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Wavelet Transforms for the Kontorovich-Lebedev-Clifford Transform and Their Applications

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ABSTRACT: In this paper, we investigate the continuous wavelet transform within the framework of the convolution theory of the Kontorovich-Lebedev-Clifford transform, We derive the reconstruction formula together with the Plancherel and Parseval relations. Additionally, we introduce localization operators associated with the Kontorovich-Lebedev-Clifford wavelet transform (KLC-wavelet). Basic properties concerning these operators are proven to illustrate their boundedness and compactness, and their belonging to the Schatten-von Neumann class. Furthermore, their corresponding trace formula is determined.

Key Words: Continuous wavelet transform, Kontorovich-Lebedev-Clifford, localization operator, Schatten-von Neumann class, reconstruction formula, translation operators.

Contents

1. Introduction

A wavelet is a small oscillation that, which, when translated and dilated, forms a family of derived from functions based on a single mother wavelet, ψ. The mother wavelet ψ is an element of the space $L^2(\mathbb{R})$ and must satisfy the admissibility condition:

$$\int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|}\, d\omega < \infty, \tag{1.1}$$

where $\hat{\psi}$ is equal to the Fourier transform of the function ψ. This admissibility condition is important when applied with wavelet transforms that are widely used for the analysis of finite energy signals over minor periods. By scaling and translating the mother wavelet $\psi_{b,a}(x)$ which is defined as follows:

$$\psi_{b,a}(x) = |a|^{-\frac{1}{2}}\psi\left(\frac{x-b}{a}\right), \quad a, b \in \mathbb{R}, a \neq 0. \tag{1.2}$$

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The CWT of a signal $f \in L^2(\mathbb{R})$ with regard to a wavelet, ψ is provided by:

$$W_\psi f(b,a) = \int_{-\infty}^{\infty} f(x)\overline{\psi_{b,a}(x)}\,dx. \tag{1.3}$$

The signal is represented in both time and frequency by the CWT via adjustments to the scaling parameter a and translation parameter b, which enables detailed analysis detailed analysis across both domains. While wavelet transforms have been extensively studied, their application alongside the Kontorovich-Lebedev-Clifford (KLC) transform remains relatively unexplored. We note that the Kontorovich-Lebedev-Clifford wavelet transform is a powerful tool for signal analysis that combines advanced mathematical concepts from Fourier analysis, Clifford algebras, and wavelet theory. Localization operators play a critical role in this framework by allowing for multi-scale and multi-dimensional analysis of signals.

Despite the theoretical potential of the KLC transform, the number of studies dedicated to this transform remains very limited. No more than three published articles have investigated the KLC transform in depth [8] [28] [29]. This indicates that the subject is still in an early stage of development and presents a rich opportunity for further exploration. As such, the present work aims to provide a significant contribution to the existing literature, offering researchers new tools and insights to extend and better understand the KLC transform from various mathematical and applied perspectives.

These transforms are particularly useful in areas that require high-dimensional or multi-scale analysis, such as Signal and Image Processing (The KLC wavelet transform can be applied to remove noise from signals or images, particularly in scenarios where data is noisy in both space and frequency domains. Localization allows the transform to identify regions where noise exists and separate them from the underlying signal), Quantum Mechanics (The KLC wavelet transform has been used to study wave functions, especially when these wave functions are defined in complex or multi-dimensional spaces. The ability of the KLC transform to localize both in space and momentum (or frequency) makes it a valuable tool in quantum analysis), Electromagnetic Field Theory (The KLC transform can be used to analyze signals related to electromagnetic waves, particularly in high-dimensional or complex geometries where standard Fourier analysis might be inadequate. The transform's ability to handle local properties of fields at different scales allows for more accurate modeling and simulations), Medical Imaging (In areas like MRI or CT scanning, where the signals are often multi-dimensional, the KLC wavelet transform can assist in improving image resolution or enhancing certain features of the data. Localization operators help capture fine details while preserving global structure), and in other several fields.

In this paper, the continuous wavelet transform (CWT), which is associated with the KLC transform, is defined and examined , with an emphasis on functions defined on the positive half-line, With an emphasis on functions defined on the positive half-line, we develop and . Although this transform has been discussed in previous works such as [8], its full potential, especially in the context of wavelet transforms, has yet to be thoroughly developed.

It should be noted that localization operators are in the heart of time-frequency analysis since they give the position of a signal in time and frequency domain.These operators were introduced by Daubechies in 1988 [17]and have attracted enormous research attention in various scientific disciplines. These operators have been extensively studied due to their relevance in various scientific fields. Notably, Wong [18], [19] investigated their boundedness and compactness, while recent work by Z. P. He [20] derived their Schatten-von Neumann properties and the trace formula. Examples of localization operators include Anti-Wick operators [21]. [25]. [26] Gabor transforms [22],Mehler-Fock Wavelet Transform [24],wavelet transform associated with the modified Whittaker transform [27],Localization operators and Shapiro-type inequality for the modified Whittaker-Stockwell transform [31] ,and wave packets [23] [30].

The second part of the paper focuses on the practical application of the theoretical framework developed in the first part. Specifically, it examines the use of localization operators within the context of the Kontorovich-Lebedev-Clifford wavelet transform (KLC-wavelet) for signal analysis. This section demonstrates the utility of the KLC-wavelet transform in real-world scenarios, including image processing and signal reconstruction. Through various examples and numerical simulations, we demonstrate how the properties of the localization operators enhance the accuracy and efficiency of these applications. The findings of this section provide valuable insights into the practical impact of KLC-wavelet transforms in modern signal processing tasks.

2. Fundamentals of the KLC Transform

This work is based on the KLC-transformfor functions defined on the positive half line, as given in [8].

$$\mathbb{K}\Lambda(\lambda) = \frac{1}{2}\int_0^\infty K_{2i\sqrt{\lambda}}(2\sqrt{x_1})\Lambda(x_1)x_1^{-1}\,dx_1, \quad \lambda \in \mathbb{R}_+, \tag{2.1}$$

where $K_{2i\sqrt{\lambda}}(2\sqrt{x_1})$ is the Macdonald function as detailed in (see Ref. [8] , [3])

$$K_{2i\sqrt{\lambda}}(2\sqrt{x_1}) = \int_0^\infty e^{-2\sqrt{x}\cosh t}\cos(2\sqrt{\lambda}\,t)\,dt, \quad x_1 > 0,\ \lambda > 0. \tag{2.2}$$

In [8], the KLC-transform (2.1) inversion formula is provided as follows:

$$\Lambda(x_1) = \frac{4}{\pi^2}\int_0^\infty K_{2i\sqrt{\lambda}}(2\sqrt{x_1})\mathbb{K}\Lambda(\lambda)\sinh(2\pi\sqrt{\lambda})\,d\lambda, \quad x_1 > 0. \tag{2.3}$$

From [5], we have

$$\frac{4}{\pi^2}\int_0^\infty K_{2i\sqrt{\lambda}}(2\sqrt{x_1})K_{2i\sqrt{\lambda}}(2\sqrt{x_2})K_{2i\sqrt{\lambda}}(2\sqrt{x_3})\sinh(2\pi\sqrt{\lambda})\,d\lambda = D(x_1,x_2,x_3), \tag{2.4}$$

where $D(x_1,x_2,x_3)$ is given as

$$D(x_1,x_2,x_3) = \frac{1}{2}\exp\left[-\frac{x_1x_2+x_2x_3+x_3x_1}{\sqrt{x_1x_2x_3}}\right], \quad x,y,z \in \mathbb{R}_+, \tag{2.5}$$

which is symmetric in x_1, x_2 and x_3. From (2.1), (2.4) and (2.6), we have a product of Macdonald functions as

$$K_{2i\sqrt{\lambda}}(2\sqrt{x_1})K_{2i\sqrt{\lambda}}(2\sqrt{x_2}) = \frac{1}{2}\int_0^\infty K_{2i\sqrt{\lambda}}(2\sqrt{x_3})D(x_1,x_2,x_3)x_3^{-1}\,dx_3 = \mathbb{K}(D(x_1,x_2,.))(\lambda), \tag{2.6}$$

and

$$0 < D(x_1,x_2,x_3) < \frac{e^{-2\sqrt{x_1}}}{2}. \tag{2.7}$$

The convolution operator with respect to KLC-transform (2.1) is defined by [8]

$$\Lambda * \psi(x_1) = \frac{1}{2}\int_0^\infty \mathfrak{T}_{x_1}(\Lambda)(y)\psi(x_2)x_2^{-1}\,dx_2, \tag{2.8}$$

$$\Lambda * \psi(x_1) = \frac{1}{4}\int_0^\infty\int_0^\infty D(x_1,x_2,x_3)\Lambda(x_3)\psi(x_2)x_3^{-1}x_2^{-1}\,dx_3dx_2, \tag{2.9}$$

where $\mathfrak{T}_{x_1},\ x_1 > 0$, denotes the translation operator and it is defined as

$$\mathfrak{T}_{x_1}(\Lambda)(y) = \frac{1}{2}\int_0^\infty D(x_1,x_2,x_3)\Lambda(x_3)x_3^{-1}\,dx_3. \tag{2.10}$$

