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A New Interpolative $(\varphi, \psi)-$Type $\mathfrak{Z}-$Contraction with Application to Nonlinear Matrix Equation Systems

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ABSTRACT: In this study, we introduce a new concept of interpolative $(\varphi, \psi)-$type $\mathfrak{Z}-$contraction via quasi-triangular $\theta-$admissible mapping and prove some related fixed point theorems in the context of $b-$metric space. As an application of our results, we solve a system of nonlinear matrix equations. Finally, a numerical example is presented to demonstrate the validity and practical significance of our approach.

Key Words: Fixed point, Interpolative $(\varphi, \psi)-$type $\mathfrak{Z}-$contraction, altering distance function, comparison function, non-linear matrix equation.

Contents

1. Introduction and preliminaries

Fixed point theory has gained a very impetus due to its wide range of applications in various fields such as engineering, economics, computer science, etc. It is well known that contractive conditions are indispensable in studying the metric fixed point theory, and Banach fixed point theorem [4] is one of the key results.

Theorem 1.1 *(Banach [4]) Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping such that*

$$d(Tx, Ty) \leq kd(x, y), \text{ for all } x, y \in X, \tag{1.1}$$

then T has a unique fixed point in X.

The above inequality (1.1) is known as a contraction or, a Banach contraction. Note that the mapping satisfying the Banach contraction is necessarily continuous, so it was natural to ask whether there is a discontinuous mapping that fulfills certain contractive conditions and possesses a fixed point in the framework of complete metric spaces. In 1968, Kannan [12] first gave a favorable answer to this question by introducing a new type of contraction. Let (X, d) be a metric space and $T : X \to X$ be a self-mapping such that

$$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)], \tag{1.2}$$

where $\alpha \in [0, \frac{1}{2})$, for all $x, y \in X$. The mapping T satisfying the inequality (1.2) is known as the Kannan contraction.

In 1976, Khan ([19], [20]) first used the idea of a geometric mean of Kannan-type contraction. A mapping $T : X \to X$ is called Khan type contraction, if there exists $0 \leq k < 1$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq k(d(x, Tx)d(y, Ty))^{\frac{1}{2}}.$$

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Furthermore, in 2018, Karapinar [13] introduced an interpolative Kannan-type contraction by revisiting Kannan-type contraction. Let (X, d) be a metric space; the self-mapping $T : X \to X$ is said to be an interpolative Kannan-type contraction if there exists $k \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq k[d(x, Tx)]^{\alpha}.[d(y, Ty)]^{1-\alpha}$$

for all $x, y \in X - Fix\{T\}$, where $Fix\{T\} = \{x \in X, Tx = x\}$.

Theorem 1.2 *[16] Let (X, d) be a complete metric space. Let T be a self-mapping satisfying the interpolative Kannan-type contraction. Then, T has a fixed point in X.*

Definition 1.1 *[14] Let (X, d) be a complete partial metric space. $T : X \to X$ is said to be an interpolative Reich-Rus-Ćirić type contraction, if there exists $k \in [0, 1)$ $\alpha_1, \alpha_2 \in (0, 1)$ and $\alpha_1 + \alpha_2 < 1$ such that*

$$d(Tx, Ty) \leq k[d(x, y)]^{\alpha_1}.[d(x, Tx)]^{\alpha_2}.[d(y, Ty)]^{1-\alpha_1-\alpha_2}.$$

Theorem 1.3 *[14] Let (X, d) be a complete partial metric space and $T : X \to X$ is an interpolative Reich-Rus-Ćirić type contraction, then T has a fixed point in X.*

Definition 1.2 *[15] Let (X, d) be a metric space. $T : X \to X$ is said to be an interpolative Hardy-Rogers type contraction if there exists $k \in [0, 1)$ and $\alpha_1, \alpha_2, \alpha_3 \in (0, 1)$ and $\alpha_1 + \alpha_2 + \alpha_3 < 1$ such that*

$$d(Tx, Ty) \leq kH(x, y), \text{ for all } x, y \in X - Fix\{T\},$$

where
$H(x, y) = [d(x, y)]^{\alpha_1}.[d(x, Tx)]^{\alpha_2}.[d(y, Ty)]^{\alpha_3}.[\frac{1}{2}(d(x, Ty) + d(y, Tx))]^{1-\alpha_1-\alpha_2-\alpha_3}$

Theorem 1.4 *[15] Let (X, d) be a complete metric space and $T : X \to X$ is an interpolative Hardy-Rogers-type contraction, then T has a fixed point in X.*

Definition 1.3 *(Czerwik [8], Bakhtim [3]) Let X be a non-empty set and $s \geq 1$. Let $d : X \times X \to [0, \infty)$ be a mapping that satisfies the following: for all $x, y, z \in X$*
(B_1) $x = y$ if and only if $d(x, y) = 0$;
(B_2) $d(x, y) = d(y, x)$;
(B_3) $d(x, y) \leq s[d(x, z) + d(z, y)]$.
Then the mapping d is called a $b-$metric and (X, d) is called a $b-$metric space.

Example 1.1 *[3] Let $X = [0, 1]$ and define $d : X \to X$ as $d(x, y) = |x - y|^p$, where $p > 1$ is a fixed real number. Then d is a $b-$metric with $s = 2^{p-1}$. Indeed, conditions (B_1) and (B_2) in Definition 1.3 are satisfied, and thus we only need to show that condition (B_3) holds for d.*

It is easy to see that if $1 < p < \infty$ then the convexity of the function $f(x) = x^p$ where $x \geq 0$ implies

$$\left(\frac{a+c}{2}\right)^p \leq \frac{1}{2}\left(a^p + c^p\right) \text{ implies } (a + c)^p \leq 2^{p-1}\left(a^p + c^p\right).$$

Therefore, for each $x, y, z \in X$ we get

$$\begin{aligned} d(x, y) =&|x - y|^p \leq [|x - z| + |z - |]^p \\ \leq& 2^{p-1}\left[|x - z|^p + |z - y|^p\right] \\ \leq& 2^{p-1}[d(x, z) + d(z, y)]. \end{aligned}$$

So condition (B_3) in Definition 1.3 is fulfilled, and thus d is a $b-$metric with coefficient $s = 2^{p-1} > 1$.

Every metric is $b-$metric for $s = 1$, but the converse may not be true in general. Hence, the class of $b-$metric space is effectively larger than that of metric space.

The following example shows that $b-$metric space need not be a metric space.

Example 1.2 *Let $X = \{0, 3, 4\}$ and $d : X \times X \to [0, \infty)$ defined by*

$$\begin{aligned} &d(0,0) = d(3,3) = d(4,4) = 0; \\ &d(0,3) = d(3,0) = d(4,3) = d(3,4) = 2; \\ &d(4,0) = d(0,4) = q, \end{aligned}$$

where $q \in [4, \infty)$. It is easy to see that

$$d(x,y) \leq \frac{q}{4}[d(x,z) + d(z,y)],$$

for all $x, y, z \in X$. Therefore, (X, d) is a $b-$metric space with $s = \frac{q}{4}$. It is also seen that the ordinary triangle inequality does not hold if $q > 4$ and then (X, d) is not a metric space.

For more examples and fixed point results in $b-$metric spaces, we refer to ([1], [6], [5], [8], [11] and [28]).

The distance function used in $b-$metric spaces is not continuous in general ([5], [11]).

Example 1.3 *[23] Let $X = \mathbb{N} \cup \{\infty\}$. Define a mapping $d : X \times X \to \mathbb{R}^+$ by*

$$d(m,n) = \begin{cases} 0, & m = n, \\ |\frac{1}{n} - \frac{1}{m}|, & \text{if one of } m, n \text{ is even and the other is even or } \infty, \\ 5, & \text{if one of } m, n \text{ is odd and the other is odd or } \infty, \\ 2, & \text{otherwise.} \end{cases}$$

Obviously, conditions $(B1)$ and $(B2)$ of Definition 1.3 are satisfied. Considering all possible cases that for all $x, y, z \in X$, we have

$$d(x,y) \leq \frac{5}{2}[d(x,z) + d(z,y)].$$

This shows that condition $(B3)$ of Definition1.3 is satisfied. Thus (X, d) is a $b-$metric space with $s = \frac{5}{2}$. Let $x_m = 2m$, then $d(x_m, 1) = 2$ and $d(\infty, 1) = 5$. Therefore, $d(x_m, 1) = 2 \neq 5 = d(\infty, 1)$ as $m \to \infty$. This show that $d(x, y)$ is not continuous at the first variable. Further, we also consider $x_m = 2m$, $y_n = 2n+1$, then $d(2m, 2n+1) = 2$ and $d(\infty, \infty) = 0$, then $d(2m, 2n+1) \to 2 \neq 0 = d(\infty, \infty)$ as $m, n \to \infty$. This shows that $d(x, y)$ is also discontinuous jointly in both variables.

Definition 1.4 *[10] Let (X, d) be a $b-$metric space with $s \geq 1$. Let $\{x_n\}$ be a sequence a sequence in X. We say that*
(i) $\{x_n\}$ $b-$converges to $x_0 \in X \iff d(x_n, x_0) \to 0$ as $n \to \infty$;
(ii) $\{x_n\}$ is $b-$ Cauchy sequence in $X \iff d(x_n, x_m) \to 0$ as $m, n \to \infty$;
(iii) (X, d) is $b-$complete metric space $\iff$ every $b-$ Cauchy sequence in X is $b-$ convergent.

