop the classical model, considering more realistic biological factors such as the fear effect, toxicity, optimal harvesting, and chaos control [24,25,26]. A globally stable and persistent food web model is studied to provide general conditions under which the coexistence of species is assured. Those studies of multiple biological and behavioral factors in ecological models are approaches that we also follow. One of the clear signs in the scientific applications is the topic of persistence in dynamic systems, which plays a prominent and critical role in enabling biologists to reach throughout the length of comprehension of the assertiveness of all these creatures in various ecosystems. It has scientific implications to understand the long-term survival of all competing organisms within the proposed model that mimics what exists in nature. From a mathematical perspective, it accurately predicts and diagnoses that positive [26,27,28,29,30,31,32,33] These studies aim to attitude the border of the positive cone. Studies on this theme have a specific, diverse brew of systems. Despite the well-researched literature on switching, refuge, and Allee effect dynamics, models combining all three mechanisms together are still relatively sparse. So far, none exist in a two-prey one-predator scenario. Furthermore, studying the bifurcation structure in these models is crucial for analyzing and predicting transitions between ecological states, but it is seldom tackled properly with methods such as Sotomayor's theorem.

In this work, we fill this gap by considering a new three-species model for which one prey gains from a constant refuge, the second prey undergoes An intensive Allee effect, along with the predator, manifests a prey-change type attitude. We express the model as a set of nonlinear differential equations, and numerically study its time evolution. The primary contributions of our work are:

- Discussing the occurrence of local bifurcation in the mathematical model. It means carrying out transcritical and Hopf bifurcation through Sotomayor's theorem,
- Verification of uniform persistence of systems using Lyapunov functions and Dulac–Bendixson criterion.
- Using numerical simulations and sensitivity analysis to show the ecological effects of changing the Allee threshold, switching rate, and refuge size.

The rest of this paper can be outlined as follows: Section 2 presents the biological model with the parameter description. Section 3 discusses the biological model's local bifurcation analysis with the occurrence of the Hopf bifurcation. The persistence as a global analysis is discussed in Section 4. Finally, Section 5 draws the conclusions of this paper.

2. Brief Biological Model with Comprehensive and Useful Information

Below is a description of the proposed biological model under study for bifurcation, shown in the following equations:

$$\begin{aligned}\frac{dx_1}{dt} &= x_1 r_1\left(1-\frac{x_1}{k_1}\right)-\frac{\alpha_1(1-m)x_1^2 y}{1+(1-m)x_1+x_2+s}\\ \frac{dx_2}{dt} &= x_2 r_2\left(1-\frac{x_2}{k_2}\right)(x_2-n)-\frac{\alpha_2 x_2^2 y}{1+(1-m)x_1+x_2+s}\\ \frac{dy}{dt} &= \frac{e_1\alpha_1(1-m)x_1^2 y}{1+(1-m)x_1+x_2+s}+\frac{e_2\alpha_2 x_2^2 y}{1+(1-m)x_1+x_2+s}-wy\end{aligned} \tag{2.1}$$

The meaning of each parameter of that model is illustrated in Table 1, and as follows:

Table 1: Full biological description of the parameters in system (1)

Parameters	Description
r_1 and r_2	The essential evolution average of both types of prey species, respectively
k_1 and k_2	The carrying ability of both kinds of prey, respectively
α_1 and α_2	The average exhaustion of both types of prey by the predator, respectively
e_1 and e_2	The conveyance average of food from the prey to the predators, respectively
m	Number of first-type prey that practice burrowing to protect themselves from predation
w	Normal dying of the predator
n	Allee sill of the second prey
s	The average of fullness, throughout a predator that used to eat the prey

Worth noting, system (1) has three active equilibrium points that play a vital role $P_3(k_1, k_2, 0)$, $P_5(\overline{\overline{x}}_1, 0, \overline{\overline{y}})$, and $P_6(\check{x}_1, \check{x}_2, \check{y})$. All of these points possess both local and global stability properties. Table 2 presents the hypothetical data used to numerically simulate the above model, which verifies the effectiveness of the three active points.

Table 2: Virtually accepted biological values of the parameters in system (2.1)

r_1	r_2	k_1	k_2	α_1	α_2	e_1	e_2	m	w	s	n
25	15	30	50	0.4	0.2	0.1	0.06	0.4	0.51	0.3	10

3. Detection of the Local bifurcation of the proposed biological system

In system (2.1), the discussion of exhibition for the bifurcation needs to guarantee only satisfying the necessary condition without satisfying the sufficient condition for Sotomayor's theorem [31]. Here, already, for the local bifurcation, the nonhyperbolic property of the equilibrium point is used. Confirm for each non-zero vector

Verify that for any non-zero vector $W = (w_1, w_2, w_3)^T$ we have:

The Jacobian matrix is :

$$J = [b_{ij}]_{3\times 3} \tag{3.1}$$

Where:

$$b_{11} = r_1\left(1 - \frac{2x_1}{k_1}\right) - \left\{\frac{\alpha_1(1-m)x_1 y\left[2\left(1+(1-m)x_1+x_2+s\right)-(1-m)x_1\right]}{\left[1+(1-m)x_1+x_2+s\right]^2}\right\},$$

$$b_{12} = \frac{\alpha_1(1-m)x_1^2 y}{\left[1+(1-m)x_1+x_2+s\right]^2},$$

$$b_{13} = \frac{-\alpha_1(1-m)x_1^2}{\left[1+(1-m)x_1+x_2+s\right]},$$

$$b_{21} = \frac{\alpha_2(1-m)x_2^2 y}{\left[1+(1-m)x_1+x_2+s\right]^2},$$

$$b_{22} = r_2\left[2x_2 - n - \frac{3x_2^2}{k_2} + \frac{2nx_2}{k_2}\right] - \frac{\alpha_2 x_2 y\left[2\left(1+(1-m)x_1+x_2+s\right)-x_2\right]}{\left[1+(1-m)x_1+x_2+s\right]^2},$$

$$b_{23} = \frac{-\alpha_2 x_2^2}{\left[1+(1-m)x_1+x_2+s\right]},$$

$$b_{31} = \frac{e_1\alpha_1(1-m)x_1 y\left[2\left(1+(1-m)x_1+x_2+s\right)-(1-m)x_1\right] - e_2\alpha_2(1-m)x_2^2 y}{\left[1+(1-m)x_1+x_2+s\right]^2},$$

$$b_{32} = \frac{-e_1\alpha_1(1-m)x_1^2 y + e_2\alpha_2 x_2 y\left[2\left(1+(1-m)x_1+x_2+s\right)-x_2\right]}{\left[1+(1-m)x_1+x_2+s\right]^2},$$

$$b_{33} = \frac{e_1\alpha_1(1-m)x_1^2 + e_2\alpha_2 x_2^2}{1+(1-m)x_1+x_2+s} - w.$$

$D^2 f_\rho(Z,\rho)(W,W) = [c_{ij}]_{3\times 1}$, where ρ represent a parameter,

$$D = 1 + (1-m)x_1 + x_2 + s \tag{3.2}$$

$$\begin{aligned}
c_{11} = {} & \left[\frac{-2r_1}{k_1} + \frac{2\alpha_1(1-m)y(1+x_1+s)^2}{D^3}\right] w_1^2 - \left[\frac{2\alpha_1(1-m)x_1^2 y}{D^3}\right] w_2^2 \\
& + 2\left[\frac{2\alpha_1(1-m)x_1 y(1+x_2+s)}{D^3}\right] w_1 w_2 \\
& + 2\left\{\frac{\alpha_1(1-m)x_1\left[2(1+x_2+s)+(1-m)x_1\right]}{D^3}\right\} w_1 w_3 \\
& + 2\left[\frac{\alpha_1(1-m)x_1^2}{D^2}\right] w_2 w_3, \\
c_{21} = {} & \left[\frac{-2\alpha_2(1-m)^2 y x_2^2}{D^3}\right] w_1^2 \\
& + \left\{2r_2\left(1-\frac{3x_2}{k_2}+\frac{n}{k_2}\right) - \frac{2\alpha_2 y(1+(1-m)x_1+s)^2}{D^3}\right\} w_2^2 \\
& + 2\left\{\frac{2\alpha_2(1-m)x_2 y(1+(1-m)x_1+s)}{D^3}\right\} w_1 w_2 \\
& + 2\left\{\frac{\alpha_2(1-m)x_2^2}{D^3}\right\} w_1 w_3 \\
& + 2\left\{\frac{-\alpha_2 x_2\left[2(1+(1-m)x_1+s)+x_2\right]}{D^3}\right\} w_2 w_3, \\
c_{31} = {} & \left\{\frac{2e_1\alpha_1(1-m)y[1+x_1+s]^2}{D^3}\right\} w_1^2 \\
& + \left\{\frac{2y\left[e_1\alpha_1(1-m)x_1^2 + e_2\alpha_2(1+(1-m)x_1+s)^2\right]}{D^3}\right\} w_2^2 \\
& + 2\left\{\frac{2(1-m)y\left[e_1\alpha_1 x_1(1+x_2+s)\right] + e_1\alpha_1 x_2[1+(1-m)x_1+s]}{D^3}\right\} w_1 w_2 \\
& + 2\left\{\frac{(1-m)\left[e_1\alpha_1\left[2x_1(1+x_2+s)+(1-m)x_1^2\right] - e_2\alpha_2 x_2^2\right]}{D^2}\right\} w_1 w_3 \\
& - \left\{\frac{e_1\alpha_1(1-m)x_1^2 + e_2\alpha_2 x_2\left[2[(1-m)x_1+s]+x_2\right]}{D^2}\right\} w_2 w_3,
\end{aligned}$$