Also from [8], we have few estimates that will be useful in further calculations

$$\text{(i)}\ \left|K_{2i\sqrt{\lambda}}(2\sqrt{x_1})\right| \le K_0(2\sqrt{x_1}). \tag{2.11}$$

$$\text{(ii)}\ \int_0^\infty D(x_1,x_2,x_3)x_3^{-1}\,dx_3 = 2K_0(2\sqrt{x+y}). \tag{2.12}$$

$$\text{(iii)}\ K_0(2\sqrt{x_1+x_2}) \le K_0(2\sqrt{x_1}) \text{ or } K_0(2\sqrt{x_2}). \tag{2.13}$$

$$\text{(iv)}\ K_0(2\sqrt{x_1}) = \frac{1}{2}\int_0^{+\infty} e^{-\left(t+\frac{x}{t}\right)}t^{-1}\,dt \le \frac{1}{2}\left(e^{-1} + \frac{e^{-x_1}}{x_1}\right). \tag{2.14}$$

$$\text{(v)}\ K_0(2\sqrt{x_1}) \approx -\frac{1}{2}\ln x_1, \quad x_1 \to 0^+. \tag{2.15}$$

3. Preliminary results

In the first we collect some notations and results on the KLC-transform analysis . Let $L^p(\mathbb{R}_+, x_1^{-1}dx_1)$, $1 \le p \le \infty$, denotes the space of measurable functions f on $\mathbb{R}_+$ such that $\|f\|_{L^p(\mathbb{R}_+,x_1^{-1}dx_1)} < \infty$, with

$$\|f\|_{L^p(\mathbb{R}_+,x_1^{-1}dx_1)} = \begin{cases} \left(\int_0^\infty |f(x)|^p x_1^{-1}\,dx\right)^{\frac{1}{p}} & \text{if } 1 \le p < \infty, \\ \operatorname{ess\,sup}_{x_1>0} |f(x)| & \text{if } p = \infty. \end{cases} \tag{3.1}$$

$\left(L^p(\mathbb{R}_+, x_1^{-1}dx_1), \|.\|_{L^p(\mathbb{R}_+,x_1^{-1}dx_1)}\right)$ is a Banach space.

From [8],the following resultsare established:

Proposition 3.1 *If the translation operators and the convolution are defined as (2.10) and (2.8) respectively, then we have*

$$\text{(i) } \mathbb{K}(\lambda_{x_1}\Lambda)(\lambda) = K_{2i\sqrt{\lambda}}(2\sqrt{x_1})\mathbb{K}\Lambda(\lambda). \tag{3.2}$$

$$\text{(ii) } \mathbb{K}(\Lambda * \psi)(\lambda) = \mathbb{K}\Lambda(\lambda)\mathbb{K}\psi(\lambda). \tag{3.3}$$

Proposition 3.2 *If $\Lambda, \psi \in L^1(\mathbb{R}_+, x_1^{-1}dx_1)$, then $\mathfrak{T}_{x_1}(\Lambda)$ and $\Lambda * \psi$ belong to $L^1(\mathbb{R}_+, dx_1)$. Furthermore, we have the following estimates:*

$$\text{(i) } \|\mathfrak{T}_{x_1}(\Lambda)\|_{L^1(\mathbb{R}_+,dx)} \le \frac{1}{4}\|\Lambda\|_{L^1(\mathbb{R}_+,x_1^{-1}dx_1)}. \tag{3.4}$$

$$\text{(ii) } \|\Lambda * \psi\|_{L^1(\mathbb{R}_+,dx)} \le \frac{1}{8}\|\Lambda\|_{L^1(\mathbb{R}_+,x_1^{-1}dx_1)}\|\psi\|_{L^1(\mathbb{R}_+,x_1^{-1}dx_1)}. \tag{3.5}$$

Proposition 3.3 *Let $\Lambda \in L^p(\mathbb{R}_+, t^{-1}dt)$, $1 \le p \le \infty$. Then for all $x_1 > 0$, $\mathfrak{T}_{x_1}(\Lambda) \in L^p(\mathbb{R}_+, t^{-1}dt)$ and*

$$\|\mathfrak{T}_{x_1}(\Lambda)\|_{L^p(\mathbb{R}_+,t^{-1}dt)} \le K_0(2\sqrt{x_1})\|\Lambda\|_{L^p(\mathbb{R}_+,t^{-1}dt)}. \tag{3.6}$$

Proof: Using (2.10), Hölder's inequality, (2.12) and (2.13), we have for all $x_1 > 0$,

$$\begin{aligned} |\mathfrak{T}_{x_1}(\Lambda)(y)|^p &\le \left(\frac{1}{2}\right)^p \int_0^\infty |\Lambda(x_3)|^p D(x_1,x_2,x_3)x_3^{-1}\,dx_3 \left(\int_0^\infty D(x_1,x_2,x_3)x_3^{-1}\,dx_3\right)^{\frac{p}{q}} \\ &\le \left(\frac{1}{2}\right)^p (2K_0(2\sqrt{x_1}))^{\frac{p}{q}} \int_0^\infty |\Lambda(x_3)|^p D(x_1,x_2,x_3)x_3^{-1}\,dx_3, \quad y > 0, \end{aligned}$$

if $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Therefore, by the symmetry of $D(x_1, x_2, x_3)$,

$$\begin{aligned} \int_0^\infty |\mathfrak{T}_{x_1}(\Lambda)(y)|^p x_2^{-1}\,dx_2 &\le \left(\frac{1}{2}\right)^p (2K_0(2\sqrt{x_1}))^{\frac{p}{q}} \int_0^\infty |\Lambda(x_3)|^p \int_0^\infty D(x_1,x_2,x_3)x_2^{-1}\,dx_2 x_3^{-1}\,dx_3 \\ &\le \left(\frac{1}{2}\right)^p (2K_0(2\sqrt{x_1}))^{\frac{p}{q}+1} \int_0^\infty |\Lambda(x_3)|^p x_3^{-1}\,dx_3. \end{aligned}$$

Then

$$\|\mathfrak{T}_{x_1}(\Lambda)\|_{L^p(\mathbb{R}_+,t^{-1}dt)} \le K_0(2\sqrt{x_1})\|\Lambda\|_{L^p(\mathbb{R}_+,t^{-1}dt)}.$$

(3.3) is trivial if $p = 1$ or $p = \infty$. □

Proposition 3.4 *If $\Lambda \in L^p(\mathbb{R}_+, x_1^{-1}dx_1)$, $1 \le p < \infty$ and $\psi \in L^1(\mathbb{R}_+, x_1^{-1}dx_1)$, then $\Lambda*\psi \in L^p(\mathbb{R}_+, dx_1)$. Moreover*

$$\|\Lambda * \psi\|_{L^p(\mathbb{R}_+,dx_1)} \le C_p\|\Lambda\|_{L^p(\mathbb{R}_+,x_1^{-1}dx_1)}\|\psi\|_{L^1(\mathbb{R}_+,x_1^{-1}dx_1)}, \tag{3.7}$$

where $C_p > 0$ is a certain constant.

Proof: By using (2.9) and Hölder's inequality, we obtain

$$|\Lambda * \psi(x_1)| \leq \frac{1}{4}\left(\int_0^\infty \int_0^\infty D(x_1,x_2,x_3)|\Lambda(x_3)|^p|\psi(x_2)|x_2^{-1}x_3^{-1}\,dx_2dx_3\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^\infty \int_0^\infty D(x_1,x_2,x_3)|\psi(x_2)|x_2^{-1}x_3^{-1}\,dx_2dx_3\right)^{\frac{1}{q}},$$

if $1 < p < \infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Thus from (2.7), we obtain

$$|\Lambda * \psi(x_1)|^p \leq \left(\frac{1}{4}\right)^p \frac{1}{2}e^{-2\sqrt{x_1}}\left(\int_0^\infty \int_0^\infty |\Lambda(x_3)|^p|\psi(x_2)|x_2^{-1}x_3^{-1}\,dx_2dx_3\right)$$
$$\times\left(\int_0^\infty\left(\int_0^\infty D(x_1,x_2,x_3)x_3^{-1}\,dx_3\right)|\psi(x_2)|x_2^{-1}\,dx_2\right)^{\frac{p}{q}}.$$

