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Minimum Dominating Reduced Second Zagreb Energy of Graph

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ABSTRACT: The energy of graph which is rooted in spectral graph theory, continues to play a pivotal role in structural graph analysis. Motivated by the concept of energy and it's applications, in this paper we discuss minimum dominating reduced second Zagreb energy. Here we present the most essential upper and lower bounds for $RM_2^D E(G)$.

Key Words: Minimum dominating set, minimum dominating reduced second Zagreb energy.

Contents

1. Introduction

This paper deals with the graphs which are simple, sign-mark free and self loop free. To indicate the adjacency of two vertices v_i and v_j of G, we use $v_i \sim v_j$. In 1978, Ivan Gutman introduced the concept of energy of graph. It is a well-known graph invariant which is defined as the sum of the absolute values of the eigenvalues of the adjacency matrix. The adjacency matrix is a $(0, 1)-$square matrix whose $(i, j)-$entry is 1 and 0 for adjacency and other cases respectively. Topological indices are popular for their numerous applications. There are several topological indices exist. For more details refer [3] [4] [13] [14]. The first and second Zagreb indices are the well known molecular descriptors in the literature of toplogocal indices. Taking into the account of first and second Zagreb indices and their differences, B. Furtula et al. [2] proposed a new degree based topological index, reduced second Zagreb index which is defined as

$$RM_2(G) = \sum_{uv \in E} (d_u - 1)(d_v - 1)$$

where d_u and d_v represents the degree of the respective vertices u and v.

In 20th and 21st century, many matrices were studied by the researchers such as Randić matrix (basically derived from Randić index), sum conncetivity matrix [10], atom bond connectivity matrix [9], harmonic matrix etc., In this paper we consider domination parameters along with reduced second Zagreb (RM_2) matrix. We consider both minimum dominating and equitable dominating sets for our further results.

Definition 1.1 *[8] For a graph G, with vertex set $V = \{v_1, v_2, v_3, ..., v_n\}$, a subset D of V is called a dominating set of G if every vertex of $V - D$ is adjacent to some vertex in D. Any dominating set with minimum cardinality is called a minimum dominating set.*

Definition 1.2 *[11] A subset U of $V(G)$ is an equitable dominating set if for every $v \in V(G) - U$, there exists a vertex $u \in U$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$ where $deg(x)$ denotes the degree of vertex x in $V(G)$. Any equitable dominating set with minimum cardinality is called a minimum equitable dominating set.*

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Motivated by the reduced second Zagreb index and domination concept, we introduce minimum dominating reduced second Zagreb matrix $RM_2^D(G)$ as

$$RM_2^D(G) = \begin{cases} (d_u-1)(d_v-1) & uv \in D \\ 1 & \text{if } i=j \text{ and } v_i \in D, \\ 0 & otherwise \end{cases}$$

Where D is a minimum dominating set. Ivan Gutman introduced the concept of energy of graph. For more details on energy refer [5] [6]. Let γ_i be the eigenvalues of minimum dominating reduced second Zagreb marices, then the minimum dominating reduced second Zagreb energy is given by

$$RM_2^D E(G) = \sum_{i=1}^{n} |\gamma_i|$$

2. Some basic properties of the dominating reduced second Zagreb energy of a graph

In the current section, we give the results with respect to any dominating set. We use the notation β which may be any dominating set (viz. minimum dominating, equitable dominating sets etc.,) Here $\Phi^\beta(G,\gamma)$ represents the dominating reduced second Zagreb characteristic polynomial of $RM_2^\beta(G)$.

Proposition 2.1 *The first three coefficients of $\Phi^\beta(G,\gamma)$ are given as follows:*
(i) $a_0 = 1$,
(ii) $a_1 = -|\beta|$,
(iii) $a_2 = \binom{|\beta|}{2} - \sum_{i<j} [(d_i-1)(d_j-1)]^2$.

Proof: (i) From the definition of characteristic polynomial $\Phi^\beta(G,\gamma) = det[\gamma I - RM_2^\beta(G)]$, we get $a_0 = 1$.

(ii) Consider the sum of determinants of all 1×1 principal submatrices of $RM_2^\beta(G)$ which is equal to the trace of $RM_2^\beta(G)$.
$\Rightarrow a_1 = (-1)^1$ trace of $[RM_2^\beta(G)] = -|\beta|$.

(iii)

$$\begin{aligned}(-1)^2 a_2 &= \sum_{1\le i<j\le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ &= \sum_{1\le i<j\le n} a_{ii}a_{jj} - a_{ji}a_{ij}\end{aligned}$$

At $i = j$, the entry of the matrix will take the values 0 or 1 depends on the vertex belongs to dominating set or not.

