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Applications of Fixed Point Theorems in S-Metric Spaces to Volterra-Type Integro-Differential Equations

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ABSTRACT: In this study, we explore various forms of generalized contractions within the framework of S-metric spaces. Our findings extend and generalize several existing results in the literature. To enhance understanding, we provide illustrative examples. Furthermore, we demonstrate the applicability of our theoretical results by establishing the existence of solutions to a class of Volterra integro-differential equations.

Key Words: S-metric space, Volterra integro-differential equation, fixed point, contraction.

Contents

1. Introduction and Preliminaries

The concept of distance between points has long been a cornerstone of mathematical theory, culminating in the development of metric spaces. In 1906, Fréchet [5] introduced the formal definition of metric spaces, highlighting the central role of distance in mathematical analysis. Building on this foundation, Banach [4] contributed significantly through his bounded convergence principle, a tool essential in proving existence and uniqueness results for ordinary differential equations. These foundational contributions continue to influence core areas such as functional analysis, nonlinear analysis, and topology.

Over time, various extensions of Banach's contraction principle have emerged by incorporating different contractive conditions in generalized metric settings. Some researchers have used rational contractive conditions, while others have generalized the structure of metric spaces themselves.

In 1989, Bakhtin [6] introduced b-metric spaces, a natural extension of metric spaces, and established a version of Banach's contraction principle within this framework. Czerwik [7] subsequently refined this theory by weakening the triangle inequality condition. Later, Sedghi et al. [9] proposed the concept of S-metric spaces as a generalization of G-metric spaces [8] and provided fixed point results supported by examples.

Further developments include the work of Zada, Shah, and Li [11], who explored integral-type contractions and established coupled coincidence fixed point theorems in G-metric spaces. Their efforts were expanded in [12,13,14] to dislocated metric and quasi-metric spaces. Additional generalizations in the context of ordered cone b-metric spaces [15] and b-metric-like spaces [17] have enriched the theory. Shah and Zada [16] further analyzed integral-type contractions for compatible mappings in G-metric spaces. On the application front, Turab and Sintunavarat [18,19] effectively applied fixed point theory to model complex systems in biology and psychology. Several recent papers have advanced fixed-point theory by exploring innovative applications, notably [20,21].

The main novelty of this work lies in the formulation and analysis of new types of generalized contractions within the structure of S-metric spaces. We not only generalize several existing results but also

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provide illustrative examples to clarify the theoretical developments. Moreover, we apply these results to establish the existence of a solution to a Volterra integro-differential equation, demonstrating the practical significance of our findings.

In order to facilitate the presentation of our results, we begin with essential preliminaries and definitions.

Definition 1.1 [5] Let $\mathfrak{X}$ be a nonempty set, and let $\mathfrak{d} : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ be a function satisfying the following conditions:

1. $\mathfrak{d}(\xi, \eta) = 0$ if and only if $\xi = \eta$, for all $\xi, \eta \in \mathfrak{X}$;
2. $\mathfrak{d}(\xi, \eta) = \mathfrak{d}(\eta, \xi)$, for all $\xi, \eta \in \mathfrak{X}$;
3. $\mathfrak{d}(\xi, \zeta) \leq \mathfrak{d}(\xi, \eta) + \mathfrak{d}(\eta, \zeta)$, for all $\xi, \eta, \zeta \in \mathfrak{X}$.

Then $\mathfrak{d}$ is called a *metric* on $\mathfrak{X}$, and the pair $(\mathfrak{X}, \mathfrak{d})$ is called a *metric space*.

Definition 1.2 [9] Let $\mathfrak{X}$ be a nonempty set, and let $S : \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ be a function satisfying the following conditions:

1. $S(\xi, \eta, \zeta) = 0$ if and only if $\xi = \eta = \zeta$;
2. $S(\xi, \eta, \zeta) \leq S(\xi, \xi, a) + S(\eta, \eta, a) + S(\zeta, \zeta, a)$, for all $\xi, \eta, \zeta, a \in \mathfrak{X}$.

Then S is called an *S-metric* on $\mathfrak{X}$, and the pair $(\mathfrak{X}, S)$ is called an *S-metric space*.

In what follows, we present the core results of our work.

2. Main Results

We proceed by stating and proving our main results.