From (2.12), (2.13), (2.14) and (2.15), we have

$$\|\Lambda*\psi\|_{L^p(\mathbb{R}_+,dx_1)} \leq C_p\|\Lambda\|_{L^p(\mathbb{R}_+,x_1^{-1}dx_1)}\|\psi\|_{L^1(\mathbb{R}_+,x_1^{-1}dx_1)},$$

where

$$C_p = 2^{-2-\frac{1}{p}+\frac{1}{q}}\left(\int_0^\infty e^{-2\sqrt{x_1}}\left(K_0(2\sqrt{x_1})\right)^{\frac{p}{q}}\,dx_1\right)^{\frac{1}{p}}$$
$$= 2^{-\left(1+\frac{2}{p}\right)}\left(\int_0^\infty e^{-2\sqrt{x_1}}\left(K_0(2\sqrt{x_1})\right)^{p-1}\,dx_1\right)^{\frac{1}{p}} > 0.$$

If $p=1$, from (2.7) we obtain easily (3.7) with $C_1 = \frac{1}{16}$. □

Proposition 3.5 *If $\Lambda \in L^p(\mathbb{R}_+,x_1^{-1}dx_1)$, $1\leq p\leq\infty$ and $\psi \in L^1(\mathbb{R}_+,K_0(2\sqrt{x_1})x_1^{-1}dx_1)$, then $\Lambda*\psi \in L^p(\mathbb{R}_+,x_1^{-1}dx_1)$. Moreover*

$$\|\Lambda*\psi\|_{L^p(\mathbb{R}_+,x_1^{-1}dx_1)} \leq \frac{1}{2}\|\Lambda\|_{L^p(\mathbb{R}_+,x_1^{-1}dx_1)}\|\psi\|_{L^1(\mathbb{R}_+,K_0(2\sqrt{x_1})x_1^{-1}dx_1)}. \tag{3.8}$$

Proof: By using (2.9), Hölder's inequality, (2.12) and (2.13), we obtain

$$|\Lambda * \psi(x_1)|^p \leq \left(\frac{1}{4}\right)^p 2^{\frac{p}{q}}\int_0^\infty \int_0^\infty D(x_1,x_2,x_3)|\Lambda(x_3)|^p|\psi(x_2)|x_2^{-1}x_3^{-1}\,dx_2dx_3$$
$$\times\left(\int_0^\infty K_0(2\sqrt{x_2})|\psi(x_2)|x_2^{-1}\,dx_2\right)^{\frac{p}{q}},$$

if $1<p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Using again (2.12) and (2.13), we obtain

$$\int_0^\infty |\Lambda*\psi(x_1)|^p x_1^{-1}\,dx_1 \leq \left(\frac{1}{2}\right)^p\int_0^\infty\int_0^\infty K_0(2\sqrt{x_2})|\Lambda(x_3)|^p|\psi(x_2)|x_2^{-1}x_3^{-1}\,dx_2dx_3$$
$$\times\left(\int_0^\infty K_0(2\sqrt{x_2})|\psi(x_2)|x_2^{-1}\,dx_2\right)^{\frac{p}{q}}.$$

Then

$$\|\Lambda*\psi\|_{L^p(\mathbb{R}_+,dx_1)} \leq \frac{1}{2}\|\Lambda\|_{L^p(\mathbb{R}_+,x_1^{-1}dx_1)}\|\psi\|_{L^1(\mathbb{R}_+,K_0(2\sqrt{x_1})x_1^{-1}dx_1)}.$$

(3.5) is trivial if $p=1$ or $p=\infty$. □

The Plancherel relation for the KLC-transform can be written as:

$$\frac{1}{2}\int_0^\infty \Lambda(x_1)\overline{\psi(x_1)}x_1^{-1}\,dx_1 = \frac{4}{\pi^2}\int_0^\infty \mathbb{K}\Lambda(\lambda)\overline{\mathbb{K}\psi(\lambda)}\sinh(2\pi\sqrt{\lambda})\,d\lambda \tag{3.9}$$

and the Parseval formula is given by

$$\frac{1}{2}\int_0^\infty |\Lambda(x_1)|^2x_1^{-1}\,dx_1 = \frac{4}{\pi^2}\int_0^\infty |\mathbb{K}\Lambda(\lambda)|^2\sinh(2\pi\sqrt{\lambda})\,d\lambda. \tag{3.10}$$

4. Kontorovich-Lebedev-Clifford continuous wavelet transform

In this section we present basic facts concerning the theory of the Kontorovich-Lebedev-Clifford wavelet transform.

Definition 4.1

1. (KLC-wavelet). A function $\Lambda \in L^2(\mathbb{R}_+, x_1^{-1}dx_1)$ is called Kontorovich-Lebedev-Clifford wavelet (KLC-wavelet) if it satisfies the admissibility condition defined by
$$A_\Lambda = \int_0^\infty \frac{|\mathbb{K}\Lambda(\lambda)|^2}{\lambda}\, d\lambda < \infty. \tag{4.1}$$
2. (Dilation function). For a function $\Lambda \in L^p(\mathbb{R}_+, x_1^{-1}dx_1)$, $1 \le p \le \infty$, we define its dilation function by
$$\mathcal{D}_a(\Lambda)(x_1) = \Lambda(ax_1), \quad a, x_1 > 0. \tag{4.2}$$
3. For $a, b > 0$ and $\Lambda \in L^2(\mathbb{R}_+, x_1^{-1}dx)$, we define the function $\Lambda_{b,a}$ by
$$\Lambda_{b,a}(x_1) = \mathfrak{T}_b(\mathcal{D}_a(\Lambda))(x_1) = \mathfrak{T}_b[\Lambda(ax_1)] \tag{4.3}$$
$$= \frac{1}{2}\int_0^\infty \Lambda(ax_3)D(b, x_1, x_3)x_3^{-1}\, dx_3, \quad x_1 > 0. \tag{4.4}$$

Remark 4.1 If $\Lambda \in L^p(\mathbb{R}_+, x_1^{-1}dx_1)$, $1 \le p \le \infty$, then for all $a > 0$,
$\mathcal{D}_a(\Lambda) \in L^p(\mathbb{R}_+, x_1^{-1}dx_1)$ and $\|\mathcal{D}_a(\Lambda)\|_{L^p(\mathbb{R}_+, x_1^{-1}dx_1)} = \|\Lambda\|_{L^p(\mathbb{R}_+, x_1^{-1}dx_1)}$.
From (3.2), (2.11), Proposition 3.5 and (3.3), we easily obtain the following proposition:

Proposition 4.1 *Let $\Lambda \in L^2(\mathbb{R}_+, x_1^{-1}dx_1)$ be a KLC-wavelet.*

1. *For all $x_1 > 0$, $\mathfrak{T}_{x_1}(\Lambda)$ is a KLC-wavelet and*
$$A_{\mathfrak{T}_{x_1}(\Lambda)} \le (K_0(2\sqrt{x_1}))^2 A_\Lambda.$$
2. *For all $\psi \in L^1(\mathbb{R}_+, K_0(2\sqrt{x_1})x_1^{-1}dx_1)$, $\Lambda * \psi$ is a KLC-wavelet and*
$$A_{\Lambda*\psi} \le \|\psi\|^2_{L^1(\mathbb{R}_+, K_0(2\sqrt{x_1})x_1^{-1}dx_1)} A_\Lambda.$$

Proposition 4.2 *Let $\Lambda \in L^2(\mathbb{R}_+, x_1^{-1}dx_1)$. Then for all $a, b \in \mathbb{R}_+$, we have $\Lambda_{b,a} \in L^2(\mathbb{R}_+, x_1^{-1}dx_1)$ and*
$$\|\Lambda_{b,a}\|_{L^2(\mathbb{R}_+, x_1^{-1}dx_1)} \le K_0(2\sqrt{b})\|\Lambda\|_{L^2(\mathbb{R}_+, x_1^{-1}dx_1)}. \tag{4.5}$$

Proof: Using (4.4), Cauchy-Schwarz inequality, (2.12) and (2.13), we have for all $x_1 > 0$,
$$|\Lambda_{b,a}(x_1)|^2 \le \frac{1}{4}\int_0^\infty |\Lambda(ax_3)|^2 D(b, x_1, x_3)x_3^{-1}\, dx_3 \int_0^\infty D(b, x_1, x_3)x_3^{-1}\, dx_3$$
$$\le K_0(2\sqrt{b})\frac{1}{2}\int_0^\infty |\Lambda(ax_3)|^2 D(b, x_1, x_3)x_3^{-1}\, dx_3.$$

Therefore, by the symmetry of $D(x_1, x_2, x_3)$,
$$\int_0^\infty |\Lambda_{b,a}(x_1)|^2 x_1^{-1}\, dx_1 \le K_0(2\sqrt{b})\frac{1}{2}\int_0^\infty |\Lambda(ax_3)|^2 \int_0^\infty D(b, z, x)x_1^{-1}\, dx_1 x_3^{-1}\, dx_3$$
$$\le (K_0(2\sqrt{b}))^2 \int_0^\infty |\Lambda(ax_3)|^2 x_3^{-1}\, dx_3$$
$$\le (K_0(2\sqrt{b}))^2 \int_0^\infty |\Lambda(x_2)|^2 x_2^{-1}\, dx_2.$$

