Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44)** : 1–8.
ISSN-0037-8712
doi:10.5269/bspm.77120

On Smirnov-Type Inequalities for Polynomials

Bilal Dar and Abdul Liman

ABSTRACT: In this paper, we have studied the Smirnov operator for complex polynomials and obtained a refinement of an inequality related to its modified version. In addition to this, we have derived several results that serve as refinements of earlier findings on Bernstein-type inequalities. These results not only extend the existing work but also provide sharper estimates that may prove useful in further research.

Key Words: Bernstein inequality, Smirnov operator, polynomials, restricted zeros.

Contents

1. Introduction

After some years when the Chemist Mendeleev invented the periodic table of the elements, he made a study of the specific gravity of a solution as a function of the percentage of the dissolved substance(see [14]). This function is of some practical importance: for example it is used in testing bear and wine for alcoholic content and in testing the cooling system of an automobile for concentration of antifreeze. Mendeleev's study led to mathematical problems of great interest, some of which are even today inspiring research in mathematics. In mathematical terms Mendeleev's problem amounts,:: If the bound of a rational polynomial of positive degree n over a given interval is known, how large may be its derivative on the same interval?". In other words, Mendeleev's problem state that;
"Let $B \subset \mathbb{C}$ be a compact set, $f(z)$ be a polynomial with $deg(f) = n \geq 1$, and suppose that $|f(z)| \leq M$ for all $z \in B$. Determine an estimate for $|f'(z)|$ for $z \in B$."
Mendeleev considered only real polynomials of degree two and the compact set $B = [a,\ b]$. The problem was presented in the general form in [18]. Mendeleev was himself able to solve the problem [13], in fact he found that $-4 \leq f'(x) \leq 4$ for $x \in [-1,\ 1]$ and $f(x) = 1 - 2x^2$ is the extremal polynomial. Mendeleev conveyed this result to A.A. Markov, who [11,12] generalized this result to the polynomials of degree n and proved the following result.
Theorem 1.1. If $f \in \mathbb{P}_n$ with $|f(x)| \leq M$ for $x \in [-1, 1]$, then

$$\max_{-1\leq x\leq 1} |f'(x)| \leq n^2 M. \tag{1.1}$$

The result is best possible and the extremal polynomial is $f(x) = MT_n(x)$, where

$$T_n(x) = \cos(n \cos^{-1} x)$$

is the Chebsychev polynomial of first kind.

To proceed further, we first introduce some notations;
Let $\mathbb{P}_n$ denote the class of polynomials in $\mathbb{C}$ of degree at most $n \in \mathbb{N}$, and let $\mathbb{D}$ be the open unit disk

2020 *Mathematics Subject Classification:* 30A10, 30C10, 30C15, 30C80.
Submitted May 31, 2025. Published December 29, 2025

$\{z \in \mathbb{C} \mid |z| < 1\}$, with $\overline{\mathbb{D}}$ as its closure and $\delta\mathbb{D}$ as its boundary. For $f \in \mathbb{P}_n$, we define a polynomial $q \in \mathbb{P}_n$ by $q(z) = z^n \overline{f(\frac{1}{\bar{z}})}$, known as inversive polynomial of f.

An analogue of Markov's result for the unit disk in the complex plane instead of interval $[-1, 1]$ was formulated by S. Bernstein [3], who proved the following interesting result known as Bernstein's inequality:

Theorem 1.2. If $f \in \mathbb{P}_n$, then

$$\max_{z\in\delta D} |f'(z)| \le n \max_{z\in\delta D} |f(z)|. \tag{1.2}$$

The equality holds only if $f = \alpha z^n$, where $\alpha \neq 0$. Later in 1930, S. Bernstein [4] proved the following generalization of Theorem 1.2:

Theorem 1.3. Let $F \in \mathbb{P}_n$ have all zeros in $\overline{\mathbb{D}}$, and let $f(z)$ be a polynomial of degree not exceeding that of $F(z)$. If $|f(z)| \le |F(z)|$ for $z \in \delta D$, then

$$|f'(z)| \le |F'(z)| \quad \text{for } z \in \mathbb{C} \setminus \mathbb{D}. \tag{1.3}$$

The equality holds only if $f = e^{i\gamma} F$, where $\gamma \in \mathbb{R}$.