Now, it is appropriate to discuss the theories that achieve the types of bifurcation under the proposed biological model (1) as follows:

Theorem 3.1 *According to condition (3.3) at the parameter* $\left(\alpha_1^* = \alpha_1 = \frac{w(1+(1-m)k_1+k_2+s)-e_2\alpha_2 k_2^2}{e_1(1-m)k_1^2}\right)$ *a transcritical bifurcation occurs near the equilibrium point* P_3 *However, there is no appearance of the two kinds of bifurcations, a saddle bifurcation and a pitchfork bifurcation, at* P_3.

$$\begin{aligned}
& 2\left[\frac{(1-m)\left[e_1\left[(2k_1(1+k_2+s))+(1-m)k_1^2\right]\right]}{e_1 r_1\left[1+(1-m)k_1+k_2+s\right]^3}\right]\left[w\left[1+(1-m)k_1+k_2+s\right]-e_2\alpha_2 k_2^2\right] \neq \\
& \left[\frac{e_1(1-m)k_1^2}{r_2(k_2-n)\left[1+(1-m)k_1+k_2+s\right]^3}\right]\left(\alpha_2 k_2^2\right).
\end{aligned} \tag{3.3}$$

Proof:

By substituting $P_3 = (k_1, k_2, 0)$ with $\alpha_1^* = \alpha_1 = \frac{w[1+(1-m)k_1+k_2+s]-e_2\alpha_2 k_2^2}{e_1(1-m)k_1^2}$ in equation (3.1), we obtain the following eigenvalues ($\lambda_{0y} = 0$).

Corresponding to eigenvalue $\lambda_{0y} = 0$, the eigenvector of J_0° is

$$\widetilde{W}^{[3]} = \left(\widetilde{w}_1^{[3]}, \widetilde{w}_2^{[3]}, \widetilde{w}_3^{[3]}\right)^T.$$

The eigenvector of J_0° related to the eigenvalue $\lambda_{0y} = 0$ satisfies:

$$(J_3^{\circ} - \lambda_{0y} I)\, \widetilde{W}^{[3]} = 0, \quad \text{where } J_3^{\circ} = J\left(P_3, \alpha_1^*\right)$$

with components:

$$\begin{aligned} \widetilde{w}_1^{[3]} &= \frac{w[1+(1-m)k_1+k_2+s] - e_2\alpha_2 k_2^2}{e_1 r_1[1+(1-m)k_1+k_2+s]} \widetilde{w}_3^{[3]}, \\ \widetilde{w}_2^{[3]} &= \frac{\alpha_2 k_2^2}{r_2(k_2-n)[1+(1-m)k_1+k_2+s]} \widetilde{w}_3^{[3]} \end{aligned}$$

and $\widetilde{w}_3^{[3]}$ being any non-zero real number.

Corresponding to eigenvalue $\lambda_{0y} = 0$, the eigenvector of $J_3^{\circ T}$ is

$$\widetilde{\Psi}^{[3]} = \left(\tilde{\psi}_1^{[3]}, \tilde{\psi}_2^{[3]}, \tilde{\psi}_3^{[3]}\right)^T.$$

This satisfies:

$$\left(J_3^{\circ T} - \lambda_{0y} I\right) \widetilde{\Psi}^{[3]} = 0,$$

which gives us $\widetilde{\Psi}^{[3]} = (0, 0, \widetilde{\psi}_3^{[3]})^T$ where $\widetilde{\psi}_3^{[3]}$ is any non-zero real number.

Since

$$\frac{\partial f}{\partial \alpha_1} = f_{\alpha_1}(Z, \alpha_1) = \left(\frac{\partial f_1}{\partial \alpha_1}, \frac{\partial f_2}{\partial \alpha_1}, \frac{\partial f_3}{\partial \alpha_1}\right) = \left(\frac{-(1-m)x_1^2 y}{1+(1-m)x_1+x_2+s}, 0, \frac{e_1(1-m)x_1^2 y}{1+(1-m)x_1+x_2+s}\right)^T,$$

we have:

$$f_{\alpha_1}(P_3, \alpha_1^*) = (0, 0, 0)^T.$$

Therefore, $\widetilde{\Psi}^{[3]} f_{\alpha_1}(P_3, \alpha_1^*) = 0$.

At P_3, via applying Sotomayor's theorem, we assert that no occurrences of the saddle-node bifurcation take place.

Moreover, since

$$Df_{\alpha_1}(Z, \alpha_1) = \begin{bmatrix} 0 & 0 & \frac{-(1-m)x_1^2}{1+(1-m)x_1+x_2+s} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{e_1(1-m)x_1^2}{1+(1-m)x_1+x_2+s} \end{bmatrix},$$

where $Z = (x_1, x_2, y)^T$ and $Df_{\alpha_1}(Z, \alpha_1)$ represents the derivative of $f_{\alpha_1}(Z, \alpha_1)$, we have:

$$\begin{aligned} Df_{\alpha_1}(P_3, \alpha_1^*)\widetilde{W}^{[3]} &= \begin{bmatrix} 0 & 0 & \frac{-(1-m)k_1^2}{1+(1-m)k_1+k_2+s} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{e_1(1-m)k_1^2}{1+(1-m)k_1+k_2+s} \end{bmatrix} \begin{bmatrix} \frac{w[1+(1-m)k_1+k_2+s]-e_2\alpha_2 k_2^2}{e_1 r_1[1+(1-m)k_1+k_2+s]} \widetilde{W}_3^{[3]} \\ \frac{\alpha_2 k_2^2}{r_2(k_2-n)[1+(1-m)k_1+k_2+s]} \widetilde{W}_3^{[3]} \\ \widetilde{W}_3^{[3]} \end{bmatrix} \\ &= \begin{bmatrix} \frac{-(1-m)k_1^2}{1+(1-m)k_1+k_2+s} \widetilde{W}_3^{[3]} \\ 0 \\ \frac{e_1(1-m)k_1^2}{1+(1-m)k_1+k_2+s} \widetilde{W}_3^{[3]} \end{bmatrix}, \end{aligned}$$

so we obtain that:

$$\left(\widetilde{\Psi}^{[3]}\right)^T \left[Df_{\alpha_1}(P_3, \alpha_1^*) w^{[3]}\right] = \frac{e_1(1-m)k_1^2}{1+(1-m)k_1+k_2+s} \widetilde{W}_3^{[3]} \widetilde{\psi}_3^{[3]} \neq 0.$$

By using $w^{[3]}$ in equation (3.2), we get:

$$D^2 f_{\alpha_1}(P_3, \alpha_1^*)\left(\widetilde{W}^{[3]}, W^{[3]}\right) = \begin{bmatrix} g_{11} \\ g_{21} \\ g_{31} \end{bmatrix},$$

where

$$\begin{aligned} g_{11} &= 2\left\{\left[\frac{(1-m)k_1[(2(1+k_2+s)+(1-m)k_1)]}{e_1 r_1[1+(1-m)k_1+s]^3}\right][w[1+(1-m)k_1+k_2+s]-e_2\alpha_2 k_2^2] \right.\\ &\quad \left. +\left[\frac{(1-m)k_1^2}{[1+(1-m)k_1+k_2+s]^3 r_2(k_2-n)}\right](\alpha_2 k_2^2)\right\}[\widetilde{w}_3^{[3]}]^2,\\ g_{21} &= 0,\\ g_{31} &= \left\{2\left[\frac{(1-m)[e_1[(2k_1(1+k_2+s))+(1-m)k_1^2]]}{e_1 r_1[1+(1-m)k_1+k_2+s]^3}\right][w[1+(1-m)k_1+k_2+s] \right.\\ &\quad \left. -e_2\alpha_2 k_2^2]-\left[\frac{e_1(1-m)k_1^2}{r_2(k_2-n)[1+(1-m)k_1+k_2+s]^3}\right](\alpha_2 k_2^2)\right\}[\widetilde{w}_3^{[3]}]^2, \end{aligned}$$

By condition (3.3), we get

$$\left(\widetilde{\Psi}^{[3]}\right)^T D^2 f_{\alpha_1}(P_3,\alpha_1^*)\left(\widetilde{W}^{[3]},\widetilde{W}^{[3]}\right)=g_{31}\widetilde{\psi}_3^{[3]}\neq 0.$$

At P_3 with $(\alpha_1^*=\alpha_1)$, after applying the steps of Sotomayor's theorem, we conclude that only transcritical bifurcation occurs, without a pitchfork bifurcation.