Then

$$\|\Lambda_{b,a}\|_{L^2(\mathbb{R}_+,x_1^{-1}dx_1)} \leq K_0(2\sqrt{b})\|\Lambda\|_{L^2(\mathbb{R}_+,x_1^{-1}dx_1)}.$$

□

The relations (4.3) and (3.2), readily yield the following result:

Proposition 4.3 *If $\Lambda \in L^2(\mathbb{R}_+, x_1^{-1}dx_1)$ and $(b,a) \in \mathbb{R}_+ \times \mathbb{R}_+$, then the KLC-transform of $\Lambda_{b,a}$ is given as*

$$\mathbb{K}(\Lambda_{b,a})(\lambda) = K_{2i\sqrt{\lambda}}(2\sqrt{b})\mathbb{K}(\mathcal{D}_a(\Lambda))(\lambda), \quad \lambda > 0. \tag{4.6}$$

Definition 4.2 The continuous wavelet transform of a function $f_1 \in L^2(\mathbb{R}_+, x_1^{-1}dx_1)$ with respect to the KLC-wavelet $\Lambda \in L^2(\mathbb{R}_+, x_1^{-1}dx_1)$ is defined by

$$\mathcal{K}_\Lambda(f_1)(b,a) = \int_0^\infty f_1(x)\overline{\Lambda_{b,a}(x_1)}x_1^{-1}\,dx_1, \quad (b,a) \in \mathbb{R}_+ \times \mathbb{R}_+, \tag{4.7}$$

$$= \langle f_1, x_1^{-1}\Lambda_{b,a}\rangle \tag{4.8}$$

$$= \frac{1}{2}\int_0^\infty\int_0^\infty f_1(x)\overline{\Lambda(ax_3)}D(b,x_1,x_3)x_1^{-1}x_3^{-1}\,dx_1dx_3. \tag{4.9}$$

$\mathcal{K}_\Lambda$ is called the Kontorovich-Lebedev-Clifford continuous wavelet transform (KLCWT).

Proposition 4.4 *Let Λ, ψ be two KLC-wavelets, $\alpha, \beta \in \mathbb{C}$, $f_1, f_2 \in L^2(\mathbb{R}_+, x_1^{-1}dx_1)$ and $(b,a) \in \mathbb{R}_+ \times \mathbb{R}_+$. Then the KLCWT satisfies the following properties:*

1. *(Linearity): $\mathcal{K}_\Lambda(\alpha f_1 + \beta f_2))(b,a) = \alpha\mathcal{K}_\Lambda(f_1)(b,a) + \beta\mathcal{K}_\Lambda(f_2)(b,a)$.*
2. *(Anti-linearity): $\mathcal{K}_{\alpha\Lambda+\beta\psi}(f_1)(b,a) = \bar{\alpha}\mathcal{K}_\Lambda(f_1)(b,a) + \bar{\beta}\mathcal{K}_\psi(f_1)(b,a)$.*
3. *(Dilation): $\mathcal{K}_{\mathcal{D}_c(\Lambda)}(f_1)(b,a) = \mathcal{K}_\Lambda(f_1)(b,ac), \; c > 0$.*

Remark 4.2 By using (4.9), (2.10) and (2.8), we obtain for $\Lambda, f_1 \in L^2(\mathbb{R}_+, x_1^{-1}dx_1)$ and $(b,a) \in \mathbb{R}_+ \times \mathbb{R}_+$,

$$\begin{aligned}
\mathcal{K}_\Lambda(f_1)(b,a) &= \int_0^\infty \overline{\Lambda(ax_3)}\left(\frac{1}{2}\int_0^\infty f_1(x)D(b,x_3,x_1)x_1^{-1}\,dx\right)x_3^{-1}\,dx_3 \\
&= \int_0^\infty \mathfrak{T}_b(f_1)(x_3)\overline{\Lambda(ax_3)}x_3^{-1}\,dx_3 \\
&= 2(f_1 * \overline{\mathcal{D}_a(\Lambda)})(b).
\end{aligned} \tag{4.10}$$

Proposition 4.5 *If $f, \Lambda \in L^2(\mathbb{R}_+, x_1^{-1}dx_1)$, then*

$$\mathbb{K}\left(b \mapsto \mathcal{K}_\Lambda(f_1)(b,a)\right)(\lambda) = 2\mathbb{K}f_1(\lambda)\,\overline{\mathbb{K}(\mathcal{D}_a(\Lambda))(\lambda)}, \quad a, \lambda > 0, \tag{4.11}$$

or

$$\begin{aligned}
\mathcal{K}_\Lambda(f_1)(b,a) &= 2\mathbb{K}^{-1}\left(\mathbb{K}f_1\,\overline{\mathbb{K}(\mathcal{D}_a(\Lambda))}\right)(b), \quad a, b > 0, \\
&= \frac{8}{\pi^2}\int_0^\infty K_{2i\sqrt{\lambda}}(2\sqrt{b})\mathbb{K}f_1(\lambda)\overline{\mathbb{K}(\mathcal{D}_a(\Lambda))(\lambda)}\sinh(2\pi\sqrt{\lambda})\,d\lambda.
\end{aligned} \tag{4.12}$$

Proof: By using (4.10) and (3.3), we get (4.11). Then (2.3) gives (4.12). □

5. Reconstruction Formula

Let $\Lambda \in L^2(\mathbb{R}_+, x_1^{-1}dx_1)$ be a KLC-wavelet and q be a positive weight function. We define $C_\Lambda(\lambda),\ \lambda > 0$, by

$$C_\Lambda(\lambda) = 4\int_0^\infty |\mathbb{K}(\mathcal{D}_a(\Lambda))(\lambda)|^2 q(a)a^{-1}\,da. \tag{5.1}$$

Assume that there exist positive constants A and B such that for all $\lambda > 0$,

$$0 < A \leq C_\Lambda(\lambda) \leq B < \infty. \tag{5.2}$$

Definition 5.1 We define for $a, b > 0$, the functions Λ^a and $\Lambda^{b,a}$ by

$$\Lambda^a(x_1) = \mathbb{K}^{-1}\left(\frac{\mathbb{K}(\mathcal{D}_a(\Lambda))}{C_\Lambda}\right)(x_1), \quad x_1 > 0. \tag{5.3}$$

$$\Lambda^{b,a}(x_1) = \mathfrak{T}_b(\Lambda^a)(x_1), \quad x_1 > 0. \tag{5.4}$$

Proposition 5.1

1. *$\Lambda^a, \Lambda^{b,a} \in L^2(\mathbb{R}_+, x_1^{-1}dx_1)$.*
2. $$\mathbb{K}(\Lambda^{b,a})(\lambda) = \frac{\mathbb{K}(\Lambda_{b,a})(\lambda)}{C_\Lambda(\lambda)}, \quad \lambda > 0. \tag{5.5}$$
3. *For $x_1 > 0$ we have*
$$\Lambda^{b,a}(x_1) = \frac{4}{\pi^2}\int_0^\infty K_{2i\sqrt{\lambda}}(2\sqrt{x_1})K_{2i\sqrt{\lambda}}(2\sqrt{b})\frac{\mathbb{K}(\mathcal{D}_a(\Lambda))(\lambda)}{C_\Lambda(\lambda)}\sinh(2\pi\sqrt{\lambda})\,d\lambda. \tag{5.6}$$
4. $$\left\|\Lambda^{b,a}\right\|_{L^2(\mathbb{R}_+, x_1^{-1}dx_1)} \leq A^{-1}\left\|\Lambda_{b,a}\right\|_{L^2(\mathbb{R}_+, x_1^{-1}dx_1)}. \tag{5.7}$$

Proof: From (5.3), Remark 4.2 and the inequality $0 < \frac{1}{C_\Lambda(\lambda)} \leq \frac{1}{A}$, we obtain $\Lambda^a \in L^2(\mathbb{R}_+, x_1^{-1}dx_1)$. So by Proposition 3.3, we get $\Lambda^{b,a} \in L^2(\mathbb{R}_+, x_1^{-1}dx_1)$.
By using (5.4), (3.2), (5.3) and (4.6), we obtain (5.5). So (2.3) gives (5.6).
From (5.5) and the inequality $0 < \frac{1}{C_\Lambda(\lambda)} \leq \frac{1}{A}$, we have

$$\left|\mathbb{K}(\Lambda^{b,a})(\lambda)\right|^2 \leq A^{-2}\left|\mathbb{K}(\Lambda_{b,a})(\lambda)\right|^2, \quad \lambda > 0.$$

Then by (3.10),

$$\left\|\Lambda^{b,a}\right\|_{L^2(\mathbb{R}_+, x_1^{-1}dx_1)} \leq A^{-1}\left\|\Lambda_{b,a}\right\|_{L^2(\mathbb{R}_+, x_1^{-1}dx_1)}.$$

