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Chaotic Oscillations in a New 2-D Discrete Dynamical System with Hidden Parameter

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ABSTRACT: The main purpose of this work is to present a new 2-D chaotic discrete dynamical system. By studying its basic properties, such as determining its fixed points and their stability types based on bifurcation parameter values, and using Lyapunov exponents, we identify chaotic oscillations for certain values of the parameter a, regardless of the value of the second parameter b. This implies that b has no effect on the system's dynamics, making it a distinctive feature, henceforth referred to as the hidden parameter. The system, denoted by $B_{a,b}$, exhibits several useful characteristics, including its quadratic form, the differentiability of its corresponding function, and the absence of the parameter b in its eigenvalues, allowing for arbitrary selection of b and simplified calculations. Simulation results validate the chaotic behavior.

Key Words: Chaotic dynamical system, discrete system, fixed points, Lyapunov exponents, bifurcation analysis, Hidden parameter.

Contents

1. Introduction

Chaotic dynamical systems have proven effective in numerous fields, including information and communication security [3,8], cryptography [4,7,12], and the generation of pseudo-random number sequences (PRNG) [5]. These systems are used to obscure images and texts to prevent tracking and espionage, and to create pseudo-random sequences for codes and product series, enhancing brand protection and reducing counterfeiting.

Several studies focus on the behavior of dynamical systems [1,2,6,11,15], particularly in the search for chaotic oscillations through fundamental mathematical steps [1,13,14,18]. After establishing chaotic behavior, these systems are applied as models in the aforementioned fields by scientists and researchers.

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This paper presents a new system defined by mathematical formulas, followed by an in-depth study to demonstrate chaotic oscillations via period-doubling bifurcation. We start by identifying fixed points and their stability types through eigenvalue analysis, followed by Lyapunov exponents to characterize the behavior of trajectories for certain values of a, independent of b. This property distinguishes our model. Simulation work validates our mathematical results.

2. Formulation and Basic Properties

Consider the 2-D discrete dynamical system $B_{a,b}$ defined by:

$$B_{a,b}: \begin{cases} x_{k+1} = ax_k(1-y_k^2), \\ y_{k+1} = bx_k, \end{cases} \qquad a, b \in \mathbb{R}. \tag{2.1}$$

Proposition 2.1 *The system (2.1) has the following properties:*

1. *It is defined for all $(x, y) \in \mathbb{R}^2$.*
2. *It is symmetric for all $(x, y) \in \mathbb{R}^2$.*
3. *It has a quadratic form.*
4. *It is defined by a function of class $C^\infty(\mathbb{R}^2)$.*

Proof: The system's components are polynomial expressions, hence defined for all $(x, y) \in \mathbb{R}^2$ and quadratic in nature.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the associated function defined by:

$$f(x, y) = \begin{pmatrix} ax(1-y^2) \\ bx \end{pmatrix}, \quad a, b \in \mathbb{R}.$$

Clearly, $f \in C^\infty(\mathbb{R}^2)$, as it is a polynomial function.

To prove symmetry, compute:

$$f(-x, -y) = \begin{pmatrix} a(-x)[1-(-y)^2] \\ b(-x) \end{pmatrix} = \begin{pmatrix} -ax(1-y^2) \\ -bx \end{pmatrix} = -f(x, y).$$

Thus, f is an odd function, confirming the system's symmetry. □

3. Fixed Points and Period Doubling

3.1. Fixed Points

Let the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by:

$$f(x, y) = \begin{pmatrix} ax(1-y^2) \\ bx \end{pmatrix}.$$

The fixed points of the system (2.1) satisfy:

$$\begin{cases} x = ax(1-y^2), \\ y = bx. \end{cases} \tag{3.1}$$

For $x \neq 0$, from the first equation:

$$1 = a(1-y^2) \implies y^2 = 1 - \frac{1}{a} = \frac{a-1}{a} \implies y = \pm\sqrt{\frac{a-1}{a}}.$$

From the second equation, $y = bx \implies x = \frac{y}{b}$. Substituting $y = \pm\sqrt{\frac{a-1}{a}}$:

$$x = \pm\frac{1}{b}\sqrt{\frac{a-1}{a}}.$$

Thus, the fixed points are:

$$P_1 = \left(\frac{1}{b}\sqrt{\frac{a-1}{a}}, \sqrt{\frac{a-1}{a}}\right), \quad P_2 = \left(-\frac{1}{b}\sqrt{\frac{a-1}{a}}, -\sqrt{\frac{a-1}{a}}\right),$$

provided $a \neq 0$ and $b \neq 0$.