Lemma 1.1 *[1] Let (X, d) be a $b-$metric space with coefficient $s \geq 1$ and let $\{x_n\}$ and $\{y_n\}$ be $b-$convergent to points $x, y \in X$, respectively. Then we have*

$$\frac{1}{s^2}d(x,y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x,y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s}d(x,z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s d(x,z).$$

In 1984, Khan et al. [18] introduced the notion of altering distance function.

Definition 1.5 *[18] A continuous function $\varphi : [0, \infty) \to [0, \infty)$ is said to be an altering distance function if it is non-decreasing and $\varphi(x) = 0 \iff x = 0$.*

It is obvious that $\varphi(x) \geq 0$, for all $x \geq 0$. We denote Φ, the set of all altering distance functions.

Example 1.4 *Let $\varphi_i : [0,\infty) \to [0,\infty)$ be continuous functions. For $i = 1,2$,*
(i) $\varphi_1(t) = at^2, a > 0$;
(ii) $\varphi_2(t) = a\ln(1-t), a > 0$.

Obviously, $\varphi_{i=1,2}$ is an altering function.

Definition 1.6 *([6], [7]) A function $\psi : [0,\infty) \to [0,\infty)$ is said to be a comparison function if it is monotonically increasing and $\psi^n(x) \to 0$ as $n \to \infty$, for all $x > 0$.*

It is clear that $\psi(x) < x$, for all $x > 0$ and $\psi(0) = 0$. The symbol Ψ denotes the set of all comparison functions.

Example 1.5 *Let $\psi_i : [0,\infty) \to [0,\infty)$ where $i = 1,2$, be defined by*
(i) $\psi_1(t) = at, a \in [0,1)$;
(ii) $\psi_2(t) = \frac{t^2}{a+t^2}, a > 0$.
Obviously, $\psi_{i=1,2}$ is a comparison function.

In 2015, Khojasteh et al. [17] introduced simulation function and generalized the Banach contraction principle.

Definition 1.7 *[17] A mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is said to be a simulation function, if it satisfies the following conditions:*
(ζ_1) $\zeta(0,0) = 0$;
(ζ_2) $\zeta(x,y) < y - x$, for all $x, y > 0$;
(ζ_3) if $\{x_n\}, \{y_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} y_n > 0$, then $\lim_{n\to+\infty}$ sup $\zeta(x_n, y_n) < 0$.

The set of all simulation functions is denoted by $\mathfrak{Z}$.

Definition 1.8 *[17] Let (X,d) be a metric space and $T : X \to X$ be a mapping. If there exist $\zeta \in \mathfrak{Z}$ such that*

$$\zeta(d(Tx,Ty), d(x,y)) \geq 0, \text{ for all } x, y \in X,$$

then T is called $\mathfrak{Z}-$contraction with respect to ζ.

In the same year, Argoubi et al. [2] refined the above notion of simulation function by removing the first condition (ζ_1). Note that the condition (ζ_1) is indeed obtained from (ζ_2), if T is a $\mathfrak{Z}-$contraction with respect to ζ. A basic example of $\mathfrak{Z}-$contraction is Banach contraction, which is obtained by setting $\zeta(t,s) = \alpha s - t$, where $\alpha \in [0,1)$. In the sense of Argoubi et al. [2], we have the following:

Definition 1.9 *[2] A mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is said to be a simulation function, if it satisfies the conditions (ζ_2) and (ζ_3).*

Example 1.6 *Let $\zeta_i : [0,\infty) \times [0,\infty) \to \mathbb{R}$ where $i = 1,2$, be defined by*
(i) $\zeta_1(x,y) = \frac{x-2y^2}{2}$;
(ii) $\zeta_2(x,y) = y - y^2 - x$.

In 2019, Karapinar [16] combined these two approaches (simulation functions and interpolation) and investigated the existence of fixed points that forms interpolative contractions in the framework of simulation functions in the context of complete metric spaces.

Definition 1.10 *[16] A mapping $T : X \to X$ is called an interpolative Hardy-Rogers type $\mathfrak{Z}-$contraction with respect to ζ, if there exist $\zeta \in \mathfrak{Z}, \alpha_1, \alpha_2, \alpha_3 \in (0,1)$, such that $\alpha_1+\alpha_2+\alpha_3 < 1$ satisfying the inequality*

$$\zeta\Big(d((Tx,Ty), H(x,y)\Big) \geq 0, \text{ for all } x, y \in X - Fix\{T\},$$

where

$$H(x,y) = [d(x,y)]^{\alpha_1}[d(x,Tx)]^{\alpha_2}[d(y,Ty)]^{\alpha_3}[\frac{1}{2}(d(x,Ty) + d(y,Tx))]^{1-\alpha_1-\alpha_2-\alpha_3}.$$

Theorem 1.5 *[16] Let (X, d) be a complete metric space and T be an interpolative Hardy-Rogers type $\mathfrak{Z}-$contraction with respect to ζ. Then there exists $u \in X$ such that $Tu = u$.*

If T is an interpolative Kannan type contraction(resp. Reich-Rus-Ćirić type contraction, Hardy-rogers type contraction, $\mathfrak{Z}-$contraction and Hardy-Rogers type $\mathfrak{Z}-$contraction), then T is continuous (see for details in [21]).

Let X be a nonempty set and $\theta : X \times X \to [0, \infty)$ be a function.

Definition 1.11 *[25] A mapping $T : X \to X$ is said to be $\theta-$admissible if*
(θ_1) $\theta(x, y) \geq 1 \implies \theta(Tx, Ty) \geq 1, x, y \in X.$

Definition 1.12 *[26] A mapping $T : X \to X$ is said to be triangular $\theta-$admissible if it satisfies (θ_1) and*
(θ_2) $\theta(x, y) \geq 1$ *and* $\theta(y, z) \geq 1 \implies \theta(x, z) \geq 1, x, y, z \in X.$

The concept of $\theta-$orbital admissible mappings was introduced by Popescu [27] as a refinement of $\theta-$admissibility.

Definition 1.13 *[27] A mapping $T : X \to X$ is said to be $\theta-$orbital admissible if it satisfies*
(θ_3) $\theta(x, Tx) \geq 1 \implies \theta(Tx, T^2x) \geq 1, x \in X.$

Definition 1.14 *[27] A mapping $T : X \to X$ is said to be triangular $\theta-$orbital admissible if it satisfies (θ_3) and*
(θ_4) $\theta(x, y) \geq 1$ *and* $\theta(y, Ty) \geq 1 \implies \theta(x, Ty) \geq 1, x, y, \in X.$

If we substitute $Tx = y$ in (θ_3), then $\theta(x, y) \geq 1 \implies \theta(Tx, Ty) \geq 1, x, y \in X$ which is (θ_1). So, every $\theta-$orbital admissible mapping is also $\theta-$admissible mapping.

Definition 1.15 *[21] A mapping $T : X \to X$ is said to be quasi triangular $\theta-$orbital admissible if T satisfies (θ_3) and (θ_5) $\theta(x, y) \geq 1 \implies \theta(x, Ty) \geq 1, x, y \in X.$*

Obviously, every triangular $\theta-$orbital admissible mapping is quasi triangular $\theta-$orbital admissible. But, the converse may not be true(for more details one can refer to [21]).

The concept of quasi triangular $\theta-$admissible mapping is defined as follows:

Definition 1.16 *A mapping $T : X \to X$ is said to be quasi triangular $\theta-$admissible if T satisfies θ_1 and (θ_6) $\theta(x, Tx) \geq 1 \implies \theta(x, T^2x) \geq 1, x \in X.$*

Note that every quasi triangular $\theta-$admissible mapping may not be a triangular $\theta-$admissible.

Example 1.7 *Let $X = \{0,1,2,3\}$ with usual metric $d(x, y) = | x - y |$. Let $T : X \to X$ and $\theta : X \times X \to \mathbb{R}$ be mappings defined by $T0 = 0, T1 = 2, T2 = 3, T3 = 1$ and*

$$\theta(x, y) = \begin{cases} 1, (x, y) \in A \\ 0, (x, y) \notin A \end{cases},$$

where $A = \{(1,2),(2,3),(3,1),(2,1),(3,2),(1,3)\}$.
Now, We have,

$$\begin{aligned}
\theta(1, 2) = 1 \quad &\mathit{and} \quad \theta(T1, T2) = \theta(2, 3) = 1, \\
\theta(2, 3) = 1 \quad &\mathit{and} \quad \theta(T2, T3) = \theta(3, 1) = 1, \\
\theta(3, 1) = 1 \quad &\mathit{and} \quad \theta(T3, T1) = \theta(1, 2) = 1, \\
\theta(2, 1) = 1 \quad &\mathit{and} \quad \theta(T2, T1) = \theta(3, 2) = 1, \\
\theta(3, 2) = 1 \quad &\mathit{and} \quad \theta(T3, T2) = \theta(1, 3) = 1, \\
\theta(1, 3) = 1 \quad &\mathit{and} \quad \theta(T1, T3) = \theta(2, 1) = 1.
\end{aligned}$$

So, T satisfies (θ_1). Also, we have

$$\begin{aligned} \theta(1,T1) &= \theta(1,2) = 1 \quad \text{and} \quad \theta(1,T^2 1) = \theta(1,T2) = \theta(1,3) = 1, \\ \theta(2,T2) &= \theta(2,3) = 1 \quad \text{and} \quad \theta(2,T^2 2) = \theta(2,T3) = \theta(2,1) = 1, \\ \theta(3,T3) &= \theta(3,1) = 1 \quad \text{and} \quad \theta(3,T^2 3) = \theta(3,T1) = \theta(3,2) = 1. \end{aligned}$$

So, T satisfies (θ_6). However, T does not satisfy (θ_2) because $\theta(2,1) = \theta(1,2) = 1$, but $\theta(2,2) = 0$. So, T does not satisfy (θ_2). Hence, T is quasi triangular $\theta-$admissible but not triangular $\theta-$admissible mapping.