□

Theorem 5.1 *If $f_1 \in L^2(\mathbb{R}_+, x_1^{-1}dx_1)$, then*

$$f_1(x_1) = \int_0^\infty\int_0^\infty \mathcal{K}_\Lambda(f_1)(b,a)\Lambda^{b,a}(x)q(a)a^{-1}b^{-1}\,da db. \tag{5.8}$$

Proof: By using (4.11) and (2.1), we have

$$\int_0^\infty K_{2i\sqrt{\lambda}}(2\sqrt{b})\mathcal{K}_\Lambda(f_1)(b,a)b^{-1}\,db = 4\mathbb{K}f_1(\lambda)\overline{\mathbb{K}(\mathcal{D}_a(\Lambda))(\lambda)}.$$

Multiplying both sides by $q(a)\mathbb{K}(\mathcal{D}_a(\Lambda))(\lambda)$, integrating with respect to $a^{-1}da$ and using (5.1), we obtain

$$\int_0^\infty\int_0^\infty K_{2i\sqrt{\lambda}}(2\sqrt{b})\mathcal{K}_\Lambda(f_1)(b,a)\mathbb{K}(\mathcal{D}_a(\Lambda))(\lambda)q(a)a^{-1}b^{-1}\,dadb=\frac{1}{2}\mathbb{K}f(\lambda)C_\Lambda(\lambda).$$

Thus by using (4.6), we obtain

$$\mathbb{K}f_1(\lambda)=\frac{1}{C_\Lambda(\lambda)}\int_0^\infty\int_0^\infty \mathcal{K}_\Lambda(f_1)(b,a)\mathbb{K}(\Lambda_{b,a})(\lambda)q(a)a^{-1}b^{-1}\,dadb.$$

From (5.5), we have

$$\mathbb{K}f_1(\lambda)=\int_0^\infty\int_0^\infty \mathcal{K}_\Lambda(f_1)(b,a)\mathbb{K}(\Lambda^{b,a})(\lambda)q(a)a^{-1}b^{-1}\,dadb.$$

Now by the using inversion formula (2.3), we get

$$f_1(x)=\int_0^\infty\int_0^\infty \mathcal{K}_\Lambda(f_1)(b,a)\Lambda^{b,a}(x)q(a)a^{-1}b^{-1}\,dadb.$$

□

6. Plancherel and Parseval relations for the KLCWT

Let $\Lambda,\psi\in L^2(\mathbb{R}_+,x_1^{-1}dx_1)$ be two KLC-wavelets and q be a positive weight function. We define $C_{\Lambda,\psi}(\lambda),\ \lambda>0$, by

$$C_{\Lambda,\psi}(\lambda)=4\int_0^\infty \mathbb{K}(\mathcal{D}_a(\Lambda))(\lambda)\overline{\mathbb{K}(\mathcal{D}_a(\psi))(\lambda)}q(a)a^{-1}\,da. \tag{6.1}$$

Assume that for all $\lambda>0$,

$$C_{\Lambda,\psi}(\lambda)=C_{\Lambda,\psi}<\infty. \tag{6.2}$$

Theorem 6.1 *(Plancherel relation). If $f_1,f_2\in L^2(\mathbb{R}_+,x_1^{-1}dx_1)$, then*

$$\int_0^\infty\int_0^\infty \mathcal{K}_\Lambda(f_1)(b,a)\overline{\mathcal{K}_\psi(f_2)(b,a)}q(a)a^{-1}b^{-1}\,dadb=C_{\psi,\Lambda}\langle f_1,x_1^{-1}f_2\rangle. \tag{6.3}$$

Proof: By using (3.9) and (4.11), we get

$$\begin{aligned}&\int_0^\infty \mathcal{K}_\Lambda(f_1)(b,a)\overline{\mathcal{K}_\psi(f_2)(b,a)}b^{-1}\,db\\&=2\frac{4}{\pi^2}\int_0^\infty 2\mathbb{K}f_1(\lambda)\overline{\mathbb{K}(\mathcal{D}_a(\Lambda))(\lambda)}2\overline{\mathbb{K}f_2(\lambda)}\mathbb{K}(\mathcal{D}_a(\psi))(\lambda)\sinh(2\pi\sqrt{\lambda})\,d\lambda.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_0^\infty\int_0^\infty \mathcal{K}_\Lambda(f_1)(b,a)\overline{\mathcal{K}_\psi(f_2)(b,a)}q(a)a^{-1}b^{-1}\,dadb\\&=\frac{32}{\pi^2}\int_0^\infty \mathbb{K}f_1(\lambda)\overline{\mathbb{K}f_2(\lambda)}\int_0^\infty \mathbb{K}(\mathcal{D}_a(\psi))(\lambda)\overline{\mathbb{K}(\mathcal{D}_a(\Lambda))(\lambda)}q(a)a^{-1}\,da\,\sinh(2\pi\sqrt{\lambda})\,d\lambda\\&=C_{\psi,\Lambda}\frac{8}{\pi^2}\int_0^\infty \mathbb{K}f_1(\lambda)\overline{\mathbb{K}f_2(\lambda)}\sinh(2\pi\sqrt{\lambda})\,d\lambda.\end{aligned}$$

Hence by using (3.9), we get

$$\begin{aligned}\int_0^\infty\int_0^\infty \mathcal{K}_\Lambda(f_1)(b,a)\overline{\mathcal{K}_\psi(f_2)(b,a)}q(a)a^{-1}b^{-1}\,dadb&=C_{\psi,\Lambda}\int_0^\infty f_1(x)\overline{f_1(x)}x_1^{-1}\,dx_1\\&=C_{\psi,\Lambda}\langle f_1,x_1^{-1}f_2\rangle.\end{aligned}$$

□

Remark 6.1 We deduce the following relations:

1. If $\Lambda = \psi$, then

$$\int_0^\infty \int_0^\infty \mathcal{K}_\Lambda(f_1)(b,a)\overline{\mathcal{K}_\Lambda(f_2)(b,a)}q(a)a^{-1}b^{-1}\, dadb = C_\Lambda \langle f_1, x_1^{-1} f_2\rangle.$$

2. If $f_1 = f_2$, then

$$\int_0^\infty \int_0^\infty \mathcal{K}_\Lambda(f_1)(b,a)\overline{\mathcal{K}_\psi(f_1)(b,a)}q(a)a^{-1}b^{-1}\, dadb = C_{\psi,\Lambda}\|f_1|^2_{L^2(\mathbb{R}_+, x_1^{-1}dx_1)}.$$

3. If $f_1 = f_2$ and $\Lambda = \psi$, then

$$\int_0^\infty \int_0^\infty |\mathcal{K}_\Lambda(f_1)(b,a)|^2 q(a)a^{-1}b^{-1}\, dadb = C_\Lambda\|f_1|^2_{L^2(\mathbb{R}_+, x_1^{-1}dx_1)}.$$

7. Localization operators

In section 6, we provide an explicit form of the continuized KLC-wavelet transform at time t and define localization operators.In this context, after recalling the definition of the Schatten-von Neumann classes from quantum mechanics, we point out that localization operators belong this class.The proof of this fact is provided in the same section and inspired by a work of one of the authors in collaboration with other colleagues [24]. Hereafter, Λ will denote an admissible KLCW-Let subdir such that $\|\Lambda\|_{L^2_\mu(\mathbb{R}_+)} = 1$.

7.1. Preliminaries

Notation We denote by:

- $l^p(\mathbb{N})$ the set of all infinite sequences of real (or complex) numbers $e := (e_j)_{j\in\mathbb{N}}$ such that

$$\|e\|_p := \left(\sum_{j=1}^\infty |e_j|^p\right)^{\frac{1}{p}} < \infty, \quad \text{if } 1 \le p < \infty$$
$$\|e\|_\infty := \sup_{j\in\mathbb{N}} |e_j| < \infty$$

For $p = 2$, we provide this space $l^2(\mathbb{N})$ with the scalar product

$$\langle e, v\rangle_{L^2_\mu(\mathbb{R}_+)} := \sum_{j=1}^\infty e_j\overline{v_j}$$

$B\left(L^2_\mu\left(\mathbb{R}_+\right)\right)$the space of bounded operators from $L^2_\mu\left(\mathbb{R}_+\right)$into itself.