3.2. Stability

The Jacobian matrix of the system (2.1) is:

$$Df(x, y) = \begin{bmatrix} a(1-y^2) & -2axy \\ b & 0 \end{bmatrix}. \tag{3.2}$$

At the fixed point $P_1\left(\frac{1}{b}\sqrt{\frac{a-1}{a}}, \sqrt{\frac{a-1}{a}}\right)$, the Jacobian is:

$$Df(P_1) = \begin{bmatrix} a\left(1-\frac{a-1}{a}\right) & -2a \cdot \frac{1}{b}\sqrt{\frac{a-1}{a}} \cdot \sqrt{\frac{a-1}{a}} \\ b & 0 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{2(a-1)}{b} \\ b & 0 \end{bmatrix}.$$

3.2.1. Eigenvalues for P_1. For the matrix

$$A = \begin{bmatrix} 1 & -\frac{2(a-1)}{b} \\ b & 0 \end{bmatrix},$$

the characteristic polynomial is:

$$\det(A - \lambda I) = \lambda(\lambda - 1) + 2(a-1) = \lambda^2 - \lambda + 2(a-1).$$

The discriminant is:

$$\Delta = 1 - 8(a-1) = 9 - 8a > 0, \quad \text{for } a < \frac{9}{8}. \tag{3.3}$$

The eigenvalues are:

$$\lambda_1 = \frac{1-\sqrt{9-8a}}{2}, \quad \lambda_2 = \frac{1+\sqrt{9-8a}}{2}.$$

For $|\lambda_1| < 1$:

$$-1 < \frac{1-\sqrt{9-8a}}{2} < 1 \implies -3 < -\sqrt{9-8a} < 1.$$

- $-3 < -\sqrt{9-8a} \implies 3 > \sqrt{9-8a} \implies 9 > 9-8a \implies a > 0$. - $-\sqrt{9-8a} < 1 \implies \sqrt{9-8a} > -1$, which is always true.

Thus, $|\lambda_1| < 1$ for $a \in \left(0, \frac{9}{8}\right)$.

For $|\lambda_2| < 1$:

$$-1 < \frac{1+\sqrt{9-8a}}{2} < 1 \implies -3 < \sqrt{9-8a} < 1.$$

- $\sqrt{9-8a} > -3$, which is always true. - $\sqrt{9-8a} < 1 \implies 9-8a < 1 \implies a > 1$.

Thus, $|\lambda_2| < 1$ for $a \in \left(1, \frac{9}{8}\right)$.

Remark 3.1 The parameter b does not appear in the eigenvalues, indicating that its choice is arbitrary and does not affect the stability of the fixed points.

Case 1 For $a \in \left(1, \frac{9}{8}\right)$, $|\lambda_1| < 1$ and $|\lambda_2| < 1$, so P_1 is asymptotically stable.

Case 2 For $a \in (0, 1)$, $|\lambda_1| < 1$ and $|\lambda_2| > 1$, so P_1 is a saddle.

Case 3 For $a \leq 0$, $|\lambda_1| > 1$ and $|\lambda_2| > 1$, so P_1 is unstable.

Lemma 3.1 *For all* $(x, y) \in \mathbb{R}^2$*:*

$$Df(-x, -y) = Df(x, y). \tag{3.4}$$

Proof: Compute:

$$Df(-x, -y) = \begin{bmatrix} a(1 - (-y)^2) & -2a(-x)(-y) \\ b & 0 \end{bmatrix} = \begin{bmatrix} a(1 - y^2) & -2axy \\ b & 0 \end{bmatrix} = Df(x, y).$$

□

At the fixed point $P_2\left(-\frac{1}{b}\sqrt{\frac{a-1}{a}}, -\sqrt{\frac{a-1}{a}}\right)$, the Jacobian is identical to that at P_1 by Lemma 3.1, so the stability analysis for P_2 is the same as for P_1.