$$= \binom{|\beta|}{2} - \sum_{i<j} [(d_i-1)(d_j-1)]^2$$

□

Proposition 2.2 *If $\gamma_1, \gamma_2, \ldots, \gamma_n$ are the dominating second Zagreb eigenvalues of $RM_2^\beta(G)$, then*

$$\sum_{i=1}^{n} {\gamma_i}^2 = |\beta| + 2\sum_{i<j} [(d_i-1)(d_j-1)]^2 .$$

Proof:

$$\begin{aligned}\sum_{i=1}^{n}\gamma_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}a_{ji}\\ &= 2\sum_{i<j}(a_{ij})^2+\sum_{i=1}^{n}(a_{ii})^2\\ &= 2\sum_{i<j}(a_{ij})^2+|\beta|\\ &= |\beta|+2\sum_{i<j}\left[(d_i-1)(d_j-1)\right]^2.\end{aligned}$$

□

Lemma 2.1 *[7] Cauchy - Schwartz inequality is*

$$\left(\sum_{i=1}^{n}a_ib_i\right)^2\leq\left(\sum_{i=1}^{n}{a_i}^2\right)\left(\sum_{i=1}^{n}{b_i}^2\right).$$

Theorem 2.1 *Let G be a graph with n vertices, then*

$$RM_2E^{\beta}(G)\leq\sqrt{n\left(|\beta|+2\sum_{i<j}\left[(d_i-1)(d_j-1)\right]^2\right)}.$$

Proof: Let $\gamma_1,\gamma_2,\ldots,\gamma_n$ be the eigenvalues of $P_k(G)$. By substituting $a_i=1$ and $b_i=|\gamma_i|$ in Cauchy - Schwartz inequality[Lemma 2.1], we get,

$$\left(\sum_{i=1}^{n}|\gamma_i|\right)^2\leq\left(\sum_{i=1}^{n}1\right)\left(\sum_{i=1}^{n}|\gamma_i|^2\right)$$

$$[RM_2E^{\beta}]^2\leq n\left(|\beta|+2\sum_{i<j}\left[(d_i-1)(d_j-1)\right]^2\right)$$

$$[RM_2E^{\beta}]\leq\sqrt{n\left(|\beta|+2\sum_{i<j}\left[(d_i-1)(d_j-1)\right]^2\right)}.$$

This is an upper bound for $RM_2E^{\beta}(G)$. □

Theorem 2.2 *Let G be a graph with n vertices. If $det(RM_2^{\beta}(G))$ represents determinant of the matrix $RM_2^{\beta}(G)$, then*

$$RM_2E^{\beta}(G)\geq\sqrt{\left(|\beta|+2\sum_{i<j}\left[(d_i-1)(d_j-1)\right]^2\right)+n(n-1)(det(RM_2^{\beta}(G)))^{\frac{2}{n}}}.$$

Proof: Using the definition of energy of graph,

$$\begin{aligned}\left(RM_2E^{\beta}(G)\right)^2 &= \left(\sum_{i=1}^{n} \mid \gamma_i \mid\right)^2 \\ &= \sum_{i=1}^{n} \mid \gamma_i \mid \sum_{j=1}^{n} \mid \gamma_j \mid \\ &= \left(\sum_{i=1}^{n} \mid \gamma_i \mid^2\right) + \sum_{i\neq j} \mid \gamma_i \mid\mid \gamma_j \mid .\end{aligned}$$

Employing arithmetic mean and geometric mean inequality, we get

$$RM_2E^{\beta}(G) \geq \sqrt{\left(|\beta| + 2\sum_{i<j}[(d_i-1)(d_j-1)]^2\right) + n(n-1)(det(RM_2^{\beta}(G)))^{\frac{2}{n}}}.$$

□

3. Dominating reduced second Zagreb energy of some standard graphs

Theorem 3.1 *The minimum dominating and minimum equitable dominating reduced second Zagreb energy of a complete graph K_n is*

$$RM_2E^D(K_n) = (n-2)^3 + \sqrt{n^6 - 8n^5 + 24n^4 - 34n^3 + 28n^2 - 24n + 17}.$$

Proof: Let K_n be the complete graph with vertex set $V = \{v_1,\ v_2,\ ...,\ v_n\}$ and minimum dominating and minimum equitable dominating set is $\{v_1\}$. Then, the minimum dominating reduced second Zagreb matrix (which is similar to minimum equitable dominating reduced second Zagreb matrix) is

$$RM_2^D(K_n) = \begin{bmatrix} 1 & (n-2)^2 & (n-2)^2 & \dots & (n-2)^2 & (n-2)^2 \\ (n-2)^2 & 0 & (n-2)^2 & \dots & (n-2)^2 & (n-2)^2 \\ (n-2)^2 & (n-2)^2 & 0 & \dots & (n-2)^2 & (n-2)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (n-2)^2 & (n-2)^2 & \dots & (n-2)^2 & 0 & (n-2)^2 \\ (n-2)^2 & (n-2)^2 & \dots & (n-2)^2 & (n-2)^2 & 0 \end{bmatrix}.$$

Hence the characteristic equation will be

$$\left(\gamma + (n-2)^2\right)^{n-2}\left(\gamma^2 - (n-1)(n^2-5n+7)\gamma - (n-2)^2(n^3-5n^2+7n-2)\right) = 0$$

and therefore the spectrum becomes

$$Spec_{RM_2^D}(K_n) = \begin{pmatrix} -(n-2)^2 & \frac{(n^3-6n^2+12n-7)+A}{2} & \frac{(n^3-6n^2+12n-7)-A}{2} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Here $A = \sqrt{n^6 - 8n^5 + 24n^4 - 34n^3 + 28n^2 - 24n + 17}$.
Therefore,

$$RM_2^D(K_n) = (n-2)^3 + \sqrt{n^6 - 8n^5 + 24n^4 - 34n^3 + 28n^2 - 24n + 17}.$$