- Let $L^p_v\left(\mathbb{R}_+ \times \mathbb{R}_+\right), 1 \le p \le \infty$, denotes the space of measurable functions f on $\mathbb{R}_+ \times \mathbb{R}_+$ satisfying:

$$\|f\|_{L^p_v(\mathbb{R}_+\times\mathbb{R}_+)} = \|f\|_{p,v} = \begin{cases} \left(\int_0^\infty \int_0^\infty |f(b,a)|^p dv(b,a)\right)^{1/p}, & 1 \le p < \infty \\ \text{ess } \sup_{(b,a)\in\mathbb{R}_+\times\mathbb{R}_+} |f(b,a)|, & p = \infty. \end{cases}$$

  where $dv(b,a)$ the measure defined on $\mathbb{R}_+ \times \mathbb{R}_+$ by

$$dv(b,a) = q(b)b^{-1}a^{-1}dbda$$

- $d\mu(x_1)$ the measure defined on $\mathbb{R}_+$ by

$$d\mu(x_1) = x_1^{-1}dx_1.$$

Definition 7.1 (i) The singular values $(s_n(X))_{n\in\mathbb{N}}$ of a compact operator X in $B\left(L^2_\mu(\mathbb{R}_+)\right)$are the eigenvalues of the positive self-adjoint operator $|X| = \sqrt{X^*X}$.

(ii) For $1 \leq p < \infty$, the Schatten class S_p is the space of all compact operators whose singular values lie in $l^p(\mathbb{N})$. The space S_p is equipped with the norm

$$\|X\|_{S_p} := \left(\sum_{n=1}^{\infty} (s_n(X))^p\right)^{\frac{1}{p}}$$

Remark 7.1 We notice that the space of trace class operators is S_1, and the space of Hilbert-Schmidt operators is S_2.

Definition 7.2 The definition of the trace of an operator$A$ in S_1 is

$$\operatorname{tr}(X) = \sum_{n=1}^{\infty} \langle Xv_n, v_n\rangle_{L^2_\mu(\mathbb{R}_+)} \tag{7.1}$$

where $(v_n)_n$ is any orthonormal basis of $L^2_\mu(\mathbb{R}_+)$.

Remark 7.2 If X is positive, then

$$\operatorname{tr}(X) = \|X\|_{S_1}$$

Moreover, a compact operator X on the Hilbert space $L^2_\mu(\mathbb{R}_+)$is a Hilbert-Schmidt operator if the positive operator X^*X is in the space of trace class S_1. Then

$$\|X\|^2_{HS} := \|X\|^2_{S_2} = \|X^*X\|_{S_1} = \operatorname{tr}(X^*X) = \sum_{n=1}^{\infty} \|Xv_n\|^2_{L^2_\mu(\mathbb{R}_+)}$$

for any orthonormal basis $(v_n)_n$ of $L^2_\mu(\mathbb{R}_+)$.

Definition 7.3 We define $S_\infty := B\left(L^2_\mu(\mathbb{R}_+)\right)$, equipped with the norm

$$\|X\|_{S_\infty} := \sup_{v\in L^2_\mu(\mathbb{R}_+):\|v\|_{L^2_\mu(\mathbb{R}_+)}=1} \|Xv\|_{L^2_\mu(\mathbb{R}_+)}$$

7.2. Boundedness

In this subsection we characterize the localization operators for the continuous KLCW transform and prove that they are bounded.

Definition 7.4 For the continuous KLC-wavelet transform, the localization operator with symbol ρ, represented by $\xi_\Lambda(\rho)$ is defined on $L^2_\mu(\mathbb{R}_+)$ by

$$\xi_\Lambda(\rho)(f_1)(x_2) = \frac{1}{C_\Lambda}\int_0^\infty\int_0^\infty \rho(b,a)\mathcal{K}_\Lambda f_1(b,a)\Lambda_{b,a}(x_2)dv(b,a), \quad x_2 \in \mathbb{R}_+ \tag{7.2}$$

The definition of $\xi_\Lambda(\rho)$ is frequently easier to interpret in a weak sense, meaning that for f,g in $L^2_\mu(\mathbb{R}_+)$ we have

$$\langle \xi_\Lambda(\rho)(f_1), f_2\rangle_{L^2_\mu(\mathbb{R}_+)} = \frac{1}{C_\Lambda}\int_0^\infty\int_0^\infty \rho(b,a)\mathcal{K}_\Lambda f_1(b,a)\overline{\mathcal{K}_\Lambda(f_2)(b,a)}dv(b,a) \tag{7.3}$$

In this subsection we prove that the linear operators

$$\xi_\Lambda(\rho) : L^2_\mu(\mathbb{R}_+) \to L^2_\mu(\mathbb{R}_+)$$

are bounded for all symbols ρ in $L^p_v(\mathbb{R}_+ \times \mathbb{R}_+), 1 \leq p \leq \infty$. We consider first this problem for ρ in $L^1_v(\mathbb{R}_+ \times \mathbb{R}_+)$and next in $L^\infty_v(\mathbb{R}_+ \times \mathbb{R}_+)$and we conclude using the interpolation theory.

Proposition 7.1 *Let ρ be in $L^1_v(\mathbb{R}_+ \times \mathbb{R}_+)$. Then the localization operator $\xi_\Lambda(\rho)$ is in S_∞ and we have*

$$\|\xi_\Lambda(\rho)\|_{S_\infty} \leqslant \frac{(K_0(2\sqrt{b}))^2}{C_\Lambda}\|\rho\|_{v,1}$$

Proof: For all functions f_1 and f_2 in $L^2_\mu(\mathbb{R}_+)$, we have the following relations from (4.2)(7.3)

$$\begin{aligned}\left|\langle\xi_\Lambda(\rho)(f_1), f_2\rangle_{L^2_\mu(\mathbb{R}_+)}\right| &\leqslant \frac{1}{C_\Lambda}\int_0^\infty\int_0^\infty |\rho(b,a)|\mathcal{K}_\Lambda(f_1)(b,a)\|\overline{\mathcal{K}_\Lambda(f_2)(b,a)}\Big|\, dv(b,a)\\ &\leqslant \frac{1}{C_\Lambda}\|\mathcal{K}_\Lambda(f_1)\|_{v,\infty}\|\mathcal{K}_\Lambda(f_2)\|_{v,\infty}\|\rho\|_{v,1}\\ &\leqslant \frac{(K_0(2\sqrt{b}))^2}{C_\Lambda}\|f_1\|_{L^2_\mu(\mathbb{R}_+)}\|f_2\|_{L^2_\mu(\mathbb{R}_+)}\|\rho\|_{v,1}.\end{aligned}$$

Thus,

$$\|\xi_\Lambda(\rho)\|_{S_\infty} \leqslant \frac{(K_0(2\sqrt{b}))^2}{C_\Lambda}\|\rho\|_{v,1}$$

□

Theorem 7.1 *Let ρ be in $L^\infty_v(\mathbb{R}_+ \times \mathbb{R}_+)$. Then the localization operator $\xi_\Lambda(\rho)$ is in S_∞ and we have*

$$\|\xi_\Lambda(\rho)\|_{S_\infty} \leqslant \|\rho\|_{v,\infty} \tag{7.4}$$

Proof: For all functions f_1 and f_2 in $L^2_\mu(\mathbb{R}^+)$, we have from Holder's inequality:

$$\begin{aligned}\left|\langle\xi_\Lambda(\rho)(f_1), f_2\rangle_{L^2_\mu(\mathbb{R}_+)}\right| &\leqslant \frac{1}{C_\Lambda}\int_0^\infty\int_0^\infty |\rho(b,a)\,\|\mathcal{K}_\Lambda(f_1)(b,a)\|\,\mathcal{K}_\Lambda(f_2)(b,a)|\, dv(b,a)\\ &\leqslant \frac{1}{C_\Lambda}\|\rho\|_{v,\infty}\|\mathcal{K}_\Lambda(f_1)\|_{v,2}\|\mathcal{K}_\Lambda(f_2)\|_{v,2}\end{aligned}$$

Using Plancherel's formula for $\mathcal{K}_\Lambda$, given by the relation (51), we get

$$\left|\langle\xi_\Lambda(\rho)(f_1), f_2\rangle_{L^2_\mu(\mathbb{R}_+)}\right| \leqslant \|\rho\|_{v,\infty}\|f_1\|_{L^2_\mu(\mathbb{R}_+)}\|f_2\|_{L^2_\mu(\mathbb{R}_+)}$$

Thus,

$$\|\xi_\Lambda(\rho)\|_{S_\infty} \leqslant \|\rho\|_{v,\infty}$$

□

Theorem 7.2 *Let ρ be in $L^p_v(\mathbb{R}_+ \times \mathbb{R}_+), 1 \leq p \leq \infty$. Then there exists a unique bounded linear operator $\xi_\Lambda(\rho) : L^2_\mu(\mathbb{R}_+) \to L^2_\mu(\mathbb{R}_+)$such that*

$$\|\xi_\Lambda(\rho)\|_{S_\infty} \leqslant \left(\frac{(K_0(2\sqrt{b}))^2}{C_\Lambda}\right)^{\frac{1}{p}}\|\rho\|_{v,p}$$