In 2012, Bessem Samet et al. [25] introduced the concept of $\theta - \psi-$ contractive mapping.

Definition 1.17 *[25] Let $T : X \to X$ is called an $\theta - \Psi-$contractive mapping if there exist two functions $\theta : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that*

$$\theta(x,y)d(Tx,Ty) \leq \psi(d(x,y)), \text{ for all } x,y \in X.$$

In 2021, Khan et al. [21] introduced interpolative (φ, ψ) type $\mathfrak{Z}-$contraction.

Definition 1.18 *[21] A mapping $T : X \to X$ is said to be interpolative $(\varphi, \psi)-$ Banach-Kannan-Chatterjea type $\mathfrak{Z}-$contraction with respect to ζ (in short, interpolative $(\varphi, \psi)-$ BKC type $\mathfrak{Z}-$contraction), if there exist $\theta : X \times X \to \mathbb{R}, \zeta \in \mathfrak{Z}, \varphi \in \Phi, \psi \in \Psi, \alpha_1, \alpha_2 \in (0,1)$ such that $\varphi(t) > \psi(t)$, for $t > 0$ and $\alpha_1 + \alpha_2 < 1$ satisfying the inequality*

$$\zeta(\theta(x,y)\varphi\big(d(Tx,Ty)), \psi(B(x,y))\big) \geq 0, \text{ for all } x,y \in X - Fix\{T\},$$

where

$$B(x,y) = [d(x,y)]^{\alpha_1}[\frac{1}{2}(d(x,Tx) + d(y,Ty))]^{\alpha_2}[\frac{1}{2}(d(x,Ty) + d(y,Tx))]^{1-\alpha_1-\alpha_2}.$$

Theorem 1.6 *[21] Let T be a self-mapping on a complete metric space (X,d). Suppose that T is quasi triangular $\theta-$orbital admissible and forms an interpolative $(\varphi, \psi)-$ BKC type $\mathfrak{Z}-$contraction with respect to ζ. If there exist $x_0 \in X$ such that $\theta(x_0, Tx_0) \geq 1$ and T is continuous, then T has a fixed point in X.*

Definition 1.19 *[21] A mapping $T : X \to X$ is said to be interpolative $(\varphi, \psi)-$ Hardy-Rogers type $\mathfrak{Z}-$contraction with respect to ζ (in short, interpolative $(\varphi, \psi)-$ HR type $\mathfrak{Z}-$contraction) if there exist $\theta : X \times X \to \mathbb{R}, \zeta \in \mathfrak{Z}, \varphi \in \Phi, \psi \in \Psi, \alpha_i \in (0,1)$, where $i = 1,2,3$, such that $\varphi(t) > \psi(t)$, for $t > 0$ and $\sum_{i=1}^{3} \alpha_i < 1$ satisfying the inequality*

$$\zeta(\theta(x,y)\varphi\big(d(Tx,Ty)), \psi(H(x,y))\big) \geq 0, \text{ for all } x,y \in X - Fix\{T\},$$

where

$$H(x,y) = [d(x,y)]^{\alpha_1}[d(x,Tx)]^{\alpha_2}[d(y,Ty))]^{\alpha_3}[\frac{1}{2}(d(x,Ty) + d(y,Tx))]^{1-\sum_{i=1}^{3}\alpha_i}.$$

Theorem 1.7 *[21] Let T be a self-mapping on a complete metric space (X,d). Suppose that T is quasi triangular $\theta-$orbital admissible and forms an interpolative $(\varphi, \psi) -$ HR type $\mathfrak{Z}-$contraction with respect to ζ. If there exist $x_0 \in X$ such that $\theta(x_0, Tx_0) \geq 1$ and T is continuous, then T has a fixed point in X.*

Lemma 1.2 *Let (X,d) be a $b-$metric space with $s \geq 1$. Let $T : X \to X$ be a quasi triangular $\theta-$admissible mapping. Assume that $x_0 \in X$ such that $\theta(x_0, x_1) \geq 1$. If there exist a sequence (x_n) in X such that $x_n = T^n x_0$, then $\theta(x_m, x_n) \geq 1, m > n$, for all $m,n \in \mathbb{N} \cup \{0\}$.*

Proof: Since there exist $x_0 \in X$ such that $\theta(x_0, x_1) \geq 1$, then by $\theta-$admissibility of mapping T, we have $\theta(x_1, x_2) = \theta(Tx_0, Tx_1) \geq 1$. By continuing in this process, we obtain $\theta(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$. Since T is quasi triangular θ admissible and $\theta(x_n, x_{n+1}) = \theta(x_n, Tx_n) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$, then from θ_6 we obtain $\theta(x_n, x_{n+2}) = \theta(x_n, T^2 x_n) \geq 1$. By continuing this process repeatedly with (θ_6) , we obtain that $\theta(x_n, x_m) \geq 1, m > n$, for all $m,n \in \mathbb{N} \cup \{0\}$. □

2. Interpolative $(\varphi, \psi)-S-$type and $M-$type $\mathfrak{Z}-$contraction

In this subsection, we present two variants of interpolative (φ, ψ) type $\mathfrak{Z}-$contraction namely interpolative $(\varphi, \psi)-$ $S-$type and $M-$ type $\mathfrak{Z}-$contraction.

Definition 2.1 *Let (X, d) be a $b-$complete metric space. A mapping $T : X \to X$ is called an interpolative $(\varphi, \psi)-$ $S-$ type $\mathfrak{Z}-$contraction with respect to ζ, if there exist $\theta : X \times X :\to \mathbb{R}$, $\zeta \in \mathfrak{Z}, \varphi \in \Phi, \psi \in \Psi, \alpha_i \in (0, \frac{1}{s})$, for $i = 1, 2, 3$ with $\alpha_1 + \alpha_2 < 1$ such that $\varphi(t) > \psi(t)$, for $t > 0$ satisfying the inequality*

$$0 \leq \zeta(\theta(x, y)\varphi(d(Tx, Ty)), \psi(S(x, y))), \text{ for all } x, y \in X - Fix\{T\}, \tag{2.1}$$

where

$$S(x, y) = [d(x, y)]^{\alpha_1} \left[\frac{1}{2s} \max \ \Big\{d(x, Tx), d(y, Ty)\Big\}\right]^{\alpha_2} \left[\frac{1}{2s} \max \Big\{d(x, Ty), d(y, Tx)\Big\}\right]^{1-\alpha_1-\alpha_2}.$$

Theorem 2.1 *Let T be a self-mapping on a $b-$complete metric space (X, d) with $s \geq 1$. Suppose that T is quasi triangular $\theta-$admissible and forms an interpolative $(\varphi, \psi)-$ $S-$ type type $\mathfrak{Z}-$contraction with respect to ζ. If there exists $x_0 \in X$ such that $\theta(x_0, Tx_0) \geq 1$ and T is continuous, then T has a fixed point in X.*

Proof: Let $x_0 \in X$ such that $\theta(x_0, Tx_0) \geq 1$. Consider the iterative sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0-1} = x_{n_0}$, then the prove is over. Indeed x_{n_0-1} forms a fixed point since $x_{x_0-1} = x_{n_0} = Tx_{n_0-1}$. Consequently, through out the proof we shall assume that $x_{n-1} \neq x_n$ and hence we have $d(x_{n-1}, x_n) > 0$, for all $n \in \mathbb{N}$. On the other hand, $\theta(x_0, Tx_0) \geq 1$ and T is $\theta-$admissible mapping we find that $\theta(x_1, x_2) = \theta(Tx_0, T^2x_0) \geq 1$. Recursively, we derive that $\theta(x_{n-1}, x_n) \geq 1$ for all $n \in \mathbb{N}$. From (2.1), we obtain

$$\begin{aligned} 0 &\leq \zeta(\theta(x_{n-1}, x_n)\varphi(d(Tx_{n-1}, Tx_n)), \psi(S(x_{n-1}, x_n))) \\ &= \zeta(\theta(x_{n-1}, x_n)\varphi(d(x_n, x_{n+1})), \psi(S(x_{n-1}, x_n))) \\ &< \psi(S(x_{n-1}, x_n)) - \theta(x_{n-1}, x_n)\varphi(d(x_n, x_{n+1}), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} S(x_{n-1}, x_n) =&[d(x_{n-1}, x_n)]^{\alpha_1} [\frac{1}{2s} \max\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\}]^{\alpha_2} \\ &[\frac{1}{2s} \max\{d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\}]^{1-\alpha_1-\alpha_3} \\ =&[d(x_{n-1}, x_n)]^{\alpha_1} [\frac{1}{2s} \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}]^{\alpha_2} \\ &[\frac{1}{2s} d(x_{n-1}, x_{n+1})]^{1-\alpha_1-\alpha_3} \\ \leq&[d(x_{n-1}, x_n)]^{\alpha_1} [\frac{1}{2s} \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}]^{\alpha_2} \\ &[\frac{1}{2}\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big)]^{1-\alpha_1-\alpha_2}. \end{aligned} \tag{2.3}$$