3.2.2. Eigenvalues for P_2. The characteristic polynomial and eigenvalues for P_2 are identical to those for P_1, confirming the same stability conditions.

3.3. Period Doubling

To find 2-cycles, compute $f^2(x, y) = f(f(x, y)) = (x, y)$:

$$f^2(x, y) = \begin{pmatrix} a\left(ax(1 - y^2)(1 - (bx)^2)\right) \\ b\left(ax(1 - y^2)\right) \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.5}$$

This gives:

$$\begin{cases} a^2x(1 - y^2)(1 - b^2x^2) = x, \\ bax(1 - y^2) = y. \end{cases} \tag{3.6}$$

From the second equation:

$$x = \frac{y}{ab(1 - y^2)}, \quad a \neq 0, b \neq 0, y^2 \neq 1. \tag{3.7}$$

Substitute into the first equation:

$$a^2\left(\frac{y}{ab(1 - y^2)}\right)(1 - y^2)\left(1 - b^2\left(\frac{y}{ab(1 - y^2)}\right)^2\right) = \frac{y}{ab(1 - y^2)}. \tag{3.8}$$

Simplify:

$$a\left(1 - \frac{y^2}{a^2(1 - y^2)^2}\right) = \frac{1}{ab^2(1 - y^2)}. \tag{3.9}$$

Multiply through by $ab^2(1 - y^2)$:

$$a^2b^2(1 - y^2)\left(1 - \frac{y^2}{a^2(1 - y^2)^2}\right) = 1. \tag{3.10}$$

This simplifies to:

$$a^2y^4 - 2a^2y^2 + a^2 - 1 = 0. \tag{3.11}$$

Let $t = y^2$:

$$a^2t^2 - 2a^2t + a^2 - 1 = 0. \tag{3.12}$$

Solutions are:

$$t = \frac{a - 1}{a}, \quad t = \frac{a + 1}{a}, \quad \text{for } a \neq 0. \tag{3.13}$$

Thus:

$$y_{1,2} = \pm\sqrt{\frac{a-1}{a}}, \quad y_{3,4} = \pm\sqrt{\frac{a+1}{a}}. \tag{3.14}$$

Corresponding x-values are:

- For $y_1 = \sqrt{\frac{a-1}{a}}$: $x_1 = \frac{1}{b}\sqrt{\frac{a-1}{a}}$, giving $P_1^2 = P_1$.
- For $y_2 = -\sqrt{\frac{a-1}{a}}$: $x_2 = -\frac{1}{b}\sqrt{\frac{a-1}{a}}$, giving $P_2^2 = P_2$.
- For $y_3 = \sqrt{\frac{a+1}{a}}$: $x_3 = \frac{1}{b}\sqrt{\frac{a+1}{a}}$, giving $P_3^2 = \left(\frac{1}{b}\sqrt{\frac{a+1}{a}}, \sqrt{\frac{a+1}{a}}\right)$.
- For $y_4 = -\sqrt{\frac{a+1}{a}}$: $x_4 = -\frac{1}{b}\sqrt{\frac{a+1}{a}}$, giving $P_4^2 = \left(-\frac{1}{b}\sqrt{\frac{a+1}{a}}, -\sqrt{\frac{a+1}{a}}\right)$.

Thus, the 2-cycle points are P_3^2 and P_4^2, with P_1^2 and P_2^2 being fixed points.

4. Lyapunov Exponents

Lyapunov exponents quantify sensitivity to initial conditions (SIC) and distinguish chaotic from stable behavior:

1. If a Lyapunov exponent is positive, the system exhibits SIC and is chaotic.
2. If all Lyapunov exponents are non-positive, the system is stable or periodic.