Proof: Let f be in $L^2_\mu(\mathbb{R}_+)$. We consider the operator

$$\mathcal{T}: L^1_v(\mathbb{R}_+\times\mathbb{R}_+)\cap L^\infty_v(\mathbb{R}_+\times\mathbb{R}_+)\to L^2_v(\mathbb{R}_+\times\mathbb{R}_+)$$

given by

$$\mathcal{T}(\rho) := \xi_\Lambda(\rho)(f_1)$$

Then, by Proposition 6.7 and theorem 6.8 ,

$$\|\mathcal{T}(\rho)\|_{L^2_\mu(\mathbb{R}_+)} \leq \frac{(K_0(2\sqrt{b}))^2}{C_\Lambda}\|f_1\|_{L^2_\mu(\mathbb{R}_+)}\|\rho\|_{v,1} \tag{7.5}$$

and

$$\|\mathcal{T}(\rho)\|_{L^2_\mu(\mathbb{R}_+)} \leq \|f_1\|_{L^2_\mu(\mathbb{R}_+)}\|\rho\|_{v,\infty} \tag{7.6}$$

□

Therefore, by (6.5), (6.6) and the Riesz-Thorin interpolation theorem (see Ref. [16], Theorem 2] and [[19], Theorem 2.11]), $\mathcal{T}$ may be uniquely extended to a linear transformation on $L^p_v(\mathbb{R}_+\times\mathbb{R}_+)$, and we have

$$\|\xi_\Lambda(\rho)(f_1)\|_{L^2_\mu(\mathbb{R}_+)} = \|\mathcal{T}(\rho)\|_{L^2_\mu(\mathbb{R}_+)} \leq \left(\frac{(K_0(2\sqrt{b}))^2}{C_\Lambda}\right)^{\frac{1}{p}}\|f\|_{L^2_\mu(\mathbb{R}_+)}\|\rho\|_{v,p} \tag{7.7}$$

Since (6.7) is true for arbitrary functions f in $L^2_\mu(\mathbb{R}_+)$, we obtain the desired result.

7.3. Schatten-von Neumann properties for $\xi_\Lambda(\boldsymbol{\rho})$

In this subsection we will demonstrate that, the localization operator $\xi\Lambda(\rho)$ belongs to Schatten class Sp provided $\rho\in L^p_v(\mathbb{R}+\times\mathbb{R}+), 1\leq p<+\infty$. We already know the result for $p=+\infty$ (by the theorem 3.1), so the proof for any $p\geq 1$ follows by proving it for $p=1$ and using interpolation theory.

Theorem 7.3 *Let ρ be in $L^1_v(\mathbb{R}_+\times\mathbb{R}_+)$. Then the bounded localization operator*

$$\xi_\Lambda(\rho): L^2_\mu(\mathbb{R}_+)\to L^2_\mu(\mathbb{R}_+)$$

is in S_1 and we have

$$\|\xi_\Lambda(\rho)\|_{S_1} \leqslant \frac{4(K_0(2\sqrt{b}))^2}{C_\Lambda}\|\rho\|_{v,1}$$

Proof: First let us assume that ρ is a non negative real-valued symbol, so the localization operator $\xi_\Lambda(\rho)$ is positive. Let $\{u_j, j=1,2,\ldots\}$ be any orthonormal basis for $L^2_\mu(\mathbb{R}_+)$. Then applying Fubini's theorem, the Parseval identity, and relations (3.5) and (3.7), we get

$$\begin{aligned}\sum_{j=1}^{\infty}\langle\xi_\Lambda(\rho)(u_j),u_j\rangle_{L^2_\mu(\mathbb{R}_+)} &= \sum_{j=1}^{\infty}\frac{1}{C_\Lambda}\int_0^\infty\int_0^\infty\rho(b,a)\left|\mathcal{K}_\Lambda(u_j)(b,a)\right|^2 dv(b,a)\\ &= \frac{1}{C_\Lambda}\int_0^\infty\int_0^\infty\rho(b,a)\left(\sum_{j=1}^{\infty}\left|\mathcal{K}_\Lambda(u_j)(b,a)\right|^2\right)dv(b,a)\end{aligned}$$

Thus we get

$$\sum_{j=1}^{\infty}\langle\xi_\Lambda(\rho)(u_j),u_j\rangle_{L^2_\mu(\mathbb{R}_+)} = \frac{1}{C_\Lambda}\int_0^\infty\int_0^\infty\rho(b,a)\left\|\Lambda_{b,a}\right\|^2_{L^2_\mu(\mathbb{R}_+)}dv(b,a) \tag{7.8}$$

Using now the relation (3.7), we deduce that

$$\sum_{j=1}^{\infty}\langle\xi_{\Lambda}(\rho)\,(u_j)\,,u_j\rangle_{L^2_{\mu}(\mathbb{R}_+)} \leqslant \sup_{(b,a)\in\mathbb{R}_+^{d+1}}\|\Lambda_{b,a}\|^2_{L^2_{\mu}(\mathbb{R}_+)}\frac{1}{C_{\Lambda}}\|\rho\|_{v,1}$$
$$=\frac{(K_0(2\sqrt{b}))^2}{C_{\Lambda}}\|\rho\|_{v,1}$$

Then, by [[19], Proposition 2.4], the operator $\xi_{\Lambda}(\rho)$ is in S_1.

We have $\sqrt{\xi_{\Lambda}(\rho)^*\xi_{\Lambda}(\rho)} = \xi_{\Lambda}(\rho)$, so if we consider $\{u_j, j=1,2,\ldots\}$ an orthonormal basis for $L^2_{\mu}(\mathbb{R}_+)$consisting of eigenvectors of the positive compact operator $\sqrt{\xi_{\Lambda}(\rho)^*\xi_{\Lambda}(\rho)}$ and let $s_j, j=1,2,\ldots$, be the eigenvalues of $|\xi_{\Lambda}(\rho)|$ corresponding to u_j, then

$$\|\xi_{\Lambda}(\rho)\|_{S_1}=\sum_{j=1}^{\infty}s_j=\sum_{j=1}^{\infty}\left\langle\sqrt{\xi_{\Lambda}(\rho)^*\xi_{\Lambda}(\rho)}\,(u_j)\,,u_j\right\rangle_{L^2_{\mu}(\mathbb{R}_+)}$$
$$=\sum_{j=1}^{\infty}\langle\xi_{\Lambda}(\rho)\,(u_j)\,,u_j\rangle_{L^2_{\mu}(\mathbb{R}_+)}\leqslant\frac{(K_0(2\sqrt{b}))^2}{C_{\Lambda}}\|\rho\|_{v,1}.$$

For ρ a real-valued function, we write $\rho=\rho_+-\rho_-$, with

$$\rho_+=\max(\rho,0),\quad \rho_-=-\min(\rho,0)$$

then $\xi_{\Lambda}(\rho)$ is in S_1 and we have

$$\|\xi_{\Lambda}(\rho)\|_{S_1}\leqslant\|\xi_{\Lambda}(\rho_+)\|_{S_1}+\|\xi_{\Lambda}(\rho_-)\|_{S_1}\leqslant\frac{2(K_0(2\sqrt{b}))^2}{C_{\Lambda}}\|\rho\|_{v,1}$$

Finally, when $\rho=\rho_1+i\rho_2$ is a complex-valued function with ρ_1 and ρ_2 the real and imaginary parts of ρ, we have that $\xi_{\Lambda}(\rho)$ is in S_1 and

$$\|\xi_{\Lambda}(\rho)\|_{S_1}\leqslant\|\xi_{\Lambda}(\rho_1)\|_{S_1}+\|\xi_{\Lambda}(\rho_2)\|_{S_1}\leqslant\frac{4(K_0(2\sqrt{b}))^2}{C_{\Lambda}}\|\rho\|_{v,1}$$

□

Corollary 7.1 *For ρ in $L^1_v(\mathbb{R}_+\times\mathbb{R}_+)$, we have the trace formula*

$$\operatorname{tr}(\xi_{\Lambda}(\rho))=\frac{1}{C_{\Lambda}}\int_0^{\infty}\int_0^{\infty}\rho(b,a)\,\|\Lambda_{b,a}\|^2_{L^2_{\mu}(\mathbb{R}^+)}\,dv(b,a)$$

Proof: From the previous theorem, the localization operator $\xi_{\Lambda}(\rho)$ belongs to S_1; then by the definition of trace given by the relation (6.1), we have

$$\operatorname{tr}(\xi_{\Lambda}(\rho))=\sum_{j=1}^{\infty}\langle\xi_{\Lambda}(\rho)\,(u_j)\,,u_j\rangle_{L^2_{\mu}(\mathbb{R}^+)}$$

The result is obtained by the relation theorem 6.10.