□

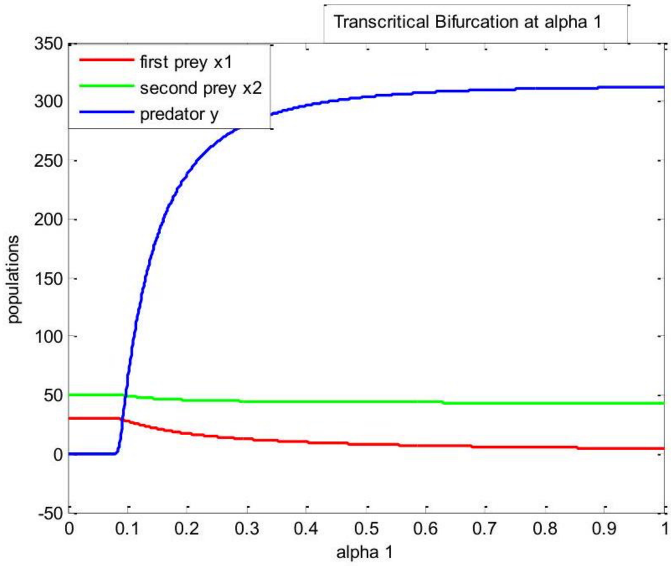

Figure 1: Beginning at (200, 120, 90) with virtual values of table 2, a transcritical bifurcation at $(\alpha_1 = 0.07)$. The solution transforms from P_3 to the point P_6.

When the parameter (α_1) crosses a threshold, a transcritical bifurcation near P_3 In the two-prey-predator model, it refers to a shift in species existence. It leads to a stable exchange between an equilibrium with the absence of the predator and one where it exists. This considers how prey-switching can allow predator incursion and exist; otherwise, as a result of a limited number of prey or poor switching action, the consumption of prey may lead to predator extinction.

Theorem 3.2 *According to conditions (3.4-3.5) at the parameter*

$$\left(e_1^*=e_1=\frac{w\,(1+(1-m)k_1+k_2+s)-\ e_2\alpha_2\ k_2^2}{\alpha_1(1-\ m)\ k_1^2}\right)$$

a transcritical bifurcation occurs near the equilibrium point P_3 in the System (2.1). However, there is no appearance of the two kinds of bifurcations, a saddle bifurcation and a pitchfork bifurcation, at P_3.

$$w(1+(1-m)k_1+k_2+s) > \ e_2\alpha_2\ k_2^2 \tag{3.4}$$

$$\begin{gathered}2\left[\frac{(1-\ m)\left[\alpha_1\left[(2k_1\left(1+k_2+s\right))+(1-m)k_1^2\right]\right]}{\left[1+(1-\ m)\ k_1+\ k_2+\ s\right]^2}\right]\left[\frac{w\left[1+(1-m)k_1+k_2+s\right]-e_2\alpha_2k_2^2}{\alpha_1r_1\left[1+(1-m)k_1+k_2+s\right]}\right]\neq\\ \left[\frac{\alpha_1(1-m)x_1^2}{[1+(1-m)k_1+k_2+s]^2}\right]\left[\frac{\alpha_2k_2^2}{r_2(k_2-n)[1+(1-m)k_1+k_2+s]}\right]\end{gathered} \tag{3.5}$$

Proof:

By substituting $P_3=(k_1,k_2,0)$ with

$$e_1^*=e_1=\frac{w[1+(1-m)k_1+k_2+s]-e_2\alpha_2k_2^2}{\alpha_1(1-m)k_1^2}$$

in the Jacobian matrix given in equation (3.1), we obtain the following eigenvalues for J_3, which has a zero eigenvalue ($\lambda_{0y}=0$).

Where

$$e_1^*=\frac{w[1+(1-m)k_1+k_2+s]-e_2\alpha_2k_2^2}{\alpha_1(1-m)k_1^2}>0,$$

according to condition (3.4).

Let $\widetilde{W}^{[3]}=(\widetilde{w}_1^{[3]},\widetilde{w}_2^{[3]},\widetilde{w}_3^{[3]})^T$ be the eigenvector of J_3° related to the eigenvalue $\lambda_{0y}=0$.

Thus: $(J_3^\circ-\lambda_{0y}I)\widetilde{W}^{[3]}=0$, where $J_3^\circ=J(P_3,e_1^*)$, with components:

$$\begin{aligned}\widetilde{w}_1^{[3]}&=\frac{w[1+(1-m)k_1+k_2+s]-e_2\alpha_2k_2^2}{\alpha_1r_1[1+(1-m)k_1+k_2+s]}\widetilde{w}_3^{[3]},\\ \widetilde{w}_2^{[3]}&=\frac{\alpha_2k_2^2}{r_2(k_2-n)[1+(1-m)k_1+k_2+s]}\widetilde{w}_3^{[3]}\end{aligned}$$

and $\widetilde{w}_3^{[3]}$ being any non-zero real number.

Corresponding to eigenvalue $\lambda_{0y}=0$, the eigenvector of $J_3^{\circ T}$ is

$$\widetilde{\Psi}^{[3]}=(\tilde{\psi}_1^{[3]},\tilde{\psi}_2^{[3]},\tilde{\psi}_3^{[3]})^T.$$

This satisfies:

$$(J_3^{\circ T}-\lambda_{0y}I)\widetilde{\Psi}^{[3]}=0,$$

which gives us $\widetilde{\Psi}^{[3]}=(0,0,\tilde{\psi}_3^{[3]})^T$ where $\tilde{\psi}_3^{[3]}$ is any non-zero real number.

Since

$$\frac{\partial f}{\partial e_1}=f_{e_1}(Z,e_1)=\left(\frac{\partial f_1}{\partial e_1},\frac{\partial f_2}{\partial e_1},\frac{\partial f_3}{\partial e_1}\right)=\left(0,0,\frac{\alpha_1(1-m)x_1^2y}{1+(1-m)x_1+x_2+s}\right)^T,$$

hence $f_{e_1}(P_3,e_1^*)=(0,0,0)^T$. Therefore $\widetilde{\Psi}^{[3]}f_{e_1}(P_3,e_1^*)=0$.

At P_3, via applying Sotomayor's theorem, we assert that no occurrences of the saddle-node bifurcation take place.

Moreover, since

$$Df_{e_1}(Z,e_1)=\begin{bmatrix}0&0&0\\0&0&0\\0&0&\frac{\alpha_1(1-m)x_1^2}{1+(1-m)x_1+x_2+s}\end{bmatrix},$$

where $Z = (x_1, x_2, y)^T$ and $Df_{e_1}(Z, e_1)$ represents the derivative of $f_{e_1}(Z, e_1)$, we have:

$$\begin{aligned} Df_{e_1}(P_3, e_1^*)\widetilde{W}^{[3]} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha_1(1-m)k_1^2}{1+(1-m)k_1+k_2+s} \end{bmatrix} \begin{bmatrix} \frac{w[1+(1-m)k_1+k_2+s]-e_2\alpha_2k_2^2}{\alpha_1 r_1[1+(1-m)k_1+k_2+s]}\widetilde{W}_3^{[3]} \\ \frac{\alpha_2k_2^2}{r_2(k_2-n)[1+(1-m)k_1+k_2+s]}\widetilde{W}_3^{[3]} \\ \widetilde{W}_3^{[3]} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ \frac{\alpha_1(1-m)k_1^2}{1+(1-m)k_1+k_2+s}\widetilde{W}_3^{[3]} \end{bmatrix}, \end{aligned}$$

so we obtain that:

$$\left(\widetilde{\Psi}^{[3]}\right)^T \left[Df_{e_1}(P_3, e_1^*)w^{[3]}\right] = \frac{\alpha_1(1-m)k_1^2}{1+(1-m)k_1+k_2+s}\widetilde{w}_3^{[3]}\widetilde{\psi}_3^{[3]} \neq 0.$$