The Jacobian matrix is:

$$J(x,y) = \begin{pmatrix} a(1-y^2) & -2axy \\ b & 0 \end{pmatrix}. \tag{4.1}$$

At P_1, the Jacobian is:

$$J(P_1) = \begin{pmatrix} 1 & -\frac{2(a-1)}{b} \\ b & 0 \end{pmatrix}. \tag{4.2}$$

The eigenvalues are:

$$\lambda_1 = \frac{1-\sqrt{9-8a}}{2}, \quad \lambda_2 = \frac{1+\sqrt{9-8a}}{2}. \tag{4.3}$$

The Lyapunov exponents are:

$$E_1 = \ln\left|\frac{1+\sqrt{9-8a}}{2}\right|, \quad E_2 = \ln\left|\frac{1-\sqrt{9-8a}}{2}\right|. \tag{4.4}$$

Theorem 4.1 *The system (2.1) is chaotic for $a < 1$.*

Proof:

$$E_1 = \ln\left|\frac{1+\sqrt{9-8a}}{2}\right| > 0 \iff \frac{1+\sqrt{9-8a}}{2} > 1 \iff \sqrt{9-8a} > 1 \iff a < 1.$$

□

Remark 4.1 $E_2 < 0$ for $a < \frac{9}{8}$, as $\left|\frac{1-\sqrt{9-8a}}{2}\right| < 1$.

The bifurcation diagram for a with $b = 0.6$ is shown in Figure 2.

The bifurcation diagram for b with $a = -1$ is shown in Figure 3.

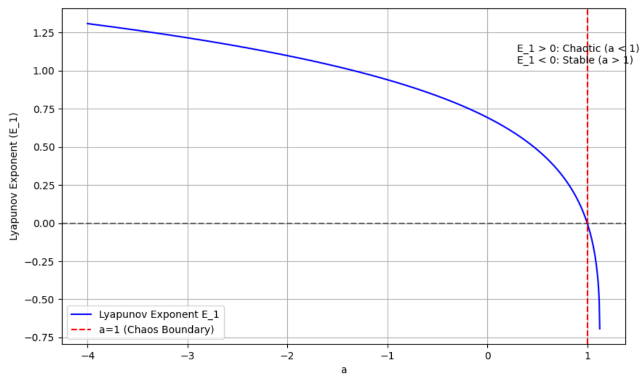

Figure 1: Evolution of the Lyapunov exponent E_1 with respect to a.

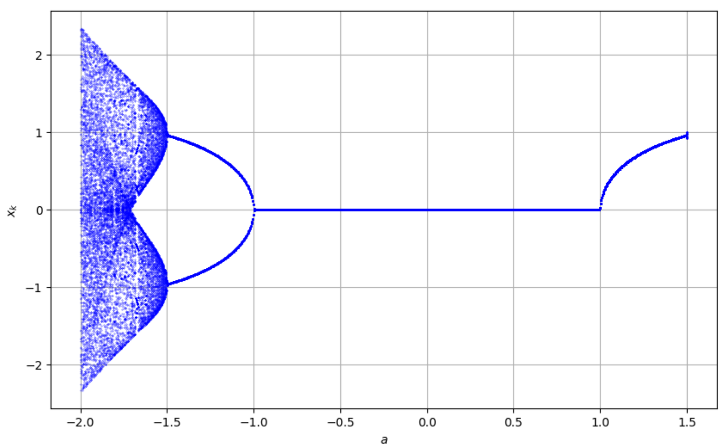

Figure 2: Bifurcation diagram of system (2.1) for a with $b = 0.6$.