Consequently, we arrive

$$\begin{aligned} \varphi(d(x_n, x_{n+1})) \leq&\theta(x_{n-1}, x_n)\varphi\Big(d(x_n, x_{n+1})\Big) \\ <&\psi\Big(S(x_{n-1}, x_n)\Big) \\ \leq&\psi\Big([d(x_{n-1}, x_n)]^{\alpha_1} [\frac{1}{2s} \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}]^{\alpha_2} \\ &[\frac{1}{2}\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big)]^{1-\alpha_1-\alpha_2}\Big). \end{aligned} \tag{2.4}$$

Suppose $d(x_{n-1}, x_n) < d(x_n, x_{n+1})$, the from (2.4),

$$\begin{aligned}\varphi(d(x_n, x_{n+1})) \leq & \psi\Big([d(x_n, x_{n+1})]^{\alpha_1}[d(x_n, x_{n+1})]^{\alpha_2}[d(x_n, x_{n+1})]^{1-\alpha_1-\alpha_2}\Big) \\ = & \psi(d(x_n, x_{n+1})) \\ < & \varphi(d(x_n, x_{n+1})).\end{aligned}$$

This is a contradiction and hence, $d(x_{n-1}, x_n) \geq d(x_n, x_{n+1})$, for all $n \geq 1$. Therefore, $\{d(x_n, x_{n+1})\}$ is a monotonic decreasing sequence of positive real numbers and bounded below by zero. So, there exists $u \geq 0$ such that $\lim_{n\to+\infty} d(x_n, x_{n+1}) = u$. We claim that $u > 0$, otherwise from (2.3) and (2.4), we obtain

$$\begin{aligned}\varphi(d(x_n, x_{n+1})) &\leq \theta(x_{n-1}, x_n)\varphi(d(x_n, x_{n+1})) \\ &\leq \psi(S(x_{n-1}, x_n)) \\ &\leq \varphi(S(x_{n-1}, x_n)) \\ &\leq \varphi(d(x_{n-1}, x_n)).\end{aligned} \tag{2.5}$$

Taking limit as $n \to +\infty$ in (2.5), we obtain

$$\lim_{n\to+\infty} \theta(x_{n-1}, x_n)\varphi(d(x_n, x_{n+1})) = \lim_{n\to+\infty} \psi(S(x_{n-1}, x_n)). \tag{2.6}$$

Setting $s_n = \theta(x_{n-1}, x_n)\varphi(d(x_n, x_{n+1}))$ and $t_n = \psi(S(x_{n-1}, x_n))$ in (2.2) and (2.3), then by (ζ_3) with (2.6), we obtain

$$0 \leq \lim_{x\to+\infty} \sup \zeta(\theta(x_{n-1}, x_n)\varphi(d(x_n, x_{n+1})), \psi(S(x_{n-1}, x_n))) < 0.$$

This is a contradiction and thus, we obtain

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0. \tag{2.7}$$

Now, we show that (x_n) is a Cauchy sequence.Suppose not,there exist $\epsilon > 0$ for which we can find two sequences $\{m_k\}$ and $\{n_k\}$, for all $k \geq 1$ with $x_{m_k} > x_{n_k} \geq k$ such that $d(x_{n_k}, x_{m_k}) \geq \epsilon$. Further, we assume that m_k is the smallest number greater than n_k, then $d(x_{n_k}, x_{m_k-1}) < \epsilon$. By triangular inequality, we obtain

$$\epsilon \leq d(x_{n_k}, x_{m_k}) \leq s[d(x_{n_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k})] < s[\epsilon + d(x_{m_k-1}, x_{m_k})]. \tag{2.8}$$

Taking limit supremum as $k \to +\infty$ in (2.7) and (2.8) , we get

$$\epsilon \leq \limsup_{k\to+\infty} d(x_{n_k}, x_{m_k}) = \epsilon s. \tag{2.9}$$

Again by triangular inequality, we obtain

$$d(x_{n_k}, x_{m_k}) \leq s[d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{m_k})] \tag{2.10}$$

and

$$d(x_{n_k+1}, x_{m_k}) \leq s[d(x_{n_k+1}, x_{n_k}) + d(x_{n_k}, x_{m_k})]. \tag{2.11}$$

Taking limit supremum as $k \to +\infty$ in (2.10) and (2.11) , from (2.7) and (2.9) we obtain

$$\epsilon \leq s\Big(\limsup_{k\to\infty} d(x_{n_k+1}, x_{m_k})\Big) \tag{2.12}$$

and

$$\limsup_{k\to\infty} d(x_{n_k+1}, x_{m_k}) \leq s^2\epsilon. \tag{2.13}$$

From (2.12) and (2.13), we obtain

$$\frac{\epsilon}{s} \leq \Big(\limsup_{k\to\infty} d(x_{n_k+1}, x_{m_k})\Big) \leq s^2\epsilon. \tag{2.14}$$

similarly, we can obtain

$$\frac{\epsilon}{s} \leq \Big(\limsup_{k\to\infty} d(x_{m_k+1}, x_{n_k})\Big) \leq s^2\epsilon. \tag{2.15}$$

Finally, we obtain

$$\begin{aligned} d(x_{n_k+1}, x_{m_k+1}) &\leq s[d(x_{n_k+1}, x_{m_k}) + d(x_{m_k}, x_{m_k+1})] \\ &\leq sd(x_{m_k+1}, x_{m_k}) + s^2[d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k+1})]. \end{aligned} \tag{2.16}$$

Taking limit supremum as $k \to +\infty$ in (2.16), we obtain

$$\limsup_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) \leq s^3\epsilon. \tag{2.17}$$

Using triangular inequality, we have

$$\begin{aligned} d(x_{m_k}, x_{n_k}) &\leq s[d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k})] \\ &\leq sd(x_{m_k}, x_{m_k+1}) + s^2[d(x_{m_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k})]. \end{aligned} \tag{2.18}$$

Taking limit supremum as $k \to +\infty$ and using (2.7) and (2.9), we obtain

$$\frac{\epsilon}{s^2} \leq \limsup_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}). \tag{2.19}$$

From (2.17) and (2.19), we obtain

$$\frac{\epsilon}{s^2} \leq \limsup_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) \leq s^3\epsilon. \tag{2.20}$$

Since T is quasi triangular $\theta-$admissible mapping, by Lemma 1.2, we obtain $\theta(x_{n_k}, x_{m_k}) \geq 1$ for all numbers m_k, n_k such that $m_k > n_k$, where $k \geq 1$. From (2.1), we obtain

$$\begin{aligned} 0 &\leq \zeta(\theta(x_{n_k}, x_{m_k})\varphi(d(Tx_{n_k}, Tx_{m_k})), \psi(S(x_{n_k}, x_{m_k}))) \\ &= \zeta\big(\theta(x_{n_k}, x_{m_k})\varphi(d(x_{n_k+1}, x_{m_k+1})), \psi(S(x_{n_k}, x_{m_k})\big) \\ &< \psi(S(x_{n_k}, x_{m_k})) - \theta(x_{n_k}, x_{m_k})\varphi(d(x_{n_k+1}, x_{m_k+1}). \end{aligned}$$

It follows that

$$\begin{aligned} \varphi(d(x_{n_k+1}, x_{m_k+1}) &\leq \theta(x_{n_k}, x_{m_k})\varphi(d(x_{n_k+1}, x_{m_k+1}) \\ &\leq \psi(S(x_{n_k}, x_{m_k})) \\ &< \varphi(S(x_{n_k}, x_{m_k})), \end{aligned}$$

where

$$\begin{aligned} S(x_{n_k}, x_{m_k}) =&[d(x_{n_k}, x_{m_k})]^{\alpha_1}.[\frac{1}{2s}\max\{d(x_{n_k}, Tx_{n_k}), d(x_{m_k}, Tx_{m_k})\}]^{\alpha_2}. \\ &[\frac{1}{2s}\max\{d(x_{n_k}, Tx_{m_k}), d(x_{m_k}, Tx_{n_k})\}]^{1-\alpha_1-\alpha_2}. \\ =&[d(x_{n_k}, x_{m_k})]^{\alpha_1}.[\frac{1}{2s}\max\{d(x_{n_k}, x_{n_k+1}), d(x_{m_k}, x_{m_k+1})\}]^{\alpha_2}. \\ &[\frac{1}{2s}\max\{d(x_{n_k}, x_{m_k+1}), d(x_{m_k}, x_{n_k+1})\}]^{1-\alpha_1-\alpha_2}. \end{aligned}$$

Taking limit supremum as $k \to +\infty$ together with (2.9),(2.14), (2.15) and (2.20), we obtain

$$0 \leq \varphi\Big(\frac{\epsilon}{s^2}\Big) < \varphi(0) = 0 \implies \varphi\Big(\frac{\epsilon}{s^2}\Big) = 0 \implies \epsilon = 0.$$

This is a contradiction and hence $\{x_n\}$ is a Cauchy sequence in X.