By using $w^{[3]}$ in equation (3.2), we get:

$$D^2 f_{e_1}(P_3, e_1^*)\left(\widetilde{W}^{[3]}, W^{[3]}\right) = \begin{bmatrix} h_{11} \\ h_{21} \\ h_{31} \end{bmatrix},$$

where

$$\begin{aligned} h_{11} &= 2\Bigg\{\left[\frac{\alpha_1(1-m)k_1[2(1+k_2+s)+(1-m)k_1]}{[1+(1-m)k_1+k_2+s]^3}\right]\left[\frac{w[1+(1-m)k_1+k_2+s]-e_2\alpha_2k_2^2}{\alpha_1 r_1[1+(1-m)k_1+k_2+s]}\right] \\ &\quad + \left[\frac{\alpha_1(1-m)k_1^2}{[1+(1-m)k_1+k_2+s]^3}\right]\left[\frac{\alpha_2k_2^2}{r_2(k_2-n)[1+(1-m)k_1+k_2+s]}\right]\Bigg\}\left(\widetilde{W}_3^{[3]}\right)^2, \\ h_{21} &= 0, \\ h_{31} &= \Bigg\{2\left[\frac{(1-m)[\alpha_1[2k_1(1+k_2+s)+(1-m)k_1^2]]}{[1+(1-m)k_1+k_2+s]^2}\right]\left[\frac{w[1+(1-m)k_1+k_2+s]-e_2\alpha_2k_2^2}{\alpha_1 r_1[1+(1-m)k_1+k_2+s]}\right] \\ &\quad - \left[\frac{\alpha_1(1-m)k_1^2}{[1+(1-m)k_1+k_2+s]^2}\right]\left[\frac{\alpha_2k_2^2}{r_2(k_2-n)[1+(1-m)k_1+k_2+s]}\right]\Bigg\}\left(\widetilde{w}_1^{[3]}\right)^2, \end{aligned}$$

According to condition (3.5), we get

$$\left(\widetilde{\Psi}^{[3]}\right)^T D^2 f_{e_1}(P_3, e_1^*)\left(\widetilde{W}^{[3]}, \widetilde{W}^{[3]}\right) = h_{31}\widetilde{\psi}_3^{[3]} \neq 0.$$

At P_3 with ($e_1^* = e_1$), after applying Sotomayor's theorem, we conclude that only transcritical bifurcation occurs, without a pitchfork bifurcation. □

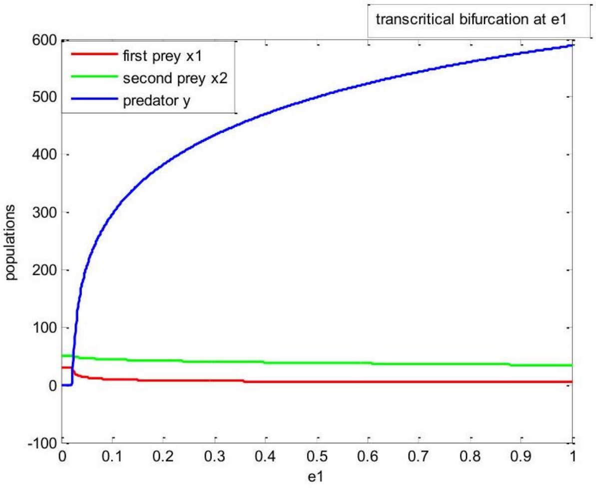

Figure 2: Beginning at (200, 120, 90) with virtual values of table 2, a transcritical bifurcation at ($e_1 = 0.01$) the solution transforms from P_3 to the point P_6.

A critical parameter (e_1 predation rate) crosses a specific threshold, where stability turns into two cases: the first has a specific type (predator) absent, and the second, where it continues. In this model, a transcritical bifurcation represents the transition where the predator's population can be less than the threshold of the parameter, resulting in the extinction of the predator due to the efficiency of switching to weak prey. Therefore, this bifurcation plays a crucial role in determining the conditions the predator requires to survive and coexist with the prey.

Theorem 3.3 *According to condition (3.6) at the parameter*

$$\left(\alpha_2^* = \alpha_2 = \frac{w\left[1+(1-m)k_1+k_2+s\right]-e_1\alpha_1(1-m)k_1^2}{e_2k_2^2}\right)$$

a transcritical bifurcation occurs near the equilibrium point P_3 in the System (2.1). However, , there are no appearance of the two kinds of bifurcations, a saddle bifurcation and a pitchfork bifurcation, at P_3.

$$w\left[1+(1-m)k_1+k_2+s\right] > e_1\alpha_1(1-m)k_1^2 \tag{3.6}$$

Proof:

$$\alpha_2^* = \frac{w[1+(1-m)k_1+k_2+s]-e_1\alpha_1(1-m)k_1^2}{e_2k_2^2} > 0,$$

according to condition (3.6).

By substituting $P_3 = (k_1, k_2, 0)$ with

$$\alpha_2^* = \alpha_2 = \frac{w[1+(1-m)k_1+k_2+s]-e_1\alpha_1(1-m)k_1^2}{e_2k_2^2}$$

in equation (3.1), we obtain the following eigenvalue ($\lambda_{0y} = 0$).

Corresponding to eigenvalue $\lambda_{0y} = 0$, the eigenvector of J_3° is

$$\widetilde{W}^{[3]} = (\widetilde{w}_1^{[3]}, \widetilde{w}_2^{[3]}, \widetilde{w}_3^{[3]})^T.$$

This satisfies:

$$(J_3^\circ - \lambda_{0y}I)\widetilde{W}^{[3]} = 0, \quad \text{where } J_3^\circ = J(P_3, \alpha_2^*),$$

with components:

$$\widetilde{w}_1^{[3]} = \frac{-\alpha_1(1-m)k_1^2}{r_1[1+(1-m)k_1+k_2+s]}\widetilde{w}_3^{[3]},$$
$$\widetilde{w}_2^{[3]} = \frac{-e_2k_2^2}{r_2(k_2-n)[1+(1-m)k_1+k_2+s]^2}\widetilde{w}_3^{[3]}$$

and $\widetilde{w}_3^{[3]}$ being any non-zero real number.

Corresponding to eigenvalue $\lambda_{0y} = 0$, the eigenvector of $J_3^{\circ T}$ is

$$\widetilde{\Psi}^{[3]} = (\tilde{\psi}_1^{[3]}, \tilde{\psi}_2^{[3]}, \tilde{\psi}_3^{[3]})^T.$$

This satisfies:

$$(J_3^{\circ T} - \lambda_{0y}I)\widetilde{\Psi}^{[3]} = 0,$$

which gives us $\widetilde{\Psi}^{[3]} = (0, 0, \widetilde{\psi}_3^{[3]})^T$ where $\widetilde{\psi}_3^{[3]}$ is any non-zero real number.

Since

$$\frac{\partial f}{\partial \alpha_2} = f_{\alpha_2}(Z, \alpha_2) = \left(\frac{\partial f_1}{\partial \alpha_2}, \frac{\partial f_2}{\partial \alpha_2}, \frac{\partial f_3}{\partial \alpha_2}\right) = \left(0, 0, \frac{e_2x_2^2y}{1+(1-m)x_1+x_2+s}\right)^T,$$

hence

$$f_{\alpha_2}(P_3, \alpha_2^*) = \left(0, 0, \frac{e_2k_2^2y}{1+(1-m)k_1+k_2+s}\right)^T.$$

Therefore $\widetilde{\Psi}^{[3]}f_{\alpha_2}(P_3, \alpha_2^*) \neq 0$.

Then by Sotomayor's theorem, the occurrence of the saddle-node bifurcation cannot occur at P_3. Moreover, since

$$Df_{\alpha_2}(Z, \alpha_2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{-e_2(1-m)x_2^2y}{D^2} & \frac{e_2x_2y(2D-x_2)}{D^2} & 0 \\ 0 & \frac{e_2x_2^2}{D} & \end{bmatrix},$$

where $Z = (x_1, x_2, y)^T$ and $Df_{\alpha_2}(Z, \alpha_2)$ represents the derivative of $f_{\alpha_2}(Z, \alpha_2)$, we have:

$$\begin{aligned} Df_{\alpha_2}(P_3, \alpha_2^*)\widetilde{W}^{[3]} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{e_2k_2^2}{1+(1-m)k_1+k_2+s} \end{bmatrix} \begin{bmatrix} \frac{-\alpha_1(1-m)k_1^2}{r_1[1+(1-m)k_1+k_2+s]}\widetilde{w}_3^{[3]} \\ \frac{-e_2k_2^2}{r_2(k_2-n)[1+(1-m)k_1+k_2+s]^2}\widetilde{w}_3^{[3]} \\ \widetilde{W}_3^{[3]} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ \frac{e_2k_2^2}{1+(1-m)k_1+k_2+s}\widetilde{W}_3^{[3]} \end{bmatrix}, \end{aligned}$$

so we obtain that:

$$\left(\widetilde{\Psi}^{[3]}\right)^T \left[Df_{\alpha_2}(P_3, \alpha_2^*)w^{[3]}\right] = \frac{e_2k_2^2}{1+(1-m)k_1+k_2+s}\widetilde{W}_3^{[3]}\widetilde{\psi}_3^{[3]} \neq 0.$$