□

Proposition 7.2 *Let ρ be a symbol in $L^p_v(\mathbb{R}_+\times\mathbb{R}_+)\,,1\leqslant p<\infty$. Then the localization operator $\xi_{\Lambda}(\rho)$ is compact.*

Proof: Let ρ be in $L_v^p(\mathbb{R}_+ \times \mathbb{R}_+)$and let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L_v^1(\mathbb{R}_+ \times \mathbb{R}_+) \cap L_v^p(\mathbb{R}_+ \times \mathbb{R}_+)$such that $\rho_n \to \rho$ in $L_v^p(\mathbb{R}_+)$as $n \to \infty$. Then by Theorem 6.9

$$\|\xi_\Lambda(\rho_n) - \xi_\Lambda(\rho)\|_{S_\infty} \leq \left(\frac{(K_0(2\sqrt{b}))^2}{C_\Lambda}\right)^{\frac{1}{p}} \|\rho_n - \rho\|_{v,p}$$

Hence $\xi_\Lambda(\rho_n) \to \xi_\Lambda(\rho)$ in S_∞ since $n \to \infty$. On the other hand, as by Theorem 6.10 $\xi_\Lambda(\rho_n)$ is in S_1, and therefore compact, it follows that $\xi_\Lambda(\rho)$ is compact.

In the following theorem we improve the constant given in Theorem 6.10. First, we begin by investigating the case ρ in $L_v^1(\mathbb{R}_+ \times \mathbb{R}_+)$and , in addition, provide a lower bound of the norm $\|\xi_\Lambda(\rho)\|_{S_1}$. □

Theorem 7.4 *Let ρ be in $L_v^1(\mathbb{R}_+ \times \mathbb{R}_+)$. Then,*

$$\frac{1}{C_\Lambda}\|\widetilde{\rho}\|_{v,1} \leqslant \|\xi_\Lambda(\rho)\|_{S_1} \leqslant \frac{1}{C_\Lambda}\|\rho\|_{v,1}$$

where $\tilde{\rho}$ is given by

$$\tilde{\rho}(b,a) = \langle \xi_\Lambda(\rho)(\Lambda_{b,a}), \Lambda_{b,a}\rangle_{L_\mu^2(\mathbb{R}^+)}, \quad (b,a) \in \mathbb{R}_+ \times \mathbb{R}_+$$

Proof: Since ρ is in $L_v^1(\mathbb{R}_+ \times \mathbb{R}_+)$, by Theorem 3.8$\xi_\Lambda(\rho)$ is in S_1. Using [[16], Theorem 2.2], there exists an orthonormal basis $\{u_j, j = 1, 2, \ldots\}$ for $N(\xi_\Lambda(\rho))^\perp$, the orthogonal complement of the kernel of $\xi_\Lambda(\rho)$, consisting of eigenvectors of $|\xi_\Lambda(\rho)|$, and $\{v_j, j = 1, 2, \ldots\}$ an orthonormal set in $L_\mu^2(\mathbb{R}^+)$, such that

$$\xi_\Lambda(\rho)(f_1) = \sum_{j=1}^{\infty} s_j \langle f, u_j\rangle_{L_\mu^2(\mathbb{R}^+)} v_j \tag{7.9}$$

where $s_j, j = 1, 2, \ldots$, are the positive singular values of $\xi_\Lambda(\rho)$ corresponding to u_j. Then we get

$$\|\xi_\Lambda(\rho)\|_{S_1} = \sum_{j=1}^{\infty} s_j = \sum_{j=1}^{\infty} (\xi_\Lambda(\rho)(u_j), v_j)_{L_\mu^2(\mathbb{R}^+)}$$

Thus, by applying Fubini's theorem, Schwarz's inequality, Bessel's inequality, and the relations (3.7) and (3.5), we obtain

$$\begin{aligned}
&\|\xi_\Lambda(\rho)\|_{S_1} = \sum_{j=1}^{\infty} \langle \xi_\Lambda(\rho)(u_j), v_j\rangle_{L_\mu^2(\mathbb{R}^+)} \\
&= \sum_{j=1}^{\infty} \frac{1}{C_\Lambda} \int_0^\infty \int_0^\infty \rho(b,a)\mathcal{K}_\Lambda(u_j)(b,a)\overline{\mathcal{K}_\Lambda(v_j)(b,a)}dv(b,a) \\
&\leqslant \frac{1}{C_\Lambda} \int_0^\infty \int_0^\infty |\rho(b,a)| \left(\sum_{j=1}^{\infty} |\mathcal{K}_\Lambda(u_j)(b,a)|^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{\infty} |\mathcal{K}_\Lambda(v_j)(b,a)|^2\right)^{\frac{1}{2}} dv(b,a) \\
&\leqslant \frac{1}{C_\Lambda} \int_0^\infty \int_0^\infty |\rho(b,a)| \, \|\Lambda_{b,a}\|^2_{L_\mu^2(\mathbb{R}^+)} dv(b,a) \\
&\leqslant \frac{(K_0(2\sqrt{b}))^2}{C_\Lambda} \|\rho\|_{v,1}
\end{aligned}$$

It is easy to see that $\tilde{\rho}$ belongs to $L_v^1(\mathbb{R}_+ \times \mathbb{R}_+)$, and using formula (6.13) we obtain

$$
\begin{aligned}
|\widetilde{\rho}(b,a)| &= \left|\langle \xi_\Lambda(\rho)\left(\Lambda_{b,a}\right), \Lambda_{b,a}\rangle_{L^2_\mu(\mathbb{R}_+)}\right| \\
&= \left|\sum_{j=1}^{\infty} s_j \left\langle \Lambda_{b,a}, u_j\right\rangle_{L^2_\mu(\mathbb{R}_+)} \left(v_j, \Lambda_{b,a}\right\rangle_{L^2_\mu(\mathbb{R}_+)}\right| \\
&\leqslant \frac{1}{2}\sum_{j=1}^{\infty} s_j \left(\left|\left(\Lambda_{b,a}, u_j\right\rangle_{L^2_\mu(\mathbb{R}_+)}\right|^2 + \left|\left\langle \Lambda_{b,a}, v_j\right\rangle_{L^2_\mu(\mathbb{R}_+)}\right|^2\right)
\end{aligned}
$$

Then using Plancherel's identity for $\mathcal{K}_\Lambda$ and Fubini's theorem, we get

$$
\begin{aligned}
\int_0^\infty \int_0^\infty |\widetilde{\rho}(b,a)| dv(b,a) =& \frac{1}{2}\sum_{j=1}^{\infty} s_j \left(\int_0^\infty \int_0^\infty \left|\left\langle \Lambda_{b,a}, u_j\right\rangle_{L^2_\mu(\mathbb{R}_+)}\right|^2 dv(b,a)\right. \\
&\left. + \int_0^\infty \int_0^\infty \left|\left\langle \Lambda_{b,a}, v_j\right\rangle_{L^2_\mu(\mathbb{R}_+)}\right|^2 dv(b,a)\right) \\
&\leqslant \frac{C_\Lambda}{(K_0(2\sqrt{b}))^2} \sum_{j=1}^{\infty} s_j = C_\Lambda \left\|\xi_\Lambda(\rho)\right\|_{S_1}.
\end{aligned}
$$

The proof is complete.
In the following theorem we present the main result of this section. □

Theorem 7.5 *Let ρ be in $L^p_v(\mathbb{R}_+ \times \mathbb{R}_+)$, $1 \leqslant p \leqslant \infty$. Then the localization operator*

$$\xi_\Lambda(\rho) : L^2_\mu(\mathbb{R}_+) \to L^2_\mu(\mathbb{R}_+)$$

is in S_p and we have

$$\left\|\xi_\Lambda(\rho)\right\|_{S_p} \leqslant \left(\frac{(K_0(2\sqrt{b}))^2}{C_\Lambda}\right)^{\frac{1}{p}} \|\rho\|_{v,p}.$$

Moreover, $\xi_\Lambda(\rho)$ satisfies the relation (6.3).

Proof: The result follows from Proposition 6.4 , Theorem 6.13 and by interpolation (see Ref. [19], Theorem 2.10 and Theorem 2.11). □

8. Conclusion and Perspectives

In this paper, we developed a comprehensive framework for continuous wavelet transforms associated with the Kontorovich–Lebedev–Clifford (KLC) transform. Our approach built upon the convolution and translation operators arising from the KLC setting and led to several key results, including a precise reconstruction formula, Plancherel and Parseval-type identities, and admissibility conditions for wavelets in the Clifford context.

Furthermore, we introduced localization operators related to the KLC-wavelet transform, demonstrated their boundedness and compactness, and showed that they belong to the Schatten-von Neumann class. Trace formulas were also established, confirming the robustness of this framework for operator-theoretic analysis.

Several perspectives emerge from this work. The structure developed here opens the way to studying Toeplitz operators in the KLC-wavelet context, with potential applications to spectral theory and time-frequency analysis. Additionally, the framework can be extended to develop scalogram analysis for high-dimensional signal representations. Future research may also explore numerical algorithms for the implementation of KLC-wavelet transforms in signal processing, quantum mechanics, and medical imaging, especially in scenarios requiring precise localization in both space and frequency.

Finally, it would be interesting to investigate connections between the KLC-wavelet transform and other integral transforms—such as the Dunkl, Mehler–Fock, or Whittaker transforms—especially regarding localization and operator properties.

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