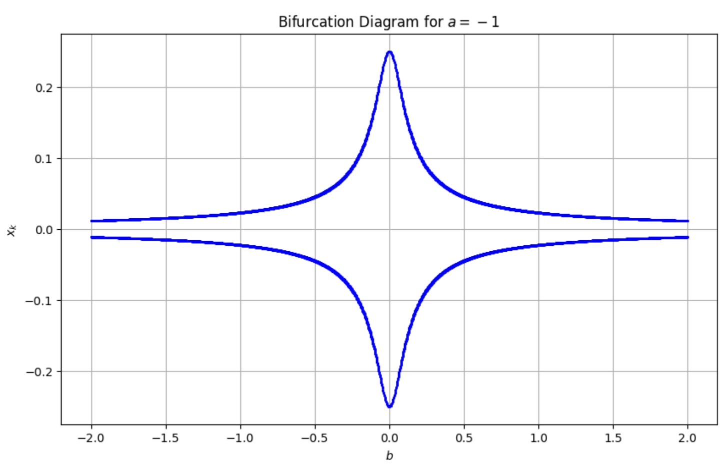

Figure 3: Bifurcation diagram of system (2.1) for b with $a = -1$.

5. Dynamics of the System

Lyapunov exponents quantify SIC and separate chaotic from stable behaviors. We numerically investigate the dynamics of system (2.1) with respect to parameters a and b.

The system is:

$$B_{a,b}: \begin{cases} x_{k+1} = a x_k (1 - y_k^2), \\ y_{k+1} = b x_k. \end{cases}$$

Simulations use initial conditions $x_0 = 0.25$, $y_0 = b x_0$.

5.1. Periodic Trajectories: $b = 0.6, 2$, $a \in \left(1, \frac{9}{8}\right)$

For $k = 0, 1, \ldots, 50$: - For $b = 0.6$, $y_0 = 0.15$, test $a = 1, 1.06, 1.12$. - For $b = 2$, $y_0 = 0.5$, test $a = 1, 1.06, 1.12$.

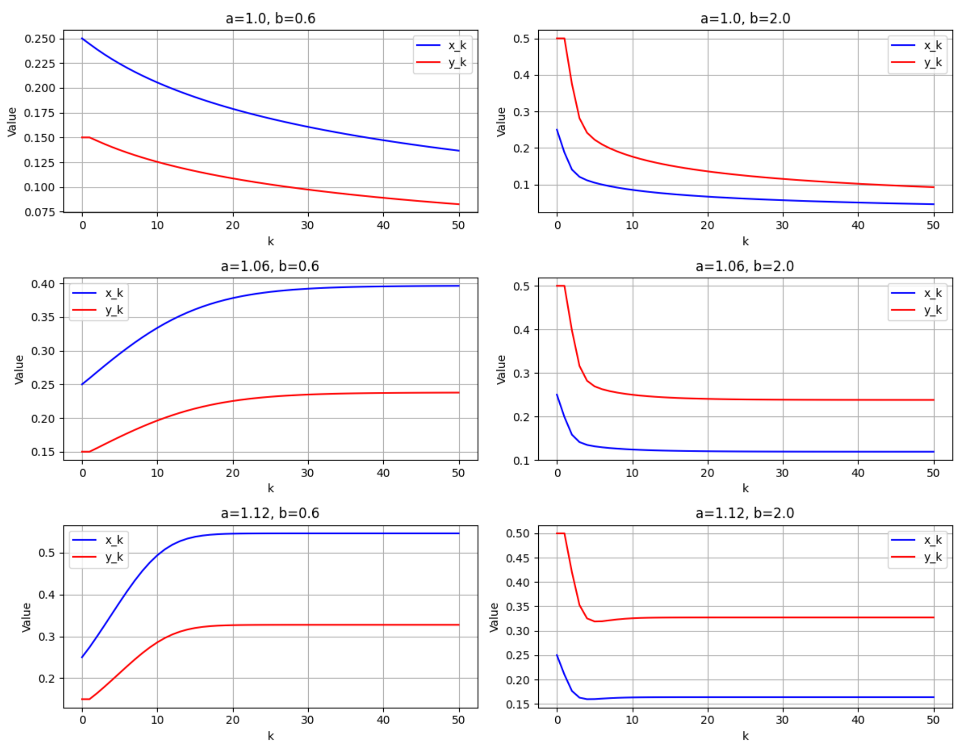

Figure 4: Periodic trajectories for $1 < a < \frac{9}{8}$, for any b.