By using $w^{[3]}$ in equation (3.2), we get:

$$D^2f_{\alpha_2}(P_3, \alpha_2^*)\left(\widetilde{W}^{[3]}, W^{[3]}\right) = \begin{bmatrix} U_{11} \\ U_{21} \\ U_{31} \end{bmatrix},$$

where

$$U_{11} = 0,$$
$$U_{21} = 0,$$
$$U_{31} = -\left\{2\left[\frac{(1-m)[\alpha_2 k_2^2]}{[1+(1-m)k_1+k_2+s]^2}\right]\left[\frac{\alpha_1(1-m)k_1^2}{r[1+(1-m)k_1+k_2+s]}\right]^2 + \left[\frac{e_2k_2[2[(1-m)k_1+s]+k_2]}{[1+(1-m)k_1+k_2+s]^2}\right]\left[\frac{-e_2k_2^2}{r_2(k_2-n)[1+(1-m)k_1+k_2+s]^2}\right]^2\right\}\left(\widetilde{W}_3^{[3]}\right)^2,$$

Also $\left(\widetilde{\Psi}^{[3]}\right)^T D^2 f_{\alpha_2}(P_3,\alpha_2^*)\left(\widetilde{W}^{[3]},\widetilde{W}^{[3]}\right) = U_{31} \neq 0$.

At P_3 with $(\alpha_2^* = \alpha_2)$, after applying the steps of Sotomayor's theorem, we conclude that only transcritical bifurcation occurs, without a pitchfork bifurcation.

□

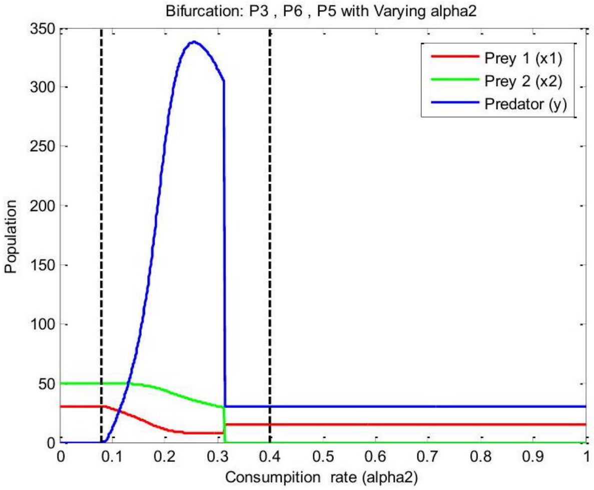

Figure 3: Beginning at (200, 120, 90) with virtual values of Table (2), a transcritical bifurcation at $(\alpha_2 = 0.08)$, the solution transforms from P_3 to the point P_6 then at $(e_2 = 0.4)$, the solution transforms from P_6 to the point P_5.

In the sight of Biological Interpretation of the Transcritical Bifurcation, where the predator transforms from extinction to existence as the parameter passes a threshold (α_2). This considers the point at which amended prey-switching or resource availability strongly affects predator invasion and live together with the two prey species.

Theorem 3.4 *According to condition (3.7) at the parameter*

$$\left(e_2{}^* = e_2 = \frac{w\,[1+(1-m)k_1+k_2+s] - e_1\alpha_1(1-m)k_1^2}{\alpha_2 k_2^2}\right)$$

a transcritical bifurcation occurs near the equilibrium point P_3 in the System (2.1). However, there is no appearance of the two kinds of bifurcations, a saddle bifurcation and a pitchfork bifurcation, at P_3.

$$w\,[1+(1-m)k_1+k_2+s] > e_1\alpha_1(1-m)k_1^2 \tag{3.7}$$

Proof:

$$e_2^* = \frac{w[1+(1-m)k_1+k_2+s] - e_1\alpha_1(1-m)k_1^2}{\alpha_2 k_2^2} > 0,$$

according to condition (8).

By substituting $P_3 = (k_1, k_2, 0)$ with

$$e_2^* = e_2 = \frac{w[1+(1-m)k_1+k_2+s] - e_1\alpha_1(1-m)k_1^2}{\alpha_2 k_2^2}$$

in equation (3.1), we obtain the following eigenvalue ($\lambda_{0y} = 0$).

Corresponding to eigenvalue $\lambda_{0y} = 0$, the eigenvector of J_3° is

$$\widetilde{W}^{[3]} = (\widetilde{w}_1^{[3]}, \widetilde{w}_2^{[3]}, \widetilde{w}_3^{[3]})^T.$$

This satisfies:

$$(J_3^\circ - \lambda_{0y} I)\widetilde{W}^{[3]} = 0, \quad \text{where } J_3^\circ = J(P_3, e_2^*),$$

with components:

$$\widetilde{w}_1^{[3]} = \frac{-\alpha_1(1-m)k_1^2}{r_1[1+(1-m)k_1+k_2+s]}\widetilde{w}_3^{[3]},$$
$$\widetilde{w}_2^{[3]} = \frac{-\alpha_2 k_2^2}{r_2(k_2-n)[1+(1-m)k_1+k_2+s]^2}\widetilde{w}_3^{[3]}$$

and $\widetilde{w}_3^{[3]}$ being any non-zero real number.

Corresponding to eigenvalue $\lambda_{0y} = 0$, the eigenvector of $J_3^{\circ T}$ is

$$\widetilde{\Psi}^{[3]} = (\tilde{\psi}_1^{[3]}, \tilde{\psi}_2^{[3]}, \tilde{\psi}_3^{[3]})^T.$$

This satisfies:

$$(J_3^{\circ T} - \lambda_{0y} I)\widetilde{\Psi}^{[3]} = 0,$$

which gives us $\widetilde{\Psi}^{[3]} = (0, 0, \widetilde{\psi}_3^{[3]})^T$ where $\widetilde{\psi}_3^{[3]}$ is any non-zero real number.

Since

$$\frac{\partial f}{\partial e_2} = f_{e_2}(Z, e_2) = \left(\frac{\partial f_1}{\partial e_2}, \frac{\partial f_2}{\partial e_2}, \frac{\partial f_3}{\partial e_2}\right) = \left(0, 0, \frac{\alpha_2 x_2^2 y}{1+(1-m)x_1+x_2+s}\right)^T,$$

hence

$$f_{e_2}(P_3, e_2^*) = \left(0, 0, \frac{\alpha_2 k_2^2 y}{1+(1-m)k_1+k_2+s}\right)^T.$$

Therefore, $\widetilde{\Psi}^{[3]} f_{e_2}(P_3, e_2^*) \neq 0$.

Then, via Sotomayor's theorem, the occurrence of the saddle-node bifurcation cannot take place at P_3. Moreover, since

$$Df_{e_2}(Z, e_2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{-\alpha_2(1-m)x_2^2 y}{D^2} & \frac{\alpha_2 x_2 y(2D-x_2)}{D^2} & 0 \\ 0 & \frac{\alpha_2 x_2^2}{D} & \end{bmatrix},$$

where $Z = (x_1, x_2, y)^T$ and $Df_{e_2}(Z, e_2)$ represents the derivative of $f_{e_2}(Z, e_2)$, we have:

$$Df_{e_2}(P_3, e_2^*)\widetilde{W}^{[3]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha_2 k_2^2}{1+(1-m)k_1+k_2+s} \end{bmatrix} \begin{bmatrix} \frac{-\alpha_1(1-m)k_1^2}{r_1[1+(1-m)k_1+k_2+s]}\widetilde{w}_3^{[3]} \\ \frac{-\alpha_2 k_2^2}{r_2(k_2-n)[1+(1-m)k_1+k_2+s]^2}\widetilde{w}_3^{[3]} \\ \widetilde{w}_3^{[3]} \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \\ \frac{\alpha_2 k_2^2}{1+(1-m)k_1+k_2+s}\widetilde{w}_3^{[3]} \end{bmatrix},$$

so we obtain that:

$$\left(\widetilde{\Psi}^{[3]}\right)^T \left[Df_{e_2}(P_3, e_2^*)w^{[3]}\right] = \frac{\alpha_2 k_2^2}{1+(1-m)k_1+k_2+s}\widetilde{w}_3^{[3]}\widetilde{\psi}_3^{[3]} \neq 0.$$

By using $w^{[3]}$ in equation (3.2), we get:

$$D^2 f_{e_2}(P_3, e_2^*)\left(\widetilde{W}^{[3]}, W^{[3]}\right) = \begin{bmatrix} S_{11} \\ S_{21} \\ S_{31} \end{bmatrix},$$

where

$$\begin{aligned} S_{11} &= 0, \\ S_{21} &= 0, \\ S_{31} &= -\left\{2\left[\frac{(1-m)[\alpha_2 k_2^2]}{[1+(1-m)k_1+k_2+s]^2}\right]\left[\frac{\alpha_1(1-m)k_1^2}{r[1+(1-m)k_1+k_2+s]}\right]^2\right. \\ &\quad \left. + \left[\frac{\alpha_2 k_2[2[(1-m)k_1+s]+k_2]}{[1+(1-m)k_1+k_2+s]^2}\right]\left[\frac{-\alpha_2 k_2^2}{r_2(k_2-n)[1+(1-m)k_1+k_2+s]^2}\right]^2\right\}\left(\widetilde{w}_3^{[3]}\right)^2, \end{aligned}$$

Also $\left(\widetilde{\Psi}^{[3]}\right)^T D^2 f_{e_2}(P_3, e_2^*)\left(\widetilde{W}^{[3]}, \widetilde{W}^{[3]}\right) = S_{31} \neq 0$.