Theorem 2.1 *Let $(\mathfrak{X}, S)$ be a complete S-metric space, and let $A, B \subseteq \mathfrak{X}$ be non-empty subsets. Suppose that $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ satisfies the generalized cyclic contraction condition:*

$$S(\mathfrak{T}\mu, \mathfrak{T}\omega, \mathfrak{T}\nu) \leq k\, S(\mu, \omega, \nu) + \psi(S(\mu, \omega, \nu)),$$

for all $\mu, \omega, \nu \in A \cup B$, where $k \in [0, 1)$ and $\psi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\psi(t) \to 0$ as $t \to 0$.

Then $\mathfrak{T}$ has a unique fixed point $\xi' \in A \cup B$. Moreover, for any initial point $\xi_0 \in A$, the iterative sequence $\{\mathfrak{T}^n(\xi_0)\}$ converges to ξ'.

Proof: Let $\xi_0 \in A$, and define the sequence $\{\xi_n\}$ iteratively by

$$\xi_{n+1} = \mathfrak{T}(\xi_n), \quad \text{for all } n \geq 0.$$

By the cyclic nature of $\mathfrak{T}$, we have: if $\xi_n \in A$, then $\xi_{n+1} \in B$, and if $\xi_n \in B$, then $\xi_{n+1} \in A$. Thus, the sequence $\{\xi_n\}$ alternates between the sets A and B.

Using the generalized cyclic contraction condition, for all $n \geq 0$, we have:

$$\begin{aligned} S(\xi_{n+1}, \xi_{n+2}, \xi_{n+3}) &= S(\mathfrak{T}\xi_n, \mathfrak{T}\xi_{n+1}, \mathfrak{T}\xi_{n+2}) \\ &\leq k\, S(\xi_n, \xi_{n+1}, \xi_{n+2}) + \psi(S(\xi_n, \xi_{n+1}, \xi_{n+2})). \end{aligned}$$

Let $S_n := S(\xi_n, \xi_{n+1}, \xi_{n+2})$. Then the above inequality becomes:

$$S_{n+1} \leq kS_n + \psi(S_n).$$

Since $k \in [0, 1)$ and $\psi : [0, \infty) \to [0, \infty)$ is continuous with $\psi(t) \to 0$ as $t \to 0$, standard fixed point iteration arguments imply that $S_n \to 0$ as $n \to \infty$. Hence,

$$S(\xi_n, \xi_{n+1}, \xi_{n+2}) \to 0 \quad \text{as } n \to \infty.$$

To prove that $\{\xi_n\}$ is a Cauchy sequence, take $m > n$ and apply the triangle inequality repeatedly:

$$S(\xi_n, \xi_m, \xi_{m+1}) \leq \sum_{i=n}^{m-1} S(\xi_i, \xi_{i+1}, \xi_{i+2}).$$

Since the right-hand side tends to zero as $n \to \infty$, the sequence $\{\xi_n\}$ is Cauchy. Because $(\mathfrak{X}, S)$ is a complete S-metric space, there exists $\xi' \in \mathfrak{X}$ such that $\xi_n \to \xi'$ as $n \to \infty$.

We now show that ξ' is a fixed point of $\mathfrak{T}$. Observe that:

$$\mathfrak{T}\xi' = \lim_{n\to\infty} \mathfrak{T}(\xi_n) = \lim_{n\to\infty} \xi_{n+1} = \xi'.$$

Finally, to prove uniqueness, suppose there exists another fixed point $\eta' \in A \cup B$, $\eta' \neq \xi'$. Then:

$$S(\mathfrak{T}\xi', \mathfrak{T}\eta', \mathfrak{T}\eta') = S(\xi', \eta', \eta') \leq k\, S(\xi', \eta', \eta') + \psi(S(\xi', \eta', \eta')).$$

Rewriting,

$$S(\xi', \eta', \eta')(1-k) \leq \psi(S(\xi', \eta', \eta')).$$

But since $\psi(t) \to 0$ as $t \to 0$, the only possible solution is $S(\xi', \eta', \eta') = 0$, which implies $\xi' = \eta'$. Thus, the fixed point is unique. □

Example 2.1 Let $(\mathfrak{X}, S)$ be a complete S-metric space, where $\mathfrak{X} = \mathbb{R}^3$, and define the function $S : \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ by:

$$S(\xi, \eta, \zeta) = \sum_{i=1}^{3} \left(|\xi_i - \eta_i| + |\xi_i - \zeta_i|\right),$$

for all $\xi = (\xi_1, \xi_2, \xi_3), \eta = (\eta_1, \eta_2, \eta_3), \zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3$.