For $M = \max_{z\in\delta\mathbb{D}} |f(z)|$, take $F(z) = Mz^n$, a polynomial of degree n with all zeros in $\overline{\mathbb{D}}$, in Theorem 1.3, and noting that

$$|f(z)| \le |Mz^n| \quad \text{for } z \in \delta\mathbb{D},$$

we obtain inequality (1.2) as a special case. Following result is a simple deduction from the maximum modulus principle [10]:

Theorem 1.4. If $f \in \mathbb{P}_n$ then for $R \ge 1$

$$\max_{z\in\delta D} |f(Rz)| \le R^n \max_{z\in\delta D} |f(z)|. \tag{1.4}$$

Equality holds for $f(z) = \alpha z^n, \quad \gamma \neq 0$.

In Bernstein's inequality (1.2), equality holds if every zero of $f(z)$ is at the origin, and the inequality becomes strict when $f(z) = 0$ has a non-zero root. This suggests that if no zero of $f(z)$ is at the origin, then the upper bound in (1.2) may be improved. This fact was deeply examined by Paul Erdös, which led him to conjecture that if $f(z)$ does not vanish in $\mathbb{D}$, then in the upper bound of (1.2), n can be replaced by $\frac{n}{2}$. In fact, if $f \in \mathbb{P}_n$ has no zeros in $\mathbb{D}$ then,

$$\max_{z\in\delta D} |f'(z)| \le \frac{n}{2} \max_{z\in\delta D} |f(z)|, \tag{1.5}$$

and for $R \ge 1$

$$\max_{z\in\delta D} |f(Rz)| \le \frac{R^n + 1}{2} \max_{z\in\delta D} |f(z)|. \tag{1.6}$$

The result in inequality (1.5) and (1.6) is best possible and equality holds for polynomials having all zeros on $\delta\mathbb{D}$.

Inequality (1.5) was conjectured by Erdős and later proved by Lax [9], whereas inequality (1.6) was proved by Ankeny and Rivlin [2]. If one has the information of $\min_{z\in\delta D} |f(z)|$ then inequalities (1.5) and (1.6) can be improved further. In fact, under the same hypothesis, Aziz and Dawood [1] proved

$$\max_{z\in\delta D} |f'(z)| \le \frac{n}{2}\Big(\max_{z\in\delta D} |f(z)| - \min_{z\in\delta D} |f(z)|\Big), \tag{1.7}$$

and for $R \ge 1$

$$\max_{z\in\delta D} |f(Rz)| \le \frac{R^n + 1}{2} \max_{z\in\delta D} |f(z)| - \frac{R^n - 1}{2} \min_{z\in\delta D} |f(Rz)|. \tag{1.8}$$

The result is best possible and equality holds for polynomial $f(z) = \alpha + \beta z^n$ where $|\beta| \ge |\alpha|$.

For $z \in \mathbb{C} \setminus \mathbb{D}$, denoting by $\Omega_{|z|}$ the image of the disk $\{w \in \mathbb{C} \mid |w| \leq |z|\}$ under the mapping $\psi(w) = \frac{w}{1+w}$, Smirnov [18] generalized Theorem 1.3 as follows:

Theorem 1.5. Let f and F be polynomials that satisfy the conditions of Theorem 1.3 then for $z \in \mathbb{C}\setminus\mathbb{D}$,

$$\Big|\mathbb{S}_\alpha[f](z)\Big| \leq \Big|\mathbb{S}_\alpha[F](z)\Big|, \tag{1.9}$$

for all $\alpha \in \overline{\Omega}_{|z|}$ where $\mathbb{S}_\alpha[f](z) = zf'(z) - n\alpha f(z)$ and α is a constant. Equality holds if $f = e^{i\gamma}F$, $\gamma \in \mathbb{R}$. We note that for fixed $z \in \mathbb{C} \setminus \mathbb{D}$ inequality (1.9) can be replaced by ([6], [7])

$$\Big|zf'(z) - n\frac{az}{1+az}f(z)\Big| \leq \Big|zF'(z) - n\frac{az}{1+az}F(z)\Big|,$$

where $a \in \overline{\mathbb{D}}$ is not the exceptional value of f. Equivalently for $z \in \mathbb{C} \setminus \mathbb{D}$,

$$|\tilde{\mathbb{S}}_a[f](z)| \leq |\tilde{\mathbb{S}}_a[F](z)|,$$

where $\tilde{\mathbb{S}}_a[f](z) = (1+az)f'(z) - naf(z)$ is known as modified Smirnov Operator.
The modified Smirnov operator $\tilde{\mathbb{S}}_a$ is more preferred than Smirnov operator $\mathbb{S}_a$, in a sense that the parameter a of $\tilde{\mathbb{S}}_a$ does not depend on z unlike parameter α of $\mathbb{S}_a$.
Marden [10] introduced a differential operator $\mathbb{M} : \mathbb{P} \to \mathbb{P}$ of mth order. This operator carries a polynomial $f \in \mathbb{P}$ into

$$\mathbb{M}[f](z) = \lambda_0 f(z) + \lambda_1 \frac{nz}{2}f'(z) + \cdots + \lambda_m\Big(\frac{nz}{2}\Big)^m f^{(m)}(z),$$

where $\lambda_0, \lambda_1, \cdots, \lambda_m$ are constants such that

$$u(z) = \lambda_0 + \binom{n}{1}\lambda_1 z + \binom{n}{2}\lambda_2 z^2 + \cdots + \binom{n}{m}\lambda_m z^m \neq 0 \quad \text{for} \quad Re(z) > \frac{n}{4}. \tag{1.10}$$

Rahman and Schmeisser [15] considered the operator of Marden for $m = 2$ and showed that this operator preserves the inequalities between polynomials and, accordingly, proved the following:

Theorem 1.6. Let f and F be polynomials that satisfy the conditions of Theorem 1.3 then,

$$\Big|\mathbb{B}[f](z)\Big| \leq \Big|\mathbb{B}[F](z)\Big| \quad \text{for } z \in \mathbb{C} \setminus \mathbb{D}, \tag{1.11}$$

where $\mathbb{B}[f](z) = \lambda_0 f(z) + \lambda_1 \frac{nz}{2} f'(z) + \lambda_2 \Big(\frac{nz}{2}\Big)^2 f''(z)$ and $\lambda_0, \lambda_1, \lambda_2$ are constants satisfying (1.10). For $z \in \mathbb{C} \setminus \mathbb{D}$ in inequality (1.11), equality holds if and only if $f(z) = \gamma z^n$, $\gamma \in \mathbb{R}$.

In order to compare the Smirnov operator and the Rahman's operator (with $\lambda_2 = 0$) $\mathbb{B}[f](z) = \lambda_0 f(z) + \lambda_1 \frac{nz}{2} f'(z)$, we require $\alpha \in \overline{\Omega}_{|z|}$ in inequality (1.9) and in inequality (1.11) the root of the polynomial $u(z) = \lambda_0 + n\lambda_1 z$ should lie in the half-plane $Re(z) \leq \frac{n}{4}$, that is $Re(\frac{-\lambda_0}{n\lambda_1}) \leq \frac{n}{4}$.
Compare the sets of parameters in Theorem 1.5 and Theorem 1.6, we see that in Theorem 1.5, the set (coefficients near $-f(z)$) is $\mathcal{X} = \{n\alpha : \alpha \in \overline{\Omega}_{|z|}$ and in Theorem 1.6, the set of such coefficients near $-f(z)$ is

$$\mathcal{Y} = \Big\{\frac{-2\lambda_0}{n\lambda_1} : Re\Big(\frac{-\lambda_0}{n\lambda_1}\Big) \leq \frac{n}{4}\Big\} = \Big\{w : Re(w) \leq \frac{n}{2}\Big\}.$$