Since X is $b-$complete, there exists $\omega \in X$ such that $\lim_{n\to+\infty} x_n = \omega$. On account of T is continuous, we find that $T\omega = \lim_{n\to+\infty} Tx_n = \lim_{n\to+\infty} x_{n+1} = \omega$. Hence, ω is the desired fixed point of T. □

Example 2.1 *Let $X = [0,\infty)$ with $b-$metric $d(x,y) = |x-y|^2$. Suppose $\theta : X \times X \to \mathbb{R}$ and $T : X \to X$ are mappings defined by*

$$\theta(x,y) = \begin{cases} 1, & 0 \leq x, y \leq 10 + ln(\frac{3}{5})x \leq y \\ 0, & otherwise \end{cases},$$

$$Tx = \begin{cases} 6, & 0 \leq x \leq 10 + ln(\frac{3}{5}) \\ 10e^{x-10}, & 10 + ln(\frac{3}{5}) < x < \infty. \end{cases}$$

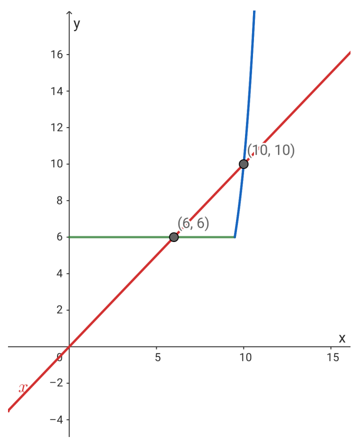

Figure 1: Figure 1. green, blue and red line indicates 6, when $0 \leq x \leq 10 + ln(\frac{3}{5})$, $10e^{x-10}$, when $10 + ln(\frac{3}{5}) < x < \infty$ and $y = x$ respectively.

It is obvious that T is quasi triangular $\theta-$admissible in X. Let $x_0 \in X$ such that $\theta(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ in X such that $x_n = T^n x_0$, for all $n \in \mathbb{N}$. By $\theta-$admissibility of T, we have $\theta(Tx_0, T^2x_0) \geq 1$. Recursively, one may obtain that $\theta(T^{n-1}x_0, T^n x_0) \geq 1$.

Taking $\zeta(t,q) = \psi(q) - t$, for all $q, t > 0$ in Theorem 2.1, we obtain

$$\theta(x,y)\varphi(d(Tx,Ty)) \leq \psi(S(x,y)), \text{ for all } x, y \in X.$$

Setting $\varphi(t) = \frac{t}{2}, \psi(t) = \frac{3t}{4}, t > 0$ where $\alpha_1 = \frac{1}{4}, \alpha_2 = \frac{1}{6}, s = 2$, then $\varphi(t) > \psi(t)$.

For $0 \leq x, y \leq 10 + ln(\frac{3}{5})$, we obtain

$$\begin{aligned} \theta(x,y)\varphi(d(Tx,Ty)) = 0 \leq& \psi(S(x,y)) \\ =& \frac{3}{4}|x-y|^{\frac{1}{2}}\Big(\frac{|x-6|^2}{4}\Big)^{\frac{1}{6}}\Big(\frac{|y-6|^2}{4}\Big)^{\frac{7}{12}}. \end{aligned}$$

For $10 + ln(\frac{3}{5}) < x < \infty$,

$$\begin{aligned} \theta(x,y)\varphi(d(Tx,Ty)) = 0 \leq \psi(S(x,y)) \\ = \frac{3}{4}|x-y|^{\frac{1}{2}}\Big(\frac{|x-10e^{(x-10)}|^2}{4}\Big)^{\frac{1}{6}}\Big(\frac{|x-10e^{(y-10)}|^2}{4}\Big)^{\frac{7}{12}}. \end{aligned}$$

Thus, all the conditions of Theorem 2.1 are satisfied. Thus, T possesses fixed points in X and $Fix\{T\} = \{6, 10\}$.

Definition 2.2 *[25] Condition (R): Let $\{x_n\}$ be a sequence in non-empty set X such that $\theta(x_n,x_{n+1})\geq 1$, for all n and $\{x_n\}\to\omega\in X$ as $n\to\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\theta(x_{n_k},\omega)\geq 1$, for each $k\in\mathbb{N}$.*

In the following, we replace the continuity condition of T by the condition (R) in Theorem 2.1 as follows.

Theorem 2.2 *Let T be a self-mapping on a $b-$complete metric space (X,d) with $s\geq 1$. Suppose that T is quasi triangular $\theta-$admissible and forms an interpolative $(\varphi,\psi)-$ $S-$ type type $\mathfrak{Z}-$contraction with respect to ζ as defined by (2.1). If there exists $x_0\in X$ such that $\theta(x_0,Tx_0)\geq 1$ satisfying condition(R), then T has a fixed point in X.*

Proof: Proceeding on the same line as in Theorem 2.1, one can prove the Cauchy sequence $\{x_n\}$ in X converges in X i.e.,

$$\lim_{n\to\infty} x_n=\omega.$$

As condition (R) holds, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\theta(x_{n(k)},\omega)\geq 1$, for all k. We claim that ω is a fixed point of T.

$$\begin{aligned}
d(\omega,T\omega) &\leq s[d(\omega,x_{n_k+1})+d(x_{n_k+1},T\omega)]\\
&= s[d(\omega,x_{n_k+1})+d(Tx_{n_k},T\omega)]\\
&\leq sd(\omega,x_{n_k+1})+s\theta(x_{n_k},\omega)\varphi\big(d(Tx_{n_k},T\omega)\big)\\
&\leq sd(\omega,x_{n_k+1})+s[d(x_{n_k},\omega)]^{\alpha_1}[\frac{1}{2s}\max\{d(x_{n_k},Tx_{n_k}),\\
&\qquad d(\omega,T\omega)\}]^{\alpha_2},[\frac{1}{2s}\max\{d(x_{n_k},T\omega),d(\omega,Tx_{n_k})\}]^{1-\alpha_1-\alpha_2}\\
&= sd(\omega,x_{n_k+1})+s[d(x_{n_k},\omega)]^{\alpha_1}[\frac{1}{2s}\max\{d(x_{n_k},x_{n_k+1}),\\
&\qquad d(\omega,T\omega)\}]^{\alpha_2},[\frac{1}{2s}\max\{d(x_{n_k},T\omega),d(\omega,Tx_{n_k})\}]^{1-\alpha_1-\alpha_2}.
\end{aligned}$$

Letting $k\to\infty$, we have

$$d(\omega,T\omega)=0\implies T\omega=\omega.$$

Hence T has a fixed point in X. □

Example 2.2 *Let $X=[0,2]$ with $b-$metric $d(x,y)=(x-y)^2$ and $T:X\to X$ be a function defined by*

$$T(x)=\begin{cases}\frac{4}{3}, & x\in[1,2];\\ \frac{3}{4}, & x\in[0,1),\end{cases}$$

Setting

$$\theta(x,y)=\begin{cases}1, & x,y\in[1,2];\\ 0, & otherwise.\end{cases}$$

Let $x,y\in X-Fix(T)$. Taking $\zeta(t,q)=\psi(q)-t$, for all $q,t>0$ in Theorem 2.2, we obtain

$$\theta(x,y)\varphi(d(Tx,Ty))\leq\psi(S(x,y)),\ \textit{for all}\ x,y\in X.$$

Setting $\varphi(t)=\frac{t}{2},\psi(t)=\frac{3t}{4},t>0$, where $,\alpha_1=\frac{1}{4},\alpha_2=\frac{1}{6},s=2$, then $\varphi(t)>\psi(t)$. Clearly T forms an interpolative $(\varphi,\psi)-$ $S-$ type type $\mathfrak{Z}-$contraction with respect to ζ . Now, $x_0=\frac{10}{6}\in X$ such that $\theta(x_0,Tx_0)=1$, and for $x,y\in[1,2]$ we have $Tx=Ty=T^2x=\frac{4}{3}$. Also, $\theta(x,y)=1,\theta(Tx,Ty)=1$ and $\theta(x,Tx)=1,\theta(x,T^2x)=1$. So, T is quasi triangular $\theta-$admissible.

Let $\{x_n\}$ be a sequence such that $\theta(x_n,x_{n+1})\geq 1$ and $\{x_n\}\to v\in X$. Since X is complete, therefore $v\in X$ and $\theta(x_{n(k)},v)=1$. Hence, the condition (R) is satisfied. Thus, T satisfies all the conditions of Theorem 2.2 and $\frac{4}{3},\frac{3}{4}$ are fixed points of T.

Definition 2.3 *Let (X,d) be a $b-$metric space with $s \geq 1$. A mapping $T: X \to X$ is called an interpolative $(\varphi,\psi)-$ $M-$ type type $\mathfrak{Z}-$contraction with respect to ζ, if there exists $\theta: X \times X :\to \mathbb{R}$, $\zeta \in \mathfrak{Z}, \varphi \in \Phi, \psi \in \Psi, \alpha_i \in (0, \frac{1}{s})$, for $i = 1,2,3$ with $\alpha_1 + \alpha_2 + \alpha_3 < 1$ such that $\varphi(t) > \psi(t)$, for $t > 0$ satisfying the inequality*

$$0 \leq \zeta(\theta(x,y)\varphi(d(Tx,Ty)), \psi(M(x,y))), \text{ for all } x,y \in X, \tag{2.21}$$

where

$$M(x,y) = [d(x,y)]^{\alpha_1}[d(x,Tx)]^{\alpha_2}[d(y,Ty)]^{\alpha_3}[\frac{1}{2s}\max\{d(x,Ty),d(y,Tx)\}]^{1-\sum_{i=1}^{3}\alpha_i}.$$

Theorem 2.3 *Let T be a self-mapping on a $b-$complete metric space (X,d). Suppose that T is quasi triangular $\theta-$admissible and forms an interpolative $(\varphi,\psi)-$ $M-$ type $\mathfrak{Z}-$contraction type with respect to ζ. If there exists $x_0 \in X$ such that $\theta(x_0, Tx_0) \geq 1$ and T is continuous (or, the condition (R) holds), then T has a fixed point in X.*

Proof: One can prove as in the line of Theorems 2.1 and 2.3 □

Let(U) be the uniqueness condition which is given as: For any distinct fixed points $\omega, \omega^* \in Fix\{T\} \neq \phi, \theta(\omega,\omega^*) \geq 1$, where $Fix\{T\} = \{x : Tx = x\}$.