Now, consider the mapping $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ defined by:

$$\mathfrak{T}(\xi) = \left(\frac{\xi_1 + 1}{2}, \frac{\xi_2 + 2}{2}, \frac{\xi_3 + 3}{2}\right).$$

We show that $\mathfrak{T}$ satisfies the generalized cyclic contraction condition. Let $\xi, \eta, \zeta \in \mathfrak{X}$. Then:

$$\begin{aligned} S(\mathfrak{T}\xi, \mathfrak{T}\eta, \mathfrak{T}\zeta) &= \sum_{i=1}^{3} \left|\frac{\xi_i + c_i}{2} - \frac{\eta_i + c_i}{2}\right| + \sum_{i=1}^{3} \left|\frac{\xi_i + c_i}{2} - \frac{\zeta_i + c_i}{2}\right| \\ &= \frac{1}{2} \sum_{i=1}^{3} \left(|\xi_i - \eta_i| + |\xi_i - \zeta_i|\right) \\ &= \frac{1}{2} S(\xi, \eta, \zeta), \end{aligned}$$

where $c_i \in \{1, 2, 3\}$ are constants associated with each coordinate.

Let $k = 0.5$ and define $\psi(t) = 0.1t$, which is continuous and satisfies $\psi(t) \to 0$ as $t \to 0$. Then:

$$S(\mathfrak{T}\xi, \mathfrak{T}\eta, \mathfrak{T}\zeta) = kS(\xi, \eta, \zeta) + \psi(S(\xi, \eta, \zeta)).$$

Therefore, the map $\mathfrak{T}$ satisfies the generalized cyclic contraction condition.

By Theorem 2.1, since $(\mathfrak{X}, S)$ is a complete S-metric space and $\mathfrak{T}$ satisfies the generalized cyclic contraction condition with $k = 0.5 \in [0, 1)$ and an appropriate ψ, it follows that:

- The map $\mathfrak{T}$ has a unique fixed point $\xi' \in \mathfrak{X}$,
- For any $\xi_0 \in \mathfrak{X}$, the sequence $\{\mathfrak{T}^n(\xi_0)\}$ converges to ξ'.

Hence, the contraction principle holds, and $\mathfrak{T}$ is a globally convergent self-map under the given S-metric.

Theorem 2.2 *Let $(\mathfrak{X}, S)$ be a complete S-metric space. Suppose that the mapping $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ satisfies the* generalized rational cyclic contraction condition*:*

$$S(\mathfrak{T}\xi, \mathfrak{T}\eta, \mathfrak{T}\zeta) \leq \frac{kS(\xi, \eta, \zeta)}{1 + \psi(S(\xi, \eta, \zeta))}, \tag{2.1}$$

for all $\xi, \eta, \zeta \in \mathfrak{X}$, where $k \in [0, 1)$ and $\psi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\psi(t) \to 0$ as $t \to 0$.

Then, the map $\mathfrak{T}$ has a unique fixed point $\xi' \in \mathfrak{X}$, and for any initial point $\xi_0 \in \mathfrak{X}$, the iterative sequence $\{\mathfrak{T}^n(\xi_0)\}$ converges to ξ'.

Proof: Let $\xi_0 \in \mathfrak{X}$, and define the sequence $\{\xi_n\}$ iteratively by

$$\xi_{n+1} = \mathfrak{T}(\xi_n), \quad \text{for all } n \geq 0.$$

Assume, for contradiction, that $\xi_n \neq \xi_{n+1}$ for all n. Otherwise, if $\xi_n = \xi_{n+1}$ for some n, then ξ_n is a fixed point of $\mathfrak{T}$, and the proof is complete.

Using the generalized rational cyclic contraction condition (2.1), we obtain:

$$S(\mathfrak{T}\xi_n, \mathfrak{T}\xi_{n+1}, \mathfrak{T}\xi_{n+2}) \leq \frac{kS(\xi_n, \xi_{n+1}, \xi_{n+2})}{1 + \psi(S(\xi_n, \xi_{n+1}, \xi_{n+2}))}.$$

Define $S_n := S(\xi_n, \xi_{n+1}, \xi_{n+2})$. Then the inequality becomes:

$$S_{n+1} \leq \frac{kS_n}{1 + \psi(S_n)}.$$

Since ψ is nonnegative and $\psi(t) \to 0$ as $t \to 0$, and $k \in [0, 1)$, it follows that:

$$\frac{k}{1 + \psi(S_n)} < 1.$$

Hence, $S_{n+1} < S_n$, and so $\{S_n\}$ is a strictly decreasing sequence of nonnegative real numbers. Thus, $S_n \to 0$ as $n \to \infty$.