The coordinates (x_k, y_k) exhibit periodicity, as shown in Figure 4.

5.2. Saddle Point

For $k = 0, 1, \ldots, 50$, with $b = 0.6$, $y_0 = 0.15$, test $a = 0.4, 0.6, 0.8$. For $b = 2$, $y_0 = 0.5$, test $a = 0.4, 0.6, 0.8$.

The trajectories tend toward a saddle point, as shown in Figure 5.

5.3. Chaotic Oscillations

For $k = 0, 1, \ldots, 50$, with $a < 0$ (where $E_1 > 0$): - For $b = 0.6$, $y_0 = 0.15$, test $a = -0.4, -0.6, -1$. - For $b = 0.8$, $y_0 = 0.2$, test $a = -0.4, -0.6, -1$. - For $b = 1$, $y_0 = 0.25$, test $a = -0.4, -0.6, -1$. - For $b = 2$, $y_0 = 0.5$, test $a = -0.4, -0.6, -1$.

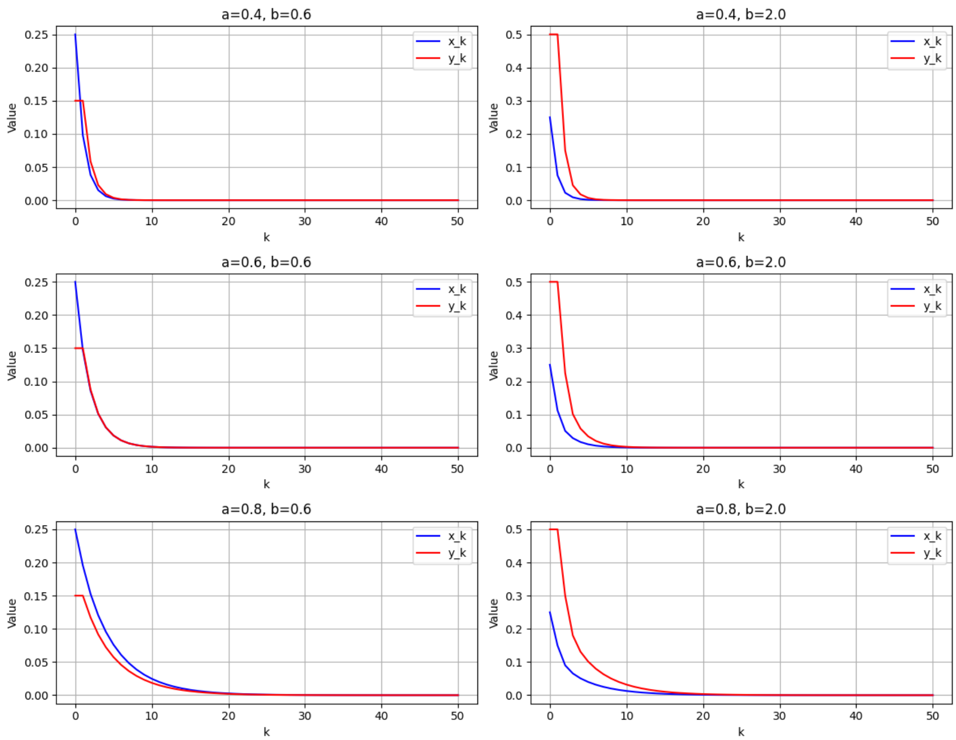

Figure 5: Saddle point for $0 < a < 1$, for any b.