At P_3 with $(e_2^* = e_2)$, after applying the steps of Sotomayor's theorem, we conclude that only transcritical bifurcation occurs, without a pitchfork bifurcation. ☐

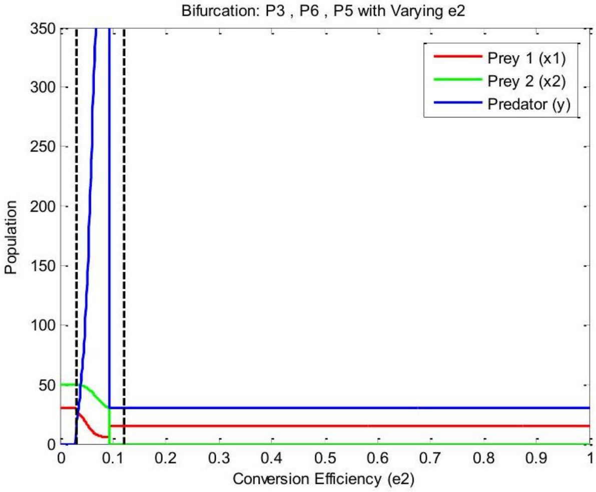

Figure 4: Beginning at (200, 120, 90) with virtual values of Table 2 a transcritical bifurcation at $(e_2 = 0.03)$, the solution transforms from P_3 to the point P_6 then at $(e_2 = 0.098)$, the solution transforms from P_6 to the point P_5.

This benefit is that a soft ecological change (e.g., in e_2) can lead to the appearance or disappearance of coexistence in the ecosystem. It reflects the weakness of biodiversity under parameter change, focusing attention on how ecological management (such as refuge or prey protection) can influence the balance toward extinction or away from it.

3.1. The Hopf-bifurcation in the biological system

The occurrence of Hopf bifurcation near the equilibrium points will be investigated using the Haque and Venturino methods [32] in the proposed system (2.1), as below.

Theorem 3.5 *Suppose that the local conditions (3.8-3.19) of the point P_6 and the following conditions hold:*

$$r_1\left(1-\frac{2\tilde{x}_1}{k_1}\right) < \frac{\alpha_1(1-m)\tilde{y}\left[2\left(1+(1-m)\tilde{x}_1+\tilde{x}_2+s\right)+(1-m)\tilde{x}_1\right]}{\left[1+(1-m)x_1+x_2+s\right]^2}, \tag{3.8}$$

$$2\tilde{x}_2+\frac{2n\tilde{x}_2}{k_2} < n+\frac{3\tilde{x}_2^2}{k_2} \tag{3.9}$$

$$\frac{e_1\alpha_1(1-m)\tilde{x}_1^2+e_2\alpha_2\tilde{x}_2^2}{1+(1-m)\tilde{x}_1+\tilde{x}_2+s} < w \tag{3.10}$$

$$\tilde{b}_{22}\tilde{b}_{33}-\tilde{b}_{23}\tilde{b}_{32}-\tilde{b}_{13}\tilde{b}_{31} > \tilde{b}_{12}\tilde{b}_{21}, \tag{3.11}$$

$$\tilde{b}_{22} > \tilde{b}_{31}, \tag{3.12}$$

$$\tilde{b}_{11}\tilde{b}_{22} < \tilde{b}_{12}\tilde{b}_{21} \tag{3.13}$$

$$\tilde{b}_{11}(\tilde{b}_{22}+\tilde{b}_{33}-\tilde{b}_{23}(\tilde{b}_{12}\tilde{b}_{31}-\tilde{b}_{11}\tilde{b}_{31})) > \tilde{b}_{13}\,\tilde{b}_{22}\left(\tilde{b}_{22}\tilde{b}_{31}\right)+\tilde{b}_{33}\left(\tilde{b}_{11}\tilde{b}_{22}-\tilde{b}_{12}\tilde{b}_{21}\right) \tag{3.14}$$

$$\tilde{b}_{13}\tilde{b}_{31}+\tilde{b}_{23}\tilde{b}_{32} > \tilde{b}_{11}+\tilde{b}_{22}^2, \tag{3.15}$$

$$\frac{1}{\tilde{b}_{22}}\left\{\left(\tilde{b}_{11}+\tilde{b}_{22}\right)\left[\tilde{b}_{12}\tilde{b}_{21}+\tilde{b}_{13}\tilde{b}_{31}+\tilde{b}_{23}\tilde{b}_{32}\right]\right\}+\tilde{b}_{31}\left(\tilde{b}_{12}-\tilde{b}_{11}\right) > \tilde{b}_{11}+\tilde{b}_{13}\left(\tilde{b}_{22}-\tilde{b}_{31}\right), \tag{3.16}$$

$$\tilde{b}_{11} < 2\tilde{b}_{11}\tilde{b}_{22}+\tilde{b}_{13}\tilde{b}_{31}+\tilde{b}_{23}\tilde{b}_{32}+\tilde{b}_{22}^2 \tag{3.17}$$

Then for the parameter (w^*), system (2.1) has a Hopf-bifurcation at P_6.

Proof: The characteristic equation of system (2.1) at P_6 is given by:

$$\lambda^3+N_1\lambda^2+N_2\lambda+N_3=0,$$

where the coefficients are:

$$\begin{aligned}
N_1 &= -(\tilde{b}_{11}+\tilde{b}_{22}+\tilde{b}_{33}) > 0,\\
N_2 &= \tilde{b}_{22}\tilde{b}_{33}-\tilde{b}_{12}\tilde{b}_{21}-\tilde{b}_{13}\tilde{b}_{31}-\tilde{b}_{23}\tilde{b}_{32},\\
N_3 &= \tilde{b}_{11}(\tilde{b}_{22}+\tilde{b}_{33})-\tilde{b}_{23}(\tilde{b}_{12}\tilde{b}_{31}-\tilde{b}_{11}\tilde{b}_{31})\\
&\quad -\tilde{b}_{13}\tilde{b}_{22}\tilde{b}_{31}-\tilde{b}_{33}(\tilde{b}_{11}\tilde{b}_{22}-\tilde{b}_{12}\tilde{b}_{21}).
\end{aligned}$$

To verify the necessary and sufficient conditions for Hopf bifurcation, we need to find the critical parameter value w^* that satisfies:

1. $N_i(w^*) > 0$ for $i = 1, 2$
2. $\Delta(w^*) = N_1N_2 - N_3 = 0$

Under conditions (3.8–3.14), the first condition $N_i(w^*) > 0$ for $i = 1, 2$ is satisfied. The second condition $\Delta = 0$ yields:

$$w^* = \frac{e_1\alpha_1(1-m)\tilde{x}_1^2+e_2\alpha_2\tilde{x}_2^2}{1+(1-m)\tilde{x}_1+\tilde{x}_2+s}+\frac{\beta_1+\sqrt{\beta_1^2+4\beta_2}}{2},$$

where

$$\begin{aligned}
\beta_1 &= \frac{1}{\tilde{b}_{22}}\left[\tilde{b}_{13}\tilde{b}_{31}+\tilde{b}_{23}\tilde{b}_{32}-\tilde{b}_{11}-\tilde{b}_{22}^2\right],\\
\beta_2 &= \frac{1}{\tilde{b}_{22}}\left\{(\tilde{b}_{11}+\tilde{b}_{22})\left[\tilde{b}_{12}\tilde{b}_{21}+\tilde{b}_{13}\tilde{b}_{31}+\tilde{b}_{23}\tilde{b}_{32}\right]\right\}\\
&\quad -\tilde{b}_{11}-\tilde{b}_{13}(\tilde{b}_{22}-\tilde{b}_{31})+\tilde{b}_{31}(\tilde{b}_{12}-\tilde{b}_{11}),
\end{aligned}$$

and $w^* > 0$ under conditions (3.8–3.14) and (3.15–3.16).