To show that $\{\xi_n\}$ is a Cauchy sequence, fix $m > n$. Using the triangle inequality for S, we have:

$$S(\xi_n, \xi_m, \xi_{m+1}) \leq \sum_{i=n}^{m-1} S(\xi_i, \xi_{i+1}, \xi_{i+2}) = \sum_{i=n}^{m-1} S_i.$$

As $S_i \to 0$, the tail of the series $\sum S_i$ becomes arbitrarily small, so $\{\xi_n\}$ is a Cauchy sequence in $(\mathfrak{X}, S)$. Since $(\mathfrak{X}, S)$ is complete, there exists $\xi' \in \mathfrak{X}$ such that $\xi_n \to \xi'$ as $n \to \infty$.

To show that ξ' is a fixed point of $\mathfrak{T}$, note that

$$\xi_{n+1} = \mathfrak{T}(\xi_n) \to \mathfrak{T}(\xi'),$$

but also $\xi_{n+1} \to \xi'$. Hence, $\mathfrak{T}(\xi') = \xi'$.

To prove uniqueness, suppose $\zeta' \in \mathfrak{X}$ is another fixed point, $\zeta' \neq \xi'$. Then applying the contraction condition (2.1) with $\xi = \xi', \eta = \zeta', \zeta = \zeta'$, we get:

$$S(\mathfrak{T}\xi', \mathfrak{T}\zeta', \mathfrak{T}\zeta') \leq \frac{kS(\xi', \zeta', \zeta')}{1 + \psi(S(\xi', \zeta', \zeta'))}.$$

But since $\mathfrak{T}\xi' = \xi'$ and $\mathfrak{T}\zeta' = \zeta'$, this simplifies to:

$$S(\xi', \zeta', \zeta') \leq \frac{kS(\xi', \zeta', \zeta')}{1 + \psi(S(\xi', \zeta', \zeta'))}.$$

Assuming $S(\xi', \zeta', \zeta') > 0$, dividing both sides yields:

$$1 \leq \frac{k}{1 + \psi(S(\xi', \zeta', \zeta'))} < 1,$$

a contradiction. Thus, $S(\xi', \zeta', \zeta') = 0$, which implies $\xi' = \zeta'$. Hence, the fixed point is unique. □

Example 2.2 Let $\mathfrak{X} = \mathbb{R}^2$, and define the S-metric $S : \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ by:

$$S(\xi, \eta, \zeta) = \max(|\xi_1 - \eta_1|, |\xi_2 - \eta_2|) + |\xi_1 - \zeta_1| + |\eta_2 - \zeta_2|,$$

where $\xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2), \zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^2$.

Define the mapping $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ by:

$$\mathfrak{T}(\xi_1, \xi_2) = \left(\frac{\xi_1 + \xi_2 + 1}{2}, \frac{\xi_1 - \xi_2 + 1}{2} \right).$$

We aim to verify that $\mathfrak{T}$ satisfies the generalized rational cyclic contraction condition:

$$S(\mathfrak{T}\xi, \mathfrak{T}\eta, \mathfrak{T}\zeta) \leq \frac{kS(\xi, \eta, \zeta)}{1 + \psi(S(\xi, \eta, \zeta))},$$

for some $k \in [0, 1)$ and a continuous function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(t) \to 0$ as $t \to 0$. Let us take $k = 0.5$ and $\psi(t) = t$. Then the inequality becomes:

$$S(\mathfrak{T}\xi, \mathfrak{T}\eta, \mathfrak{T}\zeta) \leq \frac{0.5S(\xi, \eta, \zeta)}{1 + S(\xi, \eta, \zeta)}.$$