Theorem 2.4 *In addition to the assumptions of Theorem 2.1(or, Theorem 2.2), we suppose the condition (U) holds. Then, the observed fixed point is unique.*

Proof: Taking $\omega, \omega^* \in X, \omega \neq \omega^*$ such that $T\omega = \omega$ and $T\omega^* = \omega^*$ in (2.1), we obtain

$$\begin{aligned} 0 &\leq \zeta(\theta(\omega,\omega^*)\varphi(d(T\omega,T\omega^*)), \psi(S(\omega,\omega^*))) \\ &\leq \zeta(\theta(\omega,\omega^*)\varphi(d(\omega,\omega^*)), \psi(S(\omega,\omega^*))) \\ &< \psi(S(\omega,\omega^*)) - \theta(\omega,\omega^*)\varphi(d(\omega,\omega^*) \\ &= -\theta(\omega,\omega^*)\varphi(d(\omega,\omega^*)). \end{aligned}$$

This is a contradiction and hence T has a unique fixed point in X. □

Remark 2.1 *In Example 2.1,$Fix\{T\} = \{6, 10\}$, so the condition (U) does not hold and hence Theorem 2.4 is not applicable in Example 2.1.*

Theorem 2.5 *In addition to the assumptions of Theorem 2.3, we suppose the condition (U) holds. Then, the observed fixed point is unique.*

Proof: The proof is similar to that of Theorem 2.4. □

3. Interpolative $(\varphi,\psi)-$BKC and HR type $\mathfrak{Z}_b-$contraction

We present interpolative $(\varphi,\psi)-$ Banach-Kannan-Chatterjea type $\mathfrak{Z}-$contraction with respect to ζ and interpolative $(\varphi,\psi)-$ Hardy-Rogers type $\mathfrak{Z}-$contraction with respect to ζ of [21] in the setting of $b-$metric space.

Definition 3.1 *Let (X,d) be a $b-$metric space with $s \geq 1$. A mapping $T: X \to X$ is called an interpolative $(\varphi,\psi)-$Banach-Kannan-Chatterjea type $\mathfrak{Z}_b-$contraction with respect to ζ, if there exists $\theta: X \times X :\to \mathbb{R}$, $\zeta \in \mathfrak{Z}, \varphi \in \Phi, \psi \in \Psi, \alpha_i \in (0, \frac{1}{s})$ with $\alpha_1 + \alpha_2 < 1$, for $i = 1,2$ such that $\varphi(t) > \psi(t)$, for $t > 0$ satisfying the inequality*

$$0 \leq \zeta(\theta(x,y)\varphi(d(Tx,Ty)), \psi(A(x,y))), \text{ for all } x,y \in X \tag{3.1}$$

where,

$$A(x,y) = [d(x,y)]^{\alpha_1}[\frac{1}{2s}\big(d(x,Tx) + d(y,Ty)\big)]^{\alpha_2}[\frac{1}{2s}\big(d(x,Ty) + d(y,Tx)\big)]^{1-\alpha_1-\alpha_2}.$$

Theorem 3.1 *Let (X,d) be a complete $b-$metric space with $s\geq 1$. Suppose that T is quasi triangular $\theta-$admissible and forms an interpolative $(\varphi,\psi)-$Banach-Kannan-Chatterjea type $\mathfrak{Z}_b-$contraction with respect to ζ. If there exists $x_0\in X$ such that $\theta(x_0,Tx_0)\geq 1$ and T is continuous(or, the condition (R) holds), then T has a fixed point in X.*

Proof: Let $x_0\in X$ such that $\theta(x_0,Tx_0)\geq 1$ and consider the iterative sequence $\{x_n\}$ by $x_n=Tx_{n-1}$ for all $n\in\mathbb{N}$. Following the proof Theorem 2.1, we shall consider $x_n\neq x_{n-1}$ and hence we have $d(x_{n-1},x_n)>0$, for all $n\in\mathbb{R}$. On the other hand, T is $\theta-$admissible, we obtain recursively that $\theta(x_{n-1},x_n)\geq 1$ for all $n\in\mathbb{N}$. From (3.1), we obtain

$$\begin{aligned}&0\leq\zeta(\theta(x_{n-1},x_n)\varphi(d(Tx_{n-1},Tx_n)),\psi(A(x_{n-1},x_n)))\\&0\leq\zeta(\theta(x_{n-1},x_n)\varphi(d(x_n,x_{n+1})),\psi(A(x_{n-1},x_n)))\\&\quad<\psi(A(x_{n-1},x_n))-\theta(x_{n-1},x_n)\varphi(d(x_n,x_{n+1}),\end{aligned}\tag{3.2}$$

where

$$\begin{aligned}A(x_{n-1},x_n)=&[d(x_{n-1},x_n)]^{\alpha_1}[\frac{1}{2s}\big(d(x_{n-1}+Tx_{n-1})+d(x_n,Tx_n)\big)]^{\alpha_2}\\&[\frac{1}{2s}\big(d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})\big)]^{1-\alpha_1-\alpha_3}\\=&[d(x_{n-1},x_n)]^{\alpha_1}[\frac{1}{2s}\big(d(x_{n-1},x_n)+d(x_n,x_{n+1}\big)]^{\alpha_2}\\&[\frac{1}{2s}d(x_{n-1},x_{n+1})]^{1-\alpha_1-\alpha_2}\\\leq&[d(x_{n-1},x_n)]^{\alpha_1}[\frac{1}{2s}\big(d(x_{n-1},x_n)+d(x_n,x_{n+1})\big)]^{\alpha_2}\\&[\frac{1}{2}\big(d(x_{n-1},x_n)+d(x_n,x_{n+1})\big)]^{1-\alpha_1-\alpha_2}.\end{aligned}\tag{3.3}$$

Consequently, we arrive

$$\begin{aligned}\varphi(d(x_n,x_{n+1}))\leq&\theta(x_{n-1},x_n)\varphi\Big(d(x_n,x_{n+1})\Big)\\<&\psi\left(A(x_{n-1},x_n)\right)\\=&\psi\Big([d(x_{n-1},x_n)]^{\alpha_1}[\frac{1}{2s}\big(d(x_{n-1},x_n)+d(x_n,x_{n+1})\big)]^{\alpha_2}\\&[\frac{1}{2}\big(d(x_{n-1},x_n)+d(x_n,x_{n+1})\big)]^{1-\alpha_1-\alpha_2}\Big).\end{aligned}\tag{3.4}$$

Suppose, $d(x_{n-1},x_n)<d(x_n,x_{n+1})$.
From (3.4) ,

$$\begin{aligned}\varphi(d(x_n,x_{n+1}))\leq&\psi\Big([d(x_n,x_{n+1})]^{\alpha_1}\big[d(x_n,x_{n+1})\big]^{\alpha_2}\big[d(x_n,x_{n+1})\big]^{1-\alpha_1-\alpha_2}\Big)\\=&\psi(d(x_n,x_{n+1}))\\<&\varphi(d(x_n,x_{n+1})).\end{aligned}$$

This is a contradiction. So,

$$d(x_{n-1},x_n)\geq d(x_n,x_{n+1}),\ \text{for all } n\geq 1.\tag{3.5}$$

Hence, $\{d(x_n,x_{n+1})\}$ is a monotonic decreasing sequence of positive real numbers and bounded below by zero. So, there exist $u\geq 0$ such that $\lim_{n\to+\infty}d(x_n,x_{n+1})=u$.We claim that $u>0$, otherwise from

(3.4), (3.3) together with (3.5), we obtain

$$\begin{aligned} \varphi(d(x_n,x_{n+1})) &\le \theta(x_{n-1},x_n)\varphi(d(x_n,x_{n+1})) \\ &\le \psi(A(x_{n-1},x_n)) \\ &\le \varphi(A(x_{n-1},x_n)) \\ &\le \varphi(d(x_{n-1},x_n)). \end{aligned} \tag{3.6}$$

Taking limit as $n \to +\infty$ in (3.6), we get

$$\lim_{n\to+\infty} \theta(x_{n-1},x_n)\varphi(d(x_n,x_{n+1})) = \lim_{n\to+\infty} \psi(A(x_{n-1},x_n)). \tag{3.7}$$

Setting $s_n = \theta(x_{n-1},x_n)\varphi(d(x_n,x_{n+1}))$ and $t_n = \psi(A(x_{n-1},x_n))$ in (3.2) and (3.3),then by (ζ_3) with (3.7), we get

$$0 \le \lim_{x\to+\infty} sup \zeta(\theta(x_{n-1},x_n)\varphi(d(x_n,x_{n+1})), \psi(A(x_{n-1},x_n))) < 0.$$

This is a contradiction and thus we have

$$\lim_{n\to+\infty} d(x_n,x_{n+1}) = 0, \text{ for all } n \in \mathbb{N}\cup\{0\}. \tag{3.8}$$

Moreover, T is quasi triangular $\theta-$admissible mapping, by Lemma 1.2, we obtain $\theta(x_{n(k)}, x_{m(k)}) \ge 1$ for all $n_k, m_k \in \mathbb{N}\cup\{0\}$ such that $m_k > n_k \ge k$. Following the same steps as in Theorem 2.1, we can prove that $\{x_n\}$ is a $b-$ Cauchy sequence in X. Since X is $b-$complete, there exists $w \in X$ such that $\lim_{n\to\infty} x_n = w$.