At $w = w^*$, the characteristic equation can be factored as:

$$P_3(\lambda) = (\lambda + N_1)(\lambda^2 + N_2) = 0,$$

which has roots:

$$\lambda_1 = -N_1 < 0,$$
$$\lambda_{2,3} = \pm i\sqrt{N_2}.$$

Thus, at $w = w^*$, the system has three eigenvalues: one negative real eigenvalue (λ_1) and a pair of purely imaginary eigenvalues ($\lambda_{2,3}$).

For values of w near w^*, the complex eigenvalues can be expressed as:

$$\lambda_{2,3} = \delta_1(w) \pm i\delta_2(w).$$

To verify the transversality condition for Hopf bifurcation, we need to confirm:

$$\Psi(w^*)\Theta(w^*) + \Gamma(w^*)\Phi(w^*) \neq 0.$$

At $w = w^*$, we have $\delta_1 = 0$ and $\delta_2 = \sqrt{N_2}$, which gives:

$$\begin{aligned}
\Psi(w^*) &= -2N_2, \\
\Phi(w^*) &= 2N_1\sqrt{N_2}, \\
\Theta(w^*) &= w^* \left[\tilde{b}_{11} - \tilde{b}_{11}\tilde{b}_{22} + \tilde{b}_{22}\tilde{b}_{33} - \tilde{b}_{13}\tilde{b}_{31} - \tilde{b}_{23}\tilde{b}_{32}\right], \\
\Gamma(w^*) &= w^*\sqrt{N_2}\tilde{b}_{22}.
\end{aligned}$$

Substituting these values, we obtain:

$$\begin{aligned}
\Psi(w^*)\Theta(w^*) + \Gamma(w^*)\Phi(w^*) = -2N_2 w^* \big[&\tilde{b}_{11} - 2\tilde{b}_{11}\tilde{b}_{22} \\
&- \tilde{b}_{13}\tilde{b}_{31} - \tilde{b}_{23}\tilde{b}_{32} - \tilde{b}_{22}^2\big].
\end{aligned}$$

Under conditions (3.8–3.14) and condition (3.17), this expression is non-zero:

$$\Psi(w^*)\Theta(w^*) + \Gamma(w^*)\Phi(w^*) \neq 0.$$

Therefore, all conditions for Hopf bifurcation are satisfied at P_6 with parameter value w^* in system (2.1). □

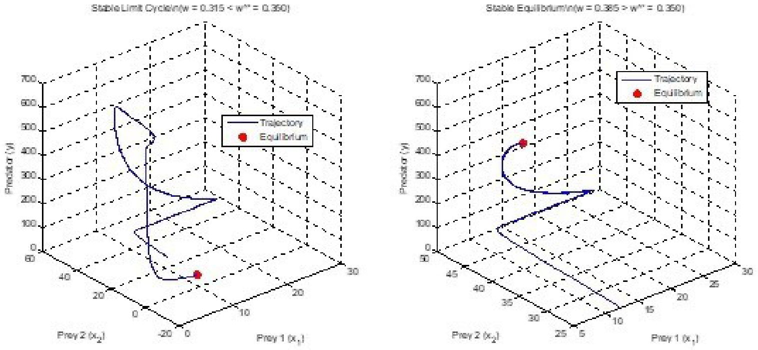

Figure 5: Beginning at $(200, 120, 90)$ with virtual values from Table 2, a trajectory approaches equilibrium at ($w = 0.315 < w^* = 0.350$), as shown on the right side of the figure. In contrast, at ($w = 0.385 > w^* = 0.350$), a limit cycle arises, illustrated on the left side of the figure.

This bifurcation explains how slight changes in the rate of death of predators can have a significant impact on ecosystem stability, focusing attention on the delicate balance between predation influence and the continuation of life for all species.

3.2. Structured Sensitivity Analysis

Studying numerically, after gaining a set of analytical results from the previous sections, and restrictions of virtual parameter variations, a sensitivity analysis was support to discuss how active ecological parameters of the proposed system influence the global dynamics of the system. The concentration occurred on three biologically significant parameters:

1. Determines the minimum population density demand for prey permanence according to how the Allee threshold impacts of the second prey (y).
2. Refuge size effect on the first prey x – indicates the rate of individuals that salvation of predation .
3. Predator natural mortality rate d – dominance of the predator persistence, and the start of oscillatory dynamics.

Diversifying values of each parameter on one side, while on the other side, still fixed the rest in (Table 2), bifurcation diagrams were structured to illustrate transitions among the ecological states. A higher Allee threshold tends to a transcritical bifurcation, such that the second prey population (y) is extinct. Similarly, refuge rate transmits the stability of equilibria: in the case of a small refuge tends the prey population is driven to overexploitation; conversely, an extravagant refuge tends to lower predator persistence. Now, Predator natural mortality exposes near critical values Hopf bifurcations: replacing stable coexistence by limit cycles, reference predator–prey oscillations.

4. Detection of the Persistence of the Proposed Biological Model

From a biological point of view, the system persists in that at all future times, the whole population will survive. Generally, persistence is considered a global property when studying the behavior of a dynamical system, and studying conditions of persistence via the method of average Lyapunov function [30]depends on solution behavior near extinction boundaries (boundary planes), which is independent of the interior solution space structure. Mathematically, it means that strictly positive solutions do not have an omega limit set on the boundary of the non-negative cone [30] It is based on a solution manner near annihilation boundaries (border planes), away from the inner solution space construction. Mathematical consideration that on the border of the non-negative cone, the precisely positive solutions do not contain an omega limit set [31]. Although the dynamical system exhibits extinction, in contrast, whenever the dynamical system of the proposed model does not persist, the solution contains an omega limit set on the border of the nonnegative cone. Firstly, it is beneficial to study the global dynamics of the boundary boundary x_1y-plane, which is illustrated analytically by the Theorem 3.6.

Theorem 4.1 *In the Int R^2_+ assume that $P_5 = (\bar{\bar{x}}_1, 0, \bar{\bar{y}})$ is locally asymptotically stable, then in the Int R^2_+ of x_1y-plane it is globally asymptotically stable.*

Proof:

When the second prey is absent, system (2.1) reduces to the following subsystem:

$$\begin{aligned} \frac{dx_1}{dt} &= x_1r_1\left(1-\frac{x_1}{k_1}\right)-\frac{\alpha_1(1-m)x_1^2y}{1+(1-m)x_1+s}, \\ \frac{dy}{dt} &= \frac{e_1\alpha_1(1-m)x_1^2y}{1+(1-m)x_1+s}-wy. \end{aligned} \tag{4.1}$$

Int $\mathbb{R}^2_+$ (the x_1y-plane), subsystem (3.18) has a unique interior equilibrium point $P_5 = (\bar{x}_1, \bar{y})$. Consider $R(x_1, y) = \frac{1}{x_1y}$. Note that $R(x_1, y)$ is positive for all $(x_1, y) \in \text{Int}(\mathbb{R}^2_+)$ and is a C^1 function.

Since,

$$\Delta(x_1,y)=\frac{\partial}{\partial x_1}(Rf_1)+\frac{\partial}{\partial y}(Rf_2)=\frac{\alpha_1(1-m)^2x_1^2+\alpha_1[1+(1-m)x_1+s]\{(1-m)+2x_1\}}{x_1[1+(1-m)x_1+s]^2}.$$

In $\mathbb{R}_+^2$, $\Delta(x_1,y)$ is neither identically zero nor does it change sign. Therefore, depending on the Bendixson-Dulac criterion [31], the generating subsystem (3.18) from system (2.1) in the inner part of the positive quadrant of $x1y-$ plane, the biological proposed model (2.1) has no periodic dynamics. Where P_5 is the only positive equilibrium point of the subsystem (3.18). Via the above Theorem asserts that P_5 is globally asymptotically stable inside the positive quadrant. □

Now, the next analysis illustrates the concept of persistence in Theorem 4.2:

Theorem 4.2 *Assume that there are no periodic dynamics of system (2.1) in the boundary x_1y-plane At the point $P_5=(\overline{\overline{x}}_1,0,\overline{\overline{y}})$ with the following conditions of locally,*

$$r_1\left(1-\frac{2\bar{x}_1}{k_1}\right)<\frac{\alpha_1(1-m)\overline{\overline{x}}_1\overline{\overline{y}}\,[2\,(1+(1-m)\overline{\overline{x}}_1+s)-(1-m)\overline{\overline{x}}_1]}{[1+(1-m)\overline{\overline{x}}_1+s]^2} \tag{4.2}$$

$$\frac{e_1\alpha_1(1-m)\overline{\overline{x}}_1^2}{1+(1-m)\overline{\overline{x}}_1+s}<w, \tag{4.3}$$

Then system (2.1) is uniformly persistent.