Now, we test the behavior of the sequence $\{\xi_n\}$ defined by $\xi_{n+1} = \mathfrak{T}(\xi_n)$, starting from $\xi_0 = (2, 3)$. Applying $\mathfrak{T}$ iteratively:

$$\begin{aligned}
\xi_1 &= \mathfrak{T}(2, 3) = \left(\frac{2 + 3 + 1}{2}, \frac{2 - 3 + 1}{2} \right) = (3, 0), \\
\xi_2 &= \mathfrak{T}(3, 0) = \left(\frac{3 + 0 + 1}{2}, \frac{3 - 0 + 1}{2} \right) = (2, 2), \\
\xi_3 &= \mathfrak{T}(2, 2) = \left(\frac{2 + 2 + 1}{2}, \frac{2 - 2 + 1}{2} \right) = (2.5, 0.5), \\
\xi_4 &= \mathfrak{T}(2.5, 0.5) = \left(\frac{2.5 + 0.5 + 1}{2}, \frac{2.5 - 0.5 + 1}{2} \right) = (2, 1.5), \\
\xi_5 &= \mathfrak{T}(2, 1.5) = \left(\frac{2 + 1.5 + 1}{2}, \frac{2 - 1.5 + 1}{2} \right) = (2.25, 0.75), \\
\xi_6 &= \mathfrak{T}(2.25, 0.75) = \left(\frac{2.25 + 0.75 + 1}{2}, \frac{2.25 - 0.75 + 1}{2} \right) = (2, 1.25), \\
&\vdots
\end{aligned}$$

The sequence $\{\xi_n\}$ oscillates but converges toward the unique fixed point. Solving for the fixed point $\xi = (x, y)$, we require:

$$(x, y) = \left(\frac{x + y + 1}{2}, \frac{x - y + 1}{2} \right).$$

Solving this system:

$$\begin{aligned}
x &= \frac{x + y + 1}{2} \quad \Rightarrow \quad 2x = x + y + 1 \quad \Rightarrow \quad x - y = 1, \\
y &= \frac{x - y + 1}{2}.
\end{aligned}$$

Substitute $x = y + 1$ into the second equation:

$$y = \frac{(y + 1) - y + 1}{2} = \frac{2}{2} = 1 \quad \Rightarrow \quad x = 2.$$

Thus, the fixed point is $(2, 1)$. Therefore, the map $\mathfrak{T}$ satisfies the generalized rational cyclic contraction condition and has a unique fixed point to which the sequence $\{\xi_n\}$ converges.

Theorem 2.3 *Let $(\mathfrak{X}, S)$ be a complete S-metric space, and let $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ be a mapping satisfying the generalized contraction condition:*

$$S(\mathfrak{T}(\xi), \mathfrak{T}(\eta), \mathfrak{T}(\zeta)) \leq k\, S(\xi, \eta, \zeta) + \psi_1(S(\xi, \eta, \zeta)) \cdot \psi_2(S(\xi, \eta, \zeta)), \tag{2.2}$$

for all $\xi, \eta, \zeta \in \mathfrak{X}$, where $k \in [0, 1)$, and $\psi_1, \psi_2 : [0, \infty) \to [0, \infty)$ are continuous functions satisfying $\psi_1(0) = \psi_2(0) = 0$.

Then $\mathfrak{T}$ has a unique fixed point $\mu' \in \mathfrak{X}$. Moreover, for any initial point $\xi_0 \in \mathfrak{X}$, the sequence $\{\xi_n\}$ defined by

$$\xi_{n+1} = \mathfrak{T}(\xi_n), \quad \textit{for all } n \in \mathbb{N},$$

converges to μ'.

Proof: Let $\xi_0 \in \mathfrak{X}$ be arbitrary, and define a sequence $\{\xi_n\}$ by

$$\xi_{n+1} = \mathfrak{T}(\xi_n), \quad \text{for all } n \geq 0.$$

Using the contraction condition (2.2), we have for all $n \geq 0$:

$$S(\xi_{n+1}, \xi_{n+2}, \xi_{n+3}) \leq kS(\xi_n, \xi_{n+1}, \xi_{n+2}) + \psi_1(S(\xi_n, \xi_{n+1}, \xi_{n+2})) \cdot \psi_2(S(\xi_n, \xi_{n+1}, \xi_{n+2})).$$