On account of T is continuous, immediately we find that $Tw = w$. Also, suppose that the condition (R) holds. Then one can show as in theorem 2.2 that $Tw = w$. Thus, T has a fixed point. This completes the proof. □

Definition 3.2 *A mapping $T: X \to X$ is said to be interpolative $(\varphi,\psi)-$ Hardy-Rogers type $\mathfrak{Z}_b-$contraction with respect to ζ if there exist $\theta: X\times X \to \mathbb{R}, \zeta \in \mathfrak{Z}, \varphi \in \Phi, \psi \in \Psi, \alpha_i \in (0, \frac{1}{s})$, where $i = 1,2,3$, such that $\varphi(t) > \psi(t)$, for $t > 0$ and $\sum_{i=1}^{3} \alpha_i < 1$ satisfying the inequality*

$$0 \le \zeta(\theta(x,y)\varphi(d(Tx,Ty)), \psi(B(x,y))), \text{ for all } x,y \in X - Fix\{T\}, \tag{3.9}$$

where

$$B(x,y) = [d(x,y)]^{\alpha_1}[d(x,Tx)]^{\alpha_2}[d(y,Ty))]^{\alpha_3}[\frac{1}{2s}(d(x,Ty)+d(y,Tx))]^{1-\sum_{i=1}^{3}\alpha_i}.$$

Theorem 3.2 *Let T be a self-mapping on a complete b metric space (X,d). Suppose that T is quasi triangular $\theta-$admissible and forms an interpolative $(\varphi,\psi)-$ HR type $\mathfrak{Z}_b-$contraction with respect to ζ. If there exist $x_0 \in X$ such that $\theta(x_0, Tx_0) \ge 1$ and T is continuous(or, the condition (R) holds), then T has a fixed point in X.*

Proof: One can prove as in the line of Theorem 3.1. □

Theorem 3.3 *In addition to the assumptions of Theorem 3.1 or 3.2, we suppose the condition (U) holds. Then, the observed fixed point is unique.*

Proof: The proof is similar to that of Theorem 2.4. □

Corollary 3.1 *Let T be a self-mapping on a complete $b-$metric space (X,d). If there exists $\psi \in \Psi, \alpha_i \in (0, \frac{1}{s})$, where $i = 1,2$ such that $\sum_{i=1}^{2} \alpha_i < 1$ satisfying the inequality*

$$d(Tx,Ty) \le \psi(S(x,y)), \text{ for all } x,y \in X,$$

then T has a unique fixed point in X.

$d(x,y)$	0	3	4
0	0	2	8
3	2	0	2
4	8	2	0

Example 3.1 *Let $X = \{0,3,4\}$ and $d : X \times X \to [0,\infty)$ defined by*
and

$$Tx = \begin{cases} 4, x = 0 \\ 3, x \neq 0 \end{cases} ;$$

Setting $\psi(t) = \frac{t}{2}, t > 0, \alpha_1 = \frac{1}{4}, \alpha_2 = \frac{2}{3}$.
Then, T satisfies all the the conditions of corollary 3.1 and $x = 3$ is the unique fixed point of T.

Corollary 3.2 (Theorem 2.1 of [21]) *Let T be a self-mapping on a complete metric space (X,d). Suppose that T is quasi triangular $\theta-$orbital admissible and forms an interpolative $(\varphi,\psi)-$ Banach-Kannan-Chatterjea type $\mathfrak{Z}-$contraction with respect to ζ. If there exists $x_0 \in X$ such that $\theta(x_0, Tx_0) \geq 1$ and T is continuous, then T has a fixed point in X. Further if the condition (U) is satisfied, then the observed fixed point is unique.*

Corollary 3.3 (Theorem 2.5 of [21]) *Let T be a self-mapping on a complete metric space (X,d) and . Suppose that T is quasi triangular $\theta-$orbital admissible and forms an interpolative $(\varphi,\psi)-$ Hardy Roger type $\mathfrak{Z}-$contraction with respect to ζ. If there exists $x_0 \in X$ such that $\theta(x_0, Tx_0) \geq 1$ and T is continuous, then T has a fixed point in X. Further if the condition (U) is satisfied, then the observed fixed point is unique.*

4. Application

Inspired by Gautam and Kaur [9]), we apply our result to solve the system of non-linear matrix equations. We shall use the following notations:
(i) $\mathcal{M}_n$ = the set of all $n \times n$ matrices;
(ii) $\mathcal{H}_n$ = the set of all $n \times n$ Hermitian matrices;
(iii) $\mathcal{P}_n$ = the set of all $n \times n$ positive definite matrices;
(iv) $\mathcal{H}_n^+$ =the set of $n \times n$ positive definite Hermitian matrices.

Further, we write $A \succ 0$, if $A \in \mathcal{P}_n$ and $A \succeq 0$, if $A \in \mathcal{H}_n^+$. We write $A \succ B$, if $A - B \succ 0$ and $A \succeq B$ if $A - B \succeq 0$. We denote by $\|A\|_{tr} = tr(A)$ the trace norm of A i.e $tr(A)$ is the sum of singular values of A. The singular values are the roots of the eigenvalues of A^*A and $\|A\|$ the spectral norm of A i.e. $\|A\| = \sqrt{\lambda^+(A^*A)}$ where $\lambda^+(A^*A)$ is the largest eigenvalue of A^*A where A^* stands for conjugate transpose of A.

Remark 4.1 *[22] $\mathcal{H}_n^+ \subseteq \mathcal{P}_n \subseteq \mathcal{H}_n \subseteq \mathcal{M}_n$ and $(\mathcal{H}_n, \preceq)$ is a partially ordered set, then $\mathcal{H}_n$ endowed with trace norm is a Banach Space.*

Lemma 4.1 *[24] If $A, B \in \mathcal{H}_n^+$, then $0 \leq tr(AB) \leq \|A\| tr(B)$.*

Lemma 4.2 *[9] If $A \in \mathcal{H}_n$ and $A \prec I$ then $\|A\| < 1$.*

Here we present an example that satisfying the above lemmas.

Example 4.1 *Consider the matrices* $A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, B = \begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix}$
$0 \leq tr(AB) = 6 \leq \|A\| tr(B) = 6$ *which validates lemma (4.1).*
And, $0 \preceq I - A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$ *with* $\|A\| = \frac{1}{\sqrt{2}} < 1$ *which validates Lemma (4.2).*

Consider the following non-linear matrix equation

$$A = Q + \sum_{j=1}^{m} D_j^* \mathcal{T}(A) D_j, \tag{4.1}$$

where
D_j is arbitrary $n \times n$ matrix for each $j = 1, 2, 3, 4, ..., m$,
A_j is an arbitrary $n \times n$ matrix for each $j = 1, 2, 3, 4, ..., m$,
Q is a positive definite Hermitian matrix of order $n \times n$.
$\mathcal{T}$ is an order-preserving continuous mapping from $\mathcal{H}_n$ into $\mathcal{P}_n$ such that $\mathcal{T}(0) = 0$.
$\mathcal{H}_n$ endowed with trace norm is a normed Banach Space, hence it is a complete matrix space.

Let $\mathcal{F} : \mathcal{H}_n \to \mathcal{H}_n$ be a continuous order preserving self mapping such that

$$\mathcal{F}(A) = Q + \sum_{j=1}^{m} D_j^* \mathcal{T}(A) D_j, \text{ for all } A \in \mathcal{H}_n. \tag{4.2}$$

Clearly, a fixed point of $\mathcal{F}$ is a solution of (4.2).

Theorem 4.1 *Consider the matrix equation (4.2), suppose there exist real numbers $k \geq 1$ and M such that for $X, Y \in \mathcal{H}_n$ with $X \preceq Y$ having following properties*

1. $\sum_{j=1}^{m} D_j^* D_j \leq M I_n$ *and* $\sum_{j=1}^{m} D_j^* \mathcal{T}(Q) D_j \succ 0$;
2. $\Big(tr\big(\mathcal{T}(Y) - \mathcal{T}(X)\big)\Big)^k \leq \frac{1}{M^k}\psi\Big(\|X - Y\|_{tr}^{\alpha_1}.[\frac{1}{2s}\max\{\|X - \mathcal{F}(X)\|_{tr}, \|Y - \mathcal{F}(Y)\|_{tr}\}]^{\alpha_2}[\frac{1}{2s}\max\{\|X - \mathcal{F}(Y)\|_{tr}, \|Y - \mathcal{F}(X)\|_{tr}\}]^{1-\alpha_1-\alpha_2}\Big)$.

where $\alpha_1, \alpha_2 \in (0, \frac{1}{s})$ with $\alpha_1 + \alpha_2 < 1$ and $\psi \in \Psi$, then the matrix equation(4.2) has a unique solution. Moreover, the iteration

$$A_n = Q + \sum_{j=1}^{m} D_j^* \mathcal{T}(A_{n-1}) D_j, \tag{4.3}$$

where $A_0 \in \mathcal{H}$ satisfies

$$A_0 \preceq Q + \sum_{j=1}^{m} D_j^* \mathcal{T}(A_0) D_j$$

converges to the solution of the matrix equation (4.1).