Consider the differential inequalities from Theorem 1.5 and Theorem 1.6 for $z \in \delta\mathbb{D}$, we have $\mathcal{X} = \mathcal{Y}$. But for $z \in \mathbb{C} \setminus \mathbb{D}$, we have $\mathcal{X} \supset \mathcal{Y}$. In other words, in Theorem 1.5 and Theorem 1.6 formally the same inequality was found but for different set of parameters. Moreover, the set of parameters in Theorem 1.5 is essentially wider than that Theorem 1.6. Consequently,

$$\Big|\mathbb{B}[f](z)\Big| \leq \frac{n\lambda_1}{2}\Big|\mathbb{S}_\alpha[f](z)\Big|. \tag{1.12}$$

These facts were first observed by Ganenkova and Starkov [6].
Recently, Shah and Fatima [16] among other results, proved the following results regarding the modified

Smirnov operator:
Theorem 1.7. Let $f \in \mathbb{P}_n$ have no zeros in $\mathbb{D}$ then for $z \in \mathbb{C} \setminus \mathbb{D}$

$$\left|\tilde{S}_a[f](z)\right| \leq \left\{\left|\tilde{S}_a[E](z)\right| + n|a|\right\}\frac{M}{2} - \left\{\left|\tilde{S}_a[E](z)\right||a|\right\}\frac{m}{2}$$

where $M = \max_{z\in\delta\mathbb{D}}|f(z)|$, $m = \min_{z\in\delta\mathbb{D}}|f(z)|$ and $E(z) = z^n$.
The result is best possible and equality holds for the polynomials having all zeros on the unit disk.

Recently various authors (see [7], [8]) have studied the Smirnov operator and provided various generalizations and refinements. In this paper, we shall provide new results concerning the modified Smirnov operator that provides a refinement of Theorem 1.7 and also prove compact generalizations of some well-known polynomial inequalities.

2. Main Results

Theorem 2.1. If $f \in \mathbb{P}_n$, then for every complex number β with $\beta \in \overline{\mathbb{D}}$ and $R \geq 1$, we have for $z \in \mathbb{C} \setminus \mathbb{D}$

$$\begin{aligned}\left|\tilde{S}_a[f](Rz) - \beta\tilde{S}_a[f](z)\right| + \left|\tilde{S}_a[q](Rz) - \beta\tilde{S}_a[q](z)\right| \\ \leq \Big(|R^n - \beta||\tilde{S}_a[E](z)| + n|a||1-\beta|\Big)\max_{z\in\delta\mathbb{D}}|f(z)|.\end{aligned} \tag{2.1}$$

Equivalently, for $z \in \mathbb{C} \setminus \mathbb{D}$

$$\begin{aligned}&\Big|(1+az)\big(Rf'(Rz) - \beta f'(z)\big) - na\big(f(Rz) - \beta f(z)\big)\Big| \\ &\quad + \Big|(1+az)\big(Rq'(Rz) - \beta q'(z)\big) - na\big(q(Rz) - \beta q(z)\big)\Big| \\ &\leq n\Big(|R^n - \beta||z|^{n-1} + |a||1-\beta|\Big)\max_{z\in\delta\mathbb{D}}|f(z)|,\end{aligned} \tag{2.2}$$

where $q(z) = z^n\overline{f(\frac{1}{\overline{z}})}$.
The result is best possible and equality holds for $f(z) = \gamma z^n, \quad \gamma \neq 0$.