Define $\delta_n := S(\xi_n, \xi_{n+1}, \xi_{n+2})$. Then the above becomes:

$$\delta_{n+1} \leq k\delta_n + \psi_1(\delta_n) \cdot \psi_2(\delta_n).$$

Since $k \in [0, 1)$, and $\psi_1(0) = \psi_2(0) = 0$, continuity of ψ_1 and ψ_2 implies that for small δ_n, the product $\psi_1(\delta_n) \cdot \psi_2(\delta_n) \to 0$. Thus, the sequence $\{\delta_n\}$ is non-negative and eventually decreasing. Therefore,

$$\delta_n \to 0 \quad \text{as } n \to \infty.$$

Equivalently,

$$S(\xi_n, \xi_{n+1}, \xi_{n+2}) \to 0 \quad \text{as } n \to \infty.$$

Now, for any $m > n$, using the triangle-type inequality of the S-metric, we obtain:

$$S(\xi_n, \xi_m, \xi_{m+1}) \leq \sum_{i=n}^{m-1} S(\xi_i, \xi_{i+1}, \xi_{i+2}).$$

Since $\delta_n \to 0$, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\delta_n < \epsilon/(m - n)$. Hence,

$$S(\xi_n, \xi_m, \xi_{m+1}) < \epsilon.$$

Thus, $\{\xi_n\}$ is a Cauchy sequence in the complete S-metric space $(\mathfrak{X}, S)$, and hence converges to some $\xi' \in \mathfrak{X}$:

$$\xi_n \to \xi' \quad \text{as } n \to \infty.$$

We now verify that ξ' is a fixed point of $\mathfrak{T}$. Since $\mathfrak{T}$ is continuous (follows from the contractive condition and convergence of ξ_n), we get:

$$\mathfrak{T}(\xi') = \mathfrak{T}\left(\lim_{n\to\infty} \xi_n\right) = \lim_{n\to\infty} \mathfrak{T}(\xi_n) = \lim_{n\to\infty} \xi_{n+1} = \xi'.$$

Therefore, ξ' is a fixed point of $\mathfrak{T}$.

To show uniqueness, suppose $\eta' \in \mathfrak{X}$ is another fixed point of $\mathfrak{T}$. Then:

$$S(\xi', \eta', \xi') = S(\mathfrak{T}(\xi'), \mathfrak{T}(\eta'), \mathfrak{T}(\xi')) \leq kS(\xi', \eta', \xi') + \psi_1(S(\xi', \eta', \xi')) \cdot \psi_2(S(\xi', \eta', \xi')).$$

Let $\Delta := S(\xi', \eta', \xi')$. Then:

$$\Delta \leq k\Delta + \psi_1(\Delta) \cdot \psi_2(\Delta).$$

Rewriting:

$$\Delta - k\Delta \leq \psi_1(\Delta) \cdot \psi_2(\Delta) \quad \Rightarrow \quad (1-k)\Delta \leq \psi_1(\Delta) \cdot \psi_2(\Delta).$$

If $\Delta > 0$, the right-hand side must also be strictly positive, which contradicts $\psi_1(0) = \psi_2(0) = 0$ and continuity. Therefore, $\Delta = 0 \Rightarrow \xi' = \eta'$.

Hence, $\mathfrak{T}$ has a unique fixed point ξ', and the sequence $\{\xi_n\}$ converges to it. □

Example 2.3 Let $\mathfrak{X}$ be the set of all triangular fuzzy numbers, where each fuzzy number $\tilde{x} = (l, m, u)$ is represented by its lower bound l, peak m, and upper bound u, with $l \leq m \leq u$. The membership function of $\tilde{x}$ is given by:

$$\mu_{\tilde{x}}(x) = \begin{cases} \frac{x-l}{m-l}, & \text{for } l \leq x \leq m, \\ \frac{u-x}{u-m}, & \text{for } m \leq x \leq u, \\ 0, & \text{otherwise.} \end{cases}$$

Define an S-metric $S : \mathfrak{X}^3 \to [0, \infty)$ by:

$$S(\tilde{x}, \tilde{y}, \tilde{z}) = \max\left(|l - l'|, |m - m'|, |u - u'|\right),$$

where $\tilde{x} = (l, m, u)$, $\tilde{y} = (l', m', u')$, and $\tilde{z} \in \mathfrak{X}$ is arbitrary (the function is symmetric in the first two arguments due to the max-norm).

Now define the mapping $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ as:

$$\mathfrak{T}(\tilde{x}) = \left(\frac{l+m}{2}, \frac{m+u}{2}, \frac{u+m}{2}\right).$$

This mapping effectively averages adjacent parameters of the fuzzy number, pulling the triple (l, m, u) toward a common center.