□

Theorem 3.2 *The minimum dominating reduced second Zagreb energy of the star graph $K_{1,n-1}$ is*

$$RM_2E^D(K_{1,n-1}) = 1.$$

Proof: Let $K_{1,n-1}$ be the star graph with vertex set $V = \{v_0, v_1 ... v_{n-1}\}$. The minimum dominating reduced second Zagreb matrix is

$$RM_2^D(K_{1,n-1}) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Therefore,

$$RM_2E^D(K_{1,n-1}) = 1.$$

□

Theorem 3.3 *The minimum dominating and minimum equitable dominating reduced second Zagreb energy of the crown graph S_n^0 is*

$$RM_2E^D(S_n^0) = \sqrt{n^6 - 8n^5 + 24n^4 - 30n^3 + 4n^2 + 24n - 15} + \sqrt{n^6 - 8n^5 + 24n^4 - 34n^3 + 28n^2 - 24n + 17} + 2(n-2)^3$$

Proof: Let S_n^0 be the crown graph of order $2n$ with vertex set $\{u_1,\ u_2,\ \cdots,\ u_n,\ v_1,\ v_2,\ \cdots,\ v_n\}$. The minimum dominating and minimum equitable dominating reduced second Zagreb matrix is

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & (n-2)^2 & \dots & (n-2)^2 & (n-2)^2 \\ 0 & 0 & 0 & \dots & 0 & (n-2)^2 & 0 & \dots & (n-2)^2 & (n-2)^2 \\ 0 & 0 & 0 & \dots & 0 & (n-2)^2 & (n-2)^2 & \dots & 0 & (n-2)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & (n-2)^2 & (n-2)^2 & \dots & (n-2)^2 & 0 \\ 0 & (n-2)^2 & (n-2)^2 & \dots & (n-2)^2 & 1 & 0 & \dots & 0 & 0 \\ (n-2)^2 & 0 & (n-2)^2 & \dots & (n-2)^2 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (n-2)^2 & (n-2)^2 & 0 & \dots & (n-2)^2 & 0 & 0 & \dots & 0 & 0 \\ (n-2)^2 & (n-2)^2 & (n-2)^2 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

In that case the characteristic equation is

$$\left(\gamma^2 + ((n-2)^3 - 1)\gamma - (n-2)^3(n^2 - 3n + 3)\right)\left(\gamma^2 - ((n-2)^3 + 1)\gamma - (n-2)^3(n^2 - 3n + 1)\right)$$

Implying that the spectrum is

$$Spec_{RM_2^D}(S_n^0) = \begin{pmatrix} \frac{(A_1 \pm C)}{2} & (n-2)^2 & -(n-2)^2 & \frac{(A_1 \pm D)}{2} \\ 1 & n-2 & n-2 & 1 \end{pmatrix}.$$

$A_1 = (n-2)^3 + 1$, $C = \sqrt{n^6 - 8n^5 + 24n^4 - 30n^3 + 4n^2 + 24n - 15}$
and $D = \sqrt{n^6 - 8n^5 + 24n^4 - 34n^3 + 28n^2 - 24n + 17}$
Therefore,

$$RM_2E^D(S_n^0) = \sqrt{n^6 - 8n^5 + 24n^4 - 30n^3 + 4n^2 + 24n - 15} + \sqrt{n^6 - 8n^5 + 24n^4 - 34n^3 + 28n^2 - 24n + 17} + 2(n-2)^3$$

□

Definition 3.1 *[12] The friendship graph, denoted by F_3^n, is the graph obtained by taking n copies of the cycle graph C_3 with a vertex in common.*

Theorem 3.4 *The minimum dominating second Zagreb energy of the friendship graph F_3^n is*

$$RM_2E^D(F_3^n) = \frac{n-1}{2} + \frac{n-3}{2} + 2\sqrt{n^3-5n^2+8n-4}.$$

Proof: Here n represents total number of vertices. The minimum dominating second Zagreb matrix is

$$\begin{bmatrix} 1 & n-2 & n-2 & n-2 & n-2 & \dots & n-2 & n-2 \\ n-2 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ n-2 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ n-2 & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ n-2 & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-2 & 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ n-2 & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

Hence, the spectrum is

$$Spec_{RM_2^D}(F_3^n) = \begin{pmatrix} -1 & 1 & \frac{1+\sqrt{n^3-5n^2+8n-4}}{2} & \frac{5-\sqrt{n^3-5n^2+8n-4}}{2} \\ \frac{n-1}{2} & \frac{n-3}{2} & 1 & 1 \end{pmatrix}.$$

Therefore,

$$RM_2E^D(F_3^n) = \frac{n-1}{2} + \frac{n-3}{2} + 2\sqrt{n^3-5n^2+8n-4}.$$

□

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