Taking $\beta = 0$ and $R = 1$ in inequality (2.1) of Theorem 2.1, we get a result of Shah and Fatima ([16] , Lemma 2.4). For $\beta = 1$ and $a = 0$ in inequality (2.2) of Theorem 2.1, we obtain the following result of Rahman [15].

If $f \in \mathbb{P}_n$ and $q(z) = z^n\overline{f(\frac{1}{\overline{z}})}$, then for $R \geq 1$, we have

$$\begin{aligned}\Big|Rf'(Rz) - f'(z)\Big| + \Big|Rq'(Rz) - q'(z))\Big| \\ \leq n(R^n - 1)|z|^{n-1}\max_{z\in\delta\mathbb{D}}|f(z)| \quad \text{for } z \in \mathbb{C} \setminus \mathbb{D}.\end{aligned}$$

The result is best possible and equality holds for $f(z) = \gamma z^n, \quad \gamma \neq 0$.
For polynomials $f \in \mathbb{P}_n$ not vanishing in $\mathbb{D}$, we prove the following result:
Theorem 2.2. Let $f \in \mathbb{P}_n$ have no zeros in $\mathbb{D}$ then for any complex number β with $|\beta| \leq 1$ and $R \geq 1$, we have for $z \in \mathbb{C} \setminus \mathbb{D}$

$$\left|\tilde{S}_a[f](Rz) - \beta\tilde{S}_a[f](z)\right| \leq \left\{|R^n - \beta|\left|\tilde{S}_a[E](z)\right| + n|a||1-\beta|\right\}\Big(\frac{M-m}{2}\Big). \tag{2.3}$$

Equivalently, for $z \in \mathbb{C} \setminus \mathbb{D}$

$$\begin{aligned}\Big|(1+az)\big(Rf'(Rz) - \beta f'(z)\big) - na\big(f(Rz) - \beta f(z)\big)\Big| \\ \leq n\Big(|R^n - \beta||z|^{n-1} + |a||1-\beta|\Big)\Big(\frac{M-m}{2}\Big),\end{aligned} \tag{2.4}$$

where $M = \max_{z\in\delta\mathbb{D}} |f(z)|$, $m = \min_{z\in\delta\mathbb{D}} |f(z)|$ and $E(z) = z^n$.
The result is sharp and equality holds for the polynomials having all zeros on the unit disk.

Taking $R = 1$ and $\beta = 0$ in inequality (2.3) of Theorem 2.2, we get the following refinement of Theorem 1.7.:

Theorem 2.3. Let $f \in \mathbb{P}$ have no zeros in $\mathbb{D}$, then for $z \in \mathbb{C} \setminus \mathbb{D}$

$$\left|\tilde{S}_a[f](z)\right| \leq \left\{\left|\tilde{S}_a[E](z)\right| + n|a|\right\}\left(\frac{M-m}{2}\right). \tag{2.5}$$

The result is best possible and equality holds for $f(z) = z^n - c,\ \ |c| = 1$.

Theorem 2.3 will be better than that of Theorem 1.7 only if

$$\left\{\left|\tilde{S}_a[E](z)\right| + n|a|\right\}\left(\frac{M-m}{2}\right) \leq \left\{\left|\tilde{S}_a[E](z)\right| + n|a|\right\}\frac{M}{2} - \left\{\left|\tilde{S}_a[E](z)\right||a|\right\}\frac{m}{2}.$$

That is, if $n|a|m \geq 0$, which is true. Further, one can see that Theorem 2.3 provides a refinement of a result of Shah and Fatima ([16], Theorem 2.7). For $a = \beta = 0$ and $R = 1$, we get from inequality (2.4) of Theorem 2.2.

$$|f'(z)| \leq n|z|^{n-1}\left(\frac{M-m}{2}\right) \quad \text{for } z \in \mathbb{C} \setminus \mathbb{D},$$

which in particular gives inequality (1.7). For $a = -\frac{1}{z}$ and $\beta = 0$, we get from inequality (2.4) of Theorem 2.2.

$$|f(Rz)| \leq \left(\frac{R^n+1}{2}\right)(M-m) \quad \text{for } z \in \mathbb{C} \setminus \mathbb{D},$$

which provides a refinement of inequality (1.6) as well as of inequality (1.8).