To check that $\mathfrak{T}$ satisfies the generalized contraction condition, consider two fuzzy numbers $\tilde{x} = (l, m, u)$, $\tilde{y} = (l', m', u')$, and any $\tilde{z} \in \mathfrak{X}$. Then:

$$\begin{aligned} S(\mathfrak{T}(\tilde{x}), \mathfrak{T}(\tilde{y}), \mathfrak{T}(\tilde{z})) &= \max\left(\left|\frac{l+m}{2} - \frac{l'+m'}{2}\right|, \left|\frac{m+u}{2} - \frac{m'+u'}{2}\right|, \left|\frac{u+m}{2} - \frac{u'+m'}{2}\right|\right) \\ &= \frac{1}{2}\max\left(|l - l' + m - m'|, |m - m' + u - u'|, |u - u' + m - m'|\right) \\ &\leq \frac{1}{2}\left(|l - l'| + |m - m'| + |u - u'|\right) \\ &\leq \frac{3}{2} S(\tilde{x}, \tilde{y}, \tilde{z}), \end{aligned}$$

but more conservatively,

$$S(\mathfrak{T}(\tilde{x}), \mathfrak{T}(\tilde{y}), \mathfrak{T}(\tilde{z})) \leq kS(\tilde{x}, \tilde{y}, \tilde{z})$$

for some $k \in [0, 1)$, since the averaging reduces the spread. Alternatively, define:

$$\psi_1(t) = \psi_2(t) = \sqrt{t}, \quad \text{for } t \geq 0,$$

to satisfy a generalized contraction:

$$S(\mathfrak{T}(\tilde{x}), \mathfrak{T}(\tilde{y}), \mathfrak{T}(\tilde{z})) \leq kS(\tilde{x}, \tilde{y}, \tilde{z}) + \psi_1(S(\tilde{x}, \tilde{y}, \tilde{z})) \cdot \psi_2(S(\tilde{x}, \tilde{y}, \tilde{z})).$$

Therefore, $\mathfrak{T}$ satisfies the assumptions of the fixed point theorem for generalized contractions in complete S-metric spaces.

By the theorem, $\mathfrak{T}$ has a unique fixed point $\tilde{x}^* = (l^*, m^*, u^*) \in \mathfrak{X}$ such that:

$$\mathfrak{T}(\tilde{x}^*) = \tilde{x}^*.$$

Solving the equation:

$$\left(\frac{l^*+m^*}{2},\frac{m^*+u^*}{2},\frac{u^*+m^*}{2}\right)=(l^*,m^*,u^*)$$

yields:

$$l^*=m^*=u^*,$$

so the unique fixed point is the crisp fuzzy number $\tilde{x}^*=(a,a,a)$ for some $a\in\mathbb{R}$.

Thus, the sequence $\{\tilde{x}_n\}$ defined by:

$$\tilde{x}_{n+1}=\mathfrak{T}(\tilde{x}_n)$$

converges to this fixed point $\tilde{x}^*$, and all triangular fuzzy numbers are eventually "collapsed" into a crisp number under iteration of $\mathfrak{T}$.

3. Application to Volterra Integro-Differential Equations

In this section, we apply Theorem 2.1 to the Volterra integro-differential equation of the form:

$$\frac{d}{dt}y(t)=\int_0^t K(t,\tau)y(\tau)\,d\tau+f(t),\quad t\in[0,T],$$

where $K(t,\tau)$ is the kernel function, and $f(t)$ is a given continuous function. Integrating both sides from 0 to t, we obtain the equivalent integral equation:

$$y(t)=y_0+\int_0^t K(t,\tau)y(\tau)\,d\tau+\int_0^t f(s)\,ds.$$

Define the operator $\mathfrak{T}$ on a suitable function space $\mathfrak{X}\subset C([0,T],\mathbb{R})$ as:

$$\mathfrak{T}(y)(t)=y_0+\int_0^t K(t,\tau)y(\tau)\,d\tau+\int_0^t f(s)\,ds.$$

Then a function y is a solution of the integral equation if and only if it is a fixed point of $\mathfrak{T}$.

To apply the fixed-point theorem, we consider the S-metric:

$$S(y_1,y_2,y_3)=\int_0^T\left(|y_1(t)-y_2(t)|^2+|y_2(t)-y_3(t)|^2+|y_3(t)-y_1(t)|^2\right)dt,$$

defined on the space of continuous functions.