Proof:

Give thought to the average Lyapunov functions illustrated below:

$$\tau\left(x_1,x_2,y\right)=x_1^{V_1}x_2^{V_2}y^{V_3},$$

Such that V_1,V_2,V_3 are positive constants. obviously, $\tau\left(x_1,x_2,y\right)$ is a nonnegative C^1 defined in R_+^3. Acquiring the outcome: $\Omega\left(x_1,x_2,y\right)=\frac{\tau'(x_1,x_2,y)}{\tau(x_1,x_2,y)}$,
$\Omega\left(x_1,x_2,y\right)=V_1\left[\,r_1\left(1-\frac{2\ x_1}{k_1}\right)-\left\{\frac{\alpha_1(1-\ m)\ x_1\ y[2(1+(1-\ m)\ x_1+\ x_2+\ s)-(1-\ m)\ x_1]}{[1+(1-\ m)\ x_1+\ x_2+\ s]^2}\right\}\right]+$
$V_2\left[\,r_2\left[2\ x_2-\ n-\frac{3\ x_2^2}{k_2}+\frac{2\ nx_2}{k_2}\right]-\frac{\alpha_2\ x_2\ y[2(1+(1-\ m)\ x_1+\ x_2+\ s)-\ x_2]}{[1+(1-\ m)\ x_1+\ x_2+\ s]^2}\right]+$
$V_3\left[\frac{e_1\alpha_1(1-\ m)\ x_1^2+\ e_2\alpha_2\ x_2^2}{1+(1-\ m)\ x_1+\ x_2+\ s}-\ w\right]$,
Right now,

- Substituted $P_3=(k_1,k_2,0)$, we get:

Violate the condition $e_1\alpha_1(1-\ m)\ k_1^2+\ e_2\alpha_2\ k_2^2<\ w\,(1+(1-m)k_1+k_2+\ s)$ therefore:

$$\Omega\left(P_3\right)=-r_1V_1+nr_2V_2+V_3\left[\frac{e_1\alpha_1(1-\ m)\ k_1^2+\ e_2\alpha_2\ k_2^2}{1+(1-\ m)\ k_1+\ k_2+\ s}-\ w\right]$$

By increasing V_2 and V_3 to a sufficiently large value, implies that $\Omega\left(P_3\right)>0$.

- Substituted $P_5=(\bar{x}_1,0,y)$, we Violate conditions of local stability (4.2-4.3) mentioned in theorem (4.2). Therefore,

$$\Omega\left(P_5\right)=V_1\left[\,r_1\left(1-\frac{2\overline{\overline{x}}_1}{k_1}\right)-\left\{\frac{\alpha_1(1-\ m)\overline{\overline{x}}_1\overline{\overline{y}}\left[2\left(1+(1-\ m)\overline{\overline{x}}_1+\ s\right)-(1-\ m)\overline{\overline{x}}_1\right]}{[1+(1-\ m)\overline{\overline{x}}_1+\ s]^2}\right\}\right]$$
$$-\,nr_2V_2+V_3\left[\frac{e_1\alpha_1(1-\ m)\overline{\overline{x}}_1^2}{1+(1-\ m)\overline{\overline{x}}_1+\ s}-\ w\right].$$

Sufficiently increasing V_1 besides V_3 large enough value, tend to $\Omega\left(P_5\right) > 0$ Therefore, system (2.1) is uniformly persistent.

□

Now, Table 3 summarizes all the necessary results :

Table 3: The analysis results

Parameters varied in system (2.1)	**Numerical behavior of system (2.1)**	**Persistence of system (2.1)**	**Bifurcation point**
$0.00001 \leq \alpha_1 < 0.07$, $0.07 \leq \alpha_1 < 1$	Approach to P_3, Approach to P_6	Not Persists, Persists	$\alpha_1 = 0.07$
$0.00001 \leq \alpha_2 < 0.08$, $0.08 \leq \alpha_2 < 0.4$, $0.4 \leq \alpha_2 < 1$	Approach to P_3, P_6, P_5	Not Persists, Persists, Not Persists	$\alpha_2 = 0.08$, $\alpha_2 = 0.4$
$0.00001 \leq e_1 < 0.01$, $0.01 \leq e_1 \leq 1$	Approach to P_3, P_6	Not Persists, Persists	$e_1 = 0.01$
$0.0001 \leq e_2 < 0.03$	Approach to P_3	Not Persists	$e_2 = 0.03$, $e_2 = 0.098$
$0.03 \leq e_2 < 0.098$	Approach to P_6	Persists	
$0.098 \leq e_2 \leq 1$	Approach to P_5	Not Persists	
$0.00001 \leq m \leq 1$	Approach to P_6	Persists	
$0.00001 \leq w < 0.35$	Approach to P_5	Not Persists	$w = 0.35$, $w = 0.75$
$0.35 \leq w < 0.75$	Approach to P_6	Persists	
$0.75 \leq w < 1$	Approach to P_3	Not Persists	
$10 \leq n < 30$	Approach to P_6	Persists	

5. Comparison with Related Models

Even though the model that contains predator–prey dynamics escorted by switching, refuge, and Allee effects has each been extensively studied, there is a restricted straight rapprochement between models that combine these mechanisms at the same time. Early interest studies on prey-switching dynamics model (e.g. [3] Krivan, 2003; [4] Abrams and Matsuda, 2004) emphasize its part in stabilizing population oscillations without considering the extra nonlinearities present by refuges or Allee thresholds. Similarly, the effect of prey refuge has been investigated in both constant and density-dependent proposed systems (e.g., [8] Mukherjee, 2016; [9] Wayesa et al., 2025), with results highlighting increased prey existence and the possibility of another stable state. So far, these models largely concentrate on single-prey systems. Neoteric predator–prey models combined the Allee effect (e.g., [11] Teixeira and Hilker, 2017; [10] Singh et al., 2025; [22] Al-Jubouri and Naji, 2024), manifestation threshold-dependent persistence and extinction results. However, these studies systematically analyze Allee effects in separation or merge with only one other ecological mechanism, to some extent than in a more complex multi-prey setting.In sight of the bifurcation view, some studies (e.g., [20] Din, 2019; [18] Arancibia-Ibarra et al., 2021) have tested Hopf or other local bifurcations in predator–prey systems. Sotomayor's theorem applies strict approaches in a few studies, such as to explain necessary and sufficient conditions for both kinds of local bifurcation (transcritical and Hopf bifurcations) .The proposed model (a two-prey, one-predator framework) amalgamates all three ecological mechanisms in the same model: prey switching, refuge, and a strong Allee which allows us to expose plentiful bifurcation structures and persistence conditions compared with those proposed in models dealing with these effective mechanisms separately. Therefore, the proposed model modified recent studies by explaining how the merge action of switching attitude, refuge size, and Allee thresholds dominates ecological stability and long-term coexistence.

6. Conclusions

The proposed system presents an inclusive analysis of one predator with two selected prey in an ecological model, including three real biological mechanisms: prey switching consumption, refuge, and a strong Allee effect concerning the first and second prey, respectively. With these effective parameters, the classical model extends to more complex impact species extinction and persistence. Sotomayor's theorem is useful for studying the core of the analysis of the local bifurcation. At several equilibrium points, we discuss the phenomenon of transcritical bifurcations. These bifurcations rely on thresholds where system stability varies between two different ecological states, incomplete and complete species coexistence. Significantly, whenever saddle-node or pitchfork bifurcations occur, which refer to a biological explanation of saddle-node bifurcation, no unexpected population extinction or explosion occurs when the parameter changes; instead, the population extent changes slowly and continuously . Furthermore, Hopf bifurcation reveals specific parameter conditions, marking the beginning of the population oscillations, which could lead to complex dynamical behavior such as cycles or instability. Via both of the Lyapunov functions and the Dulac–Bendixson criterion, confirm uniform persistence that the population's extinction cannot exist under particular conditions. Numerical simulations corroborate the analytical results, illustrating how changes in some particular ecological parameters—refuge of the first prey, switching rate, and Allee threshold of the second prey dominate population dynamics . Commonly, the study of the proposed model focuses on interactions between various ecological mechanisms that contribute to the structure, stability, and flexibility of predator–prey populations. These yield valuable insight into how focusing on ecological involvement can raise long-term coexistence, prevent species extinction, and encourage ecosystem balance.

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