3. Lemmas

In order to prove the main results, we need the following lemmas. The first lemma is due to Fatima and Shah [5].

Lemma 3.1. Let $F \in \mathbb{P}_n$ have all zeros in $\overline{\mathbb{D}}$, and let $f(z)$ be a polynomial of degree not exceeding that of $F(z)$. If $|f(z)| \leq |F(z)|$ for $z \in \delta\mathbb{D}$, then for any complex number β with $\beta \in \overline{\mathbb{D}}$ and $R \geq 1$, we have for $z \in \delta\mathbb{D}$

$$\left|\tilde{S}_a[f](Rz) - \beta\tilde{S}_a[f](z)\right| \leq \left|\tilde{S}_a[F](Rz) - \beta\tilde{S}_a[F](z)\right|.$$

The result is sharp and equality holds if $a \in \overline{\mathbb{D}}$ is not exceptional value for the polynomial $f(z) \equiv e^{i\gamma}F(z)$ where $\gamma \in \mathbb{R}$ and $F(z)$ is any polynomial having all zeros in $\overline{\mathbb{D}}$ and strict inequality holds for $z \in \mathbb{D}$, unless $f(z) \equiv e^{i\gamma}F(z),\ \ \gamma \in \mathbb{R}$.

Lemma 3.2. If $f \in \mathbb{P}_n$ and $q(z) = z^n\overline{f(\frac{1}{\bar{z}})}$, then for every complex number β with $\beta \in \overline{\mathbb{D}}$ and $R \geq 1$, we have for $z \in \mathbb{C} \setminus \mathbb{D}$

$$\left|\tilde{S}_a[q](Rz) - \beta\tilde{S}_a[q](z)\right| \leq |R^n - \beta||\tilde{S}_a[E](z)|M,$$

where $M = \max_{z\in\delta\mathbb{D}} |f(z)|$ and $E(z) = z^n$.

Proof of lemma 3.2. We have $q(z) = z^n\overline{f(\frac{1}{\bar{z}})}$, therefore $M = \max_{z\in\delta\mathbb{D}} |f(z)|$ implies for $z \in \delta\mathbb{D}$,

$$|q(z)| = |f(z)| \leq |Mz^n|$$

Applying Lemma 3.1 by taking $f(z) = q(z),\ \ F(z) = Mz^n$ so that all zeros of $F(z)$ lie in $\mathbb{D}$, we obtain for every complex number β with $\beta \in \overline{\mathbb{D}}$ and $R \geq 1$

$$\left|\tilde{S}_a[q](Rz) - \beta\tilde{S}_a[q](z)\right| \leq \left|\Big((1+az)nMR^nz^{n-1} - naMR^nz^n\Big) - \beta\Big((1+az)nMz^{n-1} - naMz^n\Big)\right|.$$

Noting that $E(z) = z^n$ so that $\tilde{S}[E](z) = nz^{n-1}$, we obtain

$$\left|\tilde{S}_a[q](Rz) - \beta\tilde{S}_a[q](z)\right| \leq M|\tilde{E}(z)|R^n - \beta|.$$

This proves the Lemma 3.2.