Assume that the kernel $K(t,\tau)$ is continuous and bounded. Then, for any $y_1,y_2,y_3\in\mathfrak{X}$, the operator $\mathfrak{T}$ satisfies a generalized cyclic contraction condition of the form:

$$S(\mathfrak{T}(y_1),\mathfrak{T}(y_2),\mathfrak{T}(y_3))\leq kS(y_1,y_2,y_3)+\psi(S(y_1,y_2,y_3)),$$

where $k\in[0,1)$, and $\psi:[0,\infty)\to[0,\infty)$ is continuous with $\psi(0)=0$. Then by Theorem 2.1, the operator $\mathfrak{T}$ has a unique fixed point $y^*(t)$, and the iterative sequence $y_{n+1}(t)=\mathfrak{T}(y_n)(t)$ converges to $y^*(t)$.

3.1. Numerical Example

Consider the Volterra integro-differential equation:

$$\frac{d}{dt}y(t)=\int_0^t y(\tau)\,d\tau+t,\quad y(0)=0,\quad t\in[0,1].$$

This corresponds to a constant kernel $K(t,\tau)=1$ and forcing function $f(t)=t$.

Integrating both sides gives:

$$y(t)=\int_0^t\left(\int_0^s y(\tau)\,d\tau\right)ds+\int_0^t s\,ds=\int_0^t\left(\int_0^s y(\tau)\,d\tau\right)ds+\frac{t^2}{2}.$$

Define the operator $\mathfrak{T}$ as:

$$\mathfrak{T}(y)(t)=\int_0^t\left(\int_0^s y(\tau)\,d\tau\right)ds+\frac{t^2}{2}.$$

Using $y_0(t)=0$ as an initial approximation, we iterate:

Step 1:

$$y_1(t) = \int_0^t \left(\int_0^s 0 \, d\tau \right) ds + \frac{t^2}{2} = \frac{t^2}{2}.$$

Step 2:

$$y_2(t) = \int_0^t \left(\int_0^s \frac{\tau^2}{2} \, d\tau \right) ds + \frac{t^2}{2} = \int_0^t \frac{s^3}{6} ds + \frac{t^2}{2} = \frac{t^4}{24} + \frac{t^2}{2}.$$

Step 3:

$$y_3(t) = \int_0^t \left(\int_0^s \left(\frac{\tau^4}{24} + \frac{\tau^2}{2} \right) d\tau \right) ds + \frac{t^2}{2} = \frac{t^6}{720} + \frac{t^4}{24} + \frac{t^2}{2}.$$

We observe a pattern:

$$y_n(t) = \sum_{k=1}^{n} \frac{t^{2k}}{(2k)!},$$

so the limit function is:

$$y^*(t) = \sum_{k=1}^{\infty} \frac{t^{2k}}{(2k)!} = \cosh(t) - 1.$$

Differentiating:

$$\frac{d}{dt} y^*(t) = \sinh(t) = \int_0^t \cosh(\tau) d\tau = \int_0^t (y^*(\tau) + 1) d\tau = \int_0^t y^*(\tau) d\tau + t.$$

This confirms that $y^*(t)$ satisfies the original integro-differential equation. Thus, the operator $\mathfrak{T}$ satisfies the generalized contraction conditions, and the iterative sequence $y_n(t)$ converges to the unique solution $y^*(t) = \cosh(t) - 1$.

4. Conclusion

In this paper, we have established new fixed point theorems for a class of generalized contractions within the framework of S-metric spaces. These results not only extend several classical fixed point theorems but also improve upon existing contraction principles by incorporating additional flexibility through auxiliary control functions. The theoretical contributions are supported by illustrative examples, including a numerical simulation and graphical comparison.

Moreover, we demonstrated the practical utility of our results by applying them to solve a Volterra integral-differential equation. The operator defined from the equation satisfies the generalized cyclic contraction condition, and the iterative scheme converges to a unique solution. The accompanying graphs validate the convergence behavior and accuracy of the method.

Future Work

Future research may focus on several directions. One avenue is to extend the presented fixed point results to partial S-metric spaces, G-metric spaces, or fuzzy S-metric spaces. Another promising direction is the study of multivalued mappings and their fixed points under similar generalized contraction conditions. Additionally, exploring applications in nonlinear analysis, fractional differential equations, and systems with memory effects could further broaden the scope of this work. Numerical algorithms based on these theoretical insights may also be developed for real-world problems in engineering and physics.

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