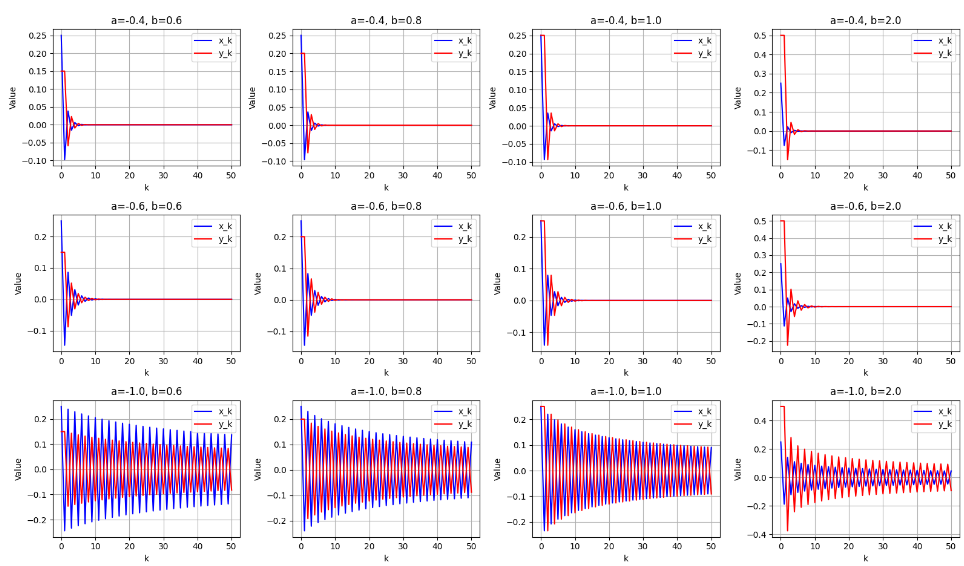

Figure 6: Chaotic oscillations for $a = -1$, $b = 0.6$.

5.4. Bounded and Unbounded Orbits

For $a \in \left(1, \frac{9}{8}\right)$, where $|\lambda_1| < 1$ and $|\lambda_2| < 1$:

$$x_{k+1} = ax_k(1 - y_k^2) = ax_k(1 - b^2 x_k^2).$$

$$y_{k+1} = bx_k = b^{k+2} x_0.$$

- If $|b| < 1$, $\lim_{k\to\infty} y_{k+1} = 0$. - If $|b| > 1$, $\lim_{k\to\infty} y_{k+1} = \infty$.

For x_k:

$$x_{k+1} = ax_k - ab^2 x_k^3.$$

This forms a polynomial $P_{3^k}(x_0, b^{3m+2})$. If $|b| < 1$, $x_k \to 0$; if $|b| > 1$, $x_k \to \infty$.

6. Basins of Attraction

The basins of attraction for $a = -1$, $b = 0.6$ are shown in Figure 7. Numerical simulations used a grid of 10^4 initial conditions with $\Delta x = \Delta y = 0.01$. The fractal-like boundaries indicate chaotic behavior.

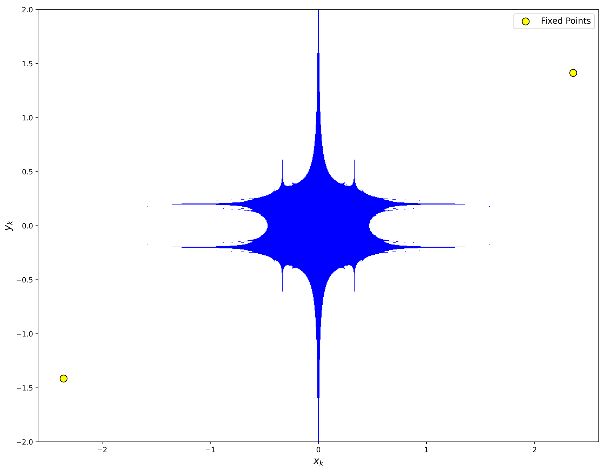

Figure 7: Basin of attraction for system (2.1) with $a = -1$, $b = 0.6$. Blue regions converge to stable orbits, red zones diverge, and green areas show transient chaos.

7. Conclusion

Discrete chaotic systems have applications in weather prediction [9], population dynamics [12], neural networks [10,16,17], and security [3,8]. This paper presents a new 2-D chaotic discrete dynamical system with a hidden parameter b, which does not affect the dynamics, distinguishing it from previous models. The system has a quadratic form and is analyzed through bifurcation diagrams and Lyapunov stability theory. Numerical simulations in Python validate the results, suggesting applications in PRNG for secure encryption and product protection.

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