4. Proof of Theorem

Proof of Theorem 2.1. Since

$$M = \max_{z\in\delta\mathbb{D}} |f(z)|,$$

we have $|f(z)| \leq M$ for $z \in \delta\mathbb{D}$. Therefore for every complex number α with $|\alpha| > 1$, the polynomial $g(z) = f(z) + \alpha M$ has no zero in $\mathbb{D}$. That is, the polynomial

$$g^*(z) = z^n\overline{g(\frac{1}{\overline{z}})} = q(z) + \overline{\alpha}Mz^n$$

has all zeros in $\overline{\mathbb{D}}$ such that $|g(z)| = |g^*(z)|$ for $z \in \delta\mathbb{D}$ where $q(z) = z^n\overline{f(\frac{1}{\overline{z}})}$. Applying Lemma 3.1 to the polynomials $|g(z)|$ and $|g^*(z)|$, we obtain for every complex number β with $\beta \in \overline{\mathbb{D}}$ and $R \geq 1$

$$\left|\tilde{S}_a[g](Rz) - \beta\tilde{S}_a[g](z)\right| \leq \left|\tilde{S}_a[g^*](Rz) - \beta\tilde{S}_a[g^*](z)\right| \quad \text{for } z \in \delta\mathbb{D}.$$

That is,

$$\begin{aligned}\Big|\tilde{S}_a[f](Rz)-&\beta\tilde{S}_a[f](z) - naM\alpha(1-\beta)\Big| \\ &\leq \left|\tilde{S}_a[q](Rz) - \beta\tilde{S}_a[q](z) + nM\overline{\alpha}z^{n-1}(R^n - \beta)\right| \quad \text{for } z \in \delta\mathbb{D}. \qquad (4.1)\end{aligned}$$

Choosing the argument of α suitably and noting that $|\alpha| > 1$, we get from (4.1) with the help of Lemma 3.2

$$\begin{aligned}\Big|\tilde{S}_a[f](Rz)-&\beta\tilde{S}_a[f](z)\Big| - n|a|M|\alpha||1-\beta| \\ &\leq M|\alpha|\tilde{S}_a[E](z)|R^n - \beta| - \left|\tilde{S}_a[q](Rz) - \beta\tilde{S}_a[q](z)\right|.\end{aligned}$$

This implies,

$$\begin{aligned}\left|\tilde{S}_a[f](Rz) - \beta\tilde{S}_a[f](z)\right| &+ \left|\tilde{S}_a[q](Rz) - \beta\tilde{S}_a[q](z)\right| \\ &\leq |\alpha|\Big\{|R^n - \beta||\tilde{S}_a[E](z)| + n|a||1-\beta|\Big\}.\end{aligned}$$

By letting $|\alpha| \to 1$, we get inequality (2.1) completely.

Proof of Theorem 2.2. By the hypothesis $m \leq |f(z)|$ for $z \in \delta\mathbb{D}$, therefore for every complex number λ with $|\lambda| < 1$, we have

$$|m\lambda z^n| < m \leq |f(z)| \quad \text{for } z \in \delta\mathbb{D}.$$

Hence by Rouche's theorem $g(z) = f(z) - m\lambda z^n \neq 0$ in $\mathbb{D}$. This implies

$$h(z) = z^n\overline{g(\frac{1}{\overline{z}})} = q(z) - m\overline{\lambda}$$

has all zeros in $\overline{\mathbb{D}}$ where $q(z) = z^n\overline{f(\frac{1}{\overline{z}})}$. Noting that $|g(z)| = |h(z)|$ for $z \in \delta\mathbb{D}$, it follows by applying Lemma 3.1 on the polynomials $g(z)$ and $h(z)$ that for every complex number β with $\beta \in \overline{\mathbb{D}}$ and $R \geq 1$,

$$\left|\tilde{S}_a[g](Rz) - \beta\tilde{g}_a[f](z)\right| \leq \left|\tilde{S}_a[h](Rz) - \beta\tilde{S}_a[h](z)\right| \quad \text{for } z \in \delta\mathbb{D}. \qquad (4.2)$$

Now

$$\begin{aligned}\tilde{S}_a[g](Rz) &= (1+az)(Rf'(Rz)m\lambda R^n z^{n-1}) - na(f(Rz) - m\lambda R^n z^n)\\ &= \tilde{S}_a[f](Rz) - nm\lambda R^n z^n \end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\tilde{S}_a[h](Rz) &= (1+az)Rq'(Rz) - na(q(Rz) - m\overline{\lambda}\\ &= \tilde{S}_a[q](Rz) + nma\overline{\lambda}, \end{aligned} \tag{4.4}$$