Proof: We define a metric $d : \mathcal{H}_n \times \mathcal{H}_n \to [0, \infty)$ as $d(A, B) = \|A - B\|_{tr}^k$. Thus, $(\mathcal{H}_n, d)$ is a complete $b-$metric space with $s = 2^{k-1}$. Let $X, Y \in \mathcal{H}_n$ with $X \succeq Y$. Now,

$$\begin{aligned} d(\mathcal{F}(Y), \mathcal{F}(X)) &= \|\sum_{j=1}^{m} D_j^* \mathcal{T}(Y) D_j - \sum_{j=1}^{m} D_j^* \mathcal{T}(X) D_j\|_{tr}^k \\ &= \|\sum_{j=1}^{m}(D_j^* \mathcal{T}(Y) D_j - D_j^* \mathcal{T}(X) D_j)\|_{tr}^k \\ &= \|\sum_{j=1}^{m}(D_j^* D_j)(\mathcal{T}(Y) - \mathcal{T}(X))\|_{tr}^k \\ &\leq \|\sum_{j=1}^{m}(D_j^* D_j)\|^k \|(\mathcal{T}(Y) - \mathcal{T}(X))\|_{tr}^k. \end{aligned}$$

Using (1) and Lemma 4.2 , we have

$$d(\mathcal{F}(Y), \mathcal{F}(X)) \leq M^k.\|(\mathcal{T}(Y) - \mathcal{T}(X))\|_{tr}^k. \tag{4.4}$$

Since $D_j^*\mathcal{T}(Q)D_j \succ 0$, therefore we have $Q \prec \mathcal{F}(Q)$. Now, using (2), (4.4), and Corollary 3.1, we obtain $\mathcal{F}$ has a fixed point. Hence the non-linear matrix equation(4.2) has a unique solution. □

Example 4.2 *Consider the non-linear matrix equation (4.1) for $m = 3, n = 2,$ with $\mathcal{T}(A) = A^{\frac{1}{2}}$ i.e.*

$$A = Q + D_1^* A^{\frac{1}{2}} D_1 + D_2^* A^{\frac{1}{2}} D_2 + D_3^* A^{\frac{1}{2}} D_3, \tag{4.5}$$

where $Q = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}, D_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, D_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, D_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$

The conditions of the Theorem (4.2) can be checked numerically, by taking various special values for matrices involved. For example,

Take $X = \begin{bmatrix} 50 & 14 \\ 14 & 50 \end{bmatrix}, Y = \begin{bmatrix} 37 & 12 \\ 12 & 37 \end{bmatrix}$, *and* $M = \frac{7}{2}$.

Now,

$$\begin{aligned}\sum_{j=1}^{m} D_j^* D_j &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\ \text{and, } MI_n &= \begin{bmatrix} \frac{7}{2} & 0 \\ 0 & \frac{7}{2} \end{bmatrix}.\end{aligned}$$

Since $0 \preceq MI_n - \sum_{j=1}^{m} D_j^* D_j (\frac{1}{2}, \frac{5}{2}$ *as eigen values),* $\sum_{j=1}^{m} D_j^* D_j \preceq MI_n$ *and* $\sum_{j=1}^{m} D_j^* \mathcal{T}(Q) D_j \succ 0$.
So, Condition (1) is satisfied.
And, taking $k = 1.5, \alpha_1 = 0.09, \alpha_2 = \frac{1}{3}$ *and* $\psi(t) = \frac{99t}{100}, s = \sqrt{2}$, *we can calculate*

$$(tr(\mathcal{T}(Y) - \mathcal{T}(X)))^k = tr(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix})^{1.5} = 2^{1.5} = 2.83.$$

Now,

$$\begin{aligned}&\frac{1}{M^k}\psi\Big(\|X - Y\|_{tr}^{\alpha_1}[\frac{1}{2s}\max\{\|X - \mathcal{F}(X)\|_{tr}, \|Y - \mathcal{F}(Y)\|_{tr}\}]^{\alpha_2} \\ &[\frac{1}{2s}\max\{\|X - \mathcal{F}(Y)\|_{tr}, \|Y - \mathcal{F}(X)\|_{tr}\}]^{1-\alpha_1-\alpha_2}\Big) \\ &= (\frac{2}{7})^{1.5}\frac{99}{100}tr\Big(\begin{bmatrix} 13 & 2 \\ 2 & 13 \end{bmatrix}\Big)^{0.09}\Big[\frac{1}{2\sqrt{2}}\max\{tr\Big(\begin{bmatrix} 31 & 3 \\ 3 & 31 \end{bmatrix}\Big), tr\Big(\begin{bmatrix} 20 & 2 \\ 2 & 20 \end{bmatrix}\Big)\}\Big]^{\frac{1}{3}} \\ &\Big[\frac{1}{2\sqrt{2}}\max\{tr\Big(\begin{bmatrix} 33 & 4 \\ 4 & 33 \end{bmatrix}\Big), tr\Big(\begin{bmatrix} 18 & 1 \\ 1 & 18 \end{bmatrix}\Big)\}\Big]^{\frac{1}{\sqrt{3}}} \\ &= (\frac{2}{7})^{1.5}\frac{99}{100}26^{0.09}[\frac{1}{2\sqrt{2}}\max\{62, 40\}]^{\frac{1}{3}}[\frac{1}{2\sqrt{2}}\max\{66, 36\}]^{\frac{1}{\sqrt{3}}} \\ &= 3.45 > 2.83 = (tr(\mathcal{T}(Y) - \mathcal{T}(X)))^k.\end{aligned}$$

So, condition (2) is also satisfied.
To see the convergence of the sequence A_n defined in (4.5), we start with three different initial values $X_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, Y_0 = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}, Z_0 = \begin{bmatrix} 7 & 3.9 \\ 3.9 & 1 \end{bmatrix}.$

We have the following approximation of the unique positive definite solution of the (4.5) after 10 iterations:

$$\hat{X} \approx X_{10} = \begin{bmatrix} 5 & 4 \\ 4 & 19.2106 \end{bmatrix}$$
$$\hat{Y} \approx Y_{10} = \begin{bmatrix} 5 & 4 \\ 4 & 19.2106 \end{bmatrix}$$
$$\hat{Z} \approx Z_{10} = \begin{bmatrix} 5 & 4 \\ 4 & 19.2106 \end{bmatrix}.$$

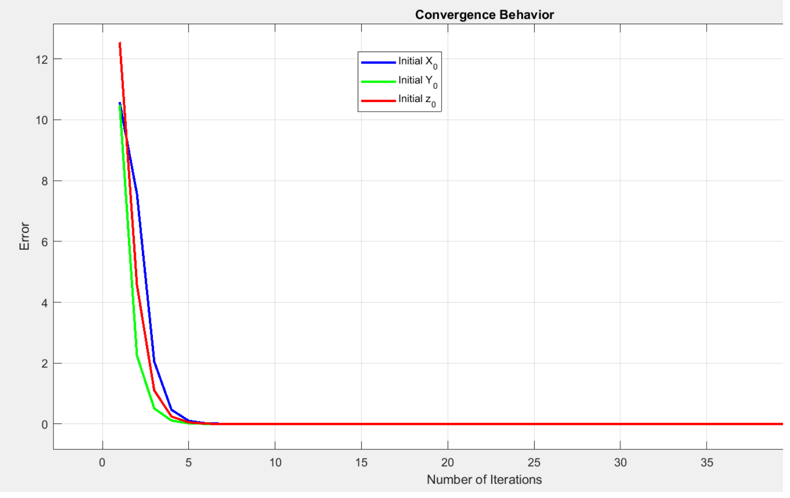

Figure 2: Figure 2. Convergence behavior.

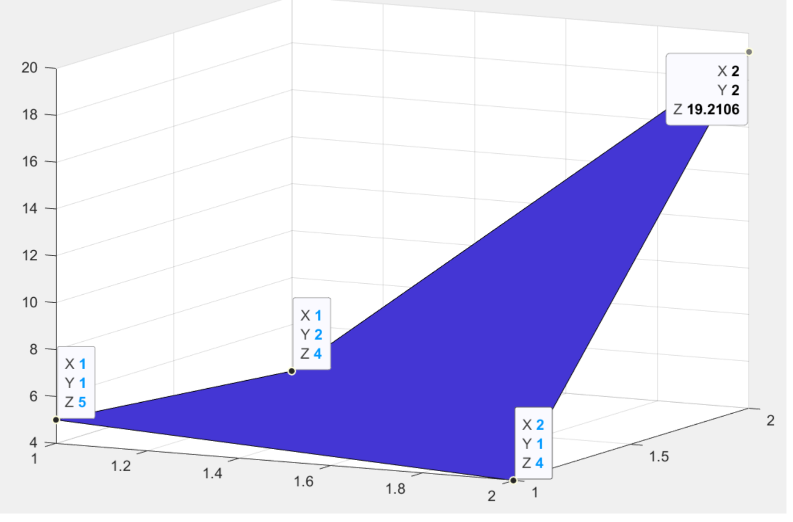

Figure 3: Figure 3. Solution graph.

The Figure 2 represents the convergence analysis of sequence and Figure 3 represents surface plot of the solution.

Declarations:

Ethics approval and consent to participate

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