Using (4.3) and (4.4) in inequality (4.2), we get

$$\begin{aligned}&\left|\tilde{S}_a[f](Rz) - \beta\tilde{S}_a[f](z) - nm\lambda(R^n - \beta)z^{n-1}\right|\\ &\qquad \le \left|\tilde{S}_a[q](Rz) - \beta\tilde{S}_a[q](z) + nma\overline{\lambda}(1-\beta)\right| \quad \text{for } z \in \mathbb{C}\setminus\mathbb{D}.\end{aligned}$$

Choosing argument of λ suitably, we obtain

$$\begin{aligned}&\left|\tilde{S}_a[f](Rz) - \beta\tilde{S}_a[f](z)\right| + nm|\lambda||R^n - \beta||z|^{n-1}\\ &\qquad \le \left|\tilde{S}_a[q](Rz) - \beta\tilde{S}_a[q](z)\right| - nm|a||\lambda||1-\beta| \quad \text{for } z \in \mathbb{C}\setminus\mathbb{D}.\end{aligned}$$

Letting $|\lambda| \to 1$, we get

$$\begin{aligned}\left|\tilde{S}_a[f](Rz) - \beta\tilde{S}_a[f](z)\right| \le& \left|\tilde{S}_a[q](Rz) - \beta\tilde{S}_a[q](z)\right|\\ &- m\Big(n|R^n - \beta||z|^{n-1} + n|a||1-\beta|\Big) \quad \text{for } z \in \mathbb{C}\setminus\mathbb{D}.\end{aligned} \tag{4.5}$$

Combining inequalities (2.1) and (4.5) and noting that $\tilde{S}_a[E](z) = nz^{n-1}$, we get

$$\begin{aligned}2\left|\tilde{S}_a[f](Rz) - \beta\tilde{S}_a[f](z)\right| \le& \left|\tilde{S}_a[f](Rz) - \beta\tilde{S}_a[f](z)\right| + \left|\tilde{S}_a[q](Rz) - \beta\tilde{S}_a[q](z)\right|\\ &- m\Big(n|R^n - \beta||z|^{n-1} + n|a||1-\beta|\Big)\\ \le& \Big(|R^n - \beta||\tilde{S}_a[E](z)| + n|a||1-\beta|\Big)M\\ &- \Big(|R^n - \beta||\tilde{S}_a[E](z)| + n|a||1-\beta|\Big)m \quad \text{for } z \in \mathbb{C}\setminus\mathbb{D}.\end{aligned}$$

That is,

$$\begin{aligned}&\left|\tilde{S}_a[f](Rz) - \beta\tilde{S}_a[f](z)\right|\\ &\qquad \le \Big(|R^n - \beta|\tilde{S}_a[E](z)| + n|a||1-\beta|\Big)\Big(\frac{M-m}{2}\Big) \quad \text{for } z \in \mathbb{C}\setminus\mathbb{D}.\end{aligned}$$

This completes the proof of Theorem 2.2 completely.

5. Conclusions

Some fresh findings on the modified Smirnov operator preserving the inequalities for polynomials, having zeros within or outside a unit disk, have been discovered. Further, new refinements and generalizations of Bernstein-type inequalities for polynomials have been proved.

Acknowledgment: The authors sincerely thank the referees for their valuable suggestions and constructive comments, which have been instrumental in enhancing the quality and clarity of this work.

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Bilal Dar
Department of Mathematics
National Institute of Technology, Srinagar-190006
The author is Assistant Professor in Higher Education Department, J and K Govt., India and is presently working at Govt. Degree College, Pampore.
E-mail address: darbilal67@gmail.com

and

Abdul Liman
Department of Mathematics
National Institute of Technology, Srinagar-190006
E-mail address: abliman@rediffmail.com