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# Novel analytical approaches for generalized Burgers-KdV equation: A Comparative study

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ABSTRACT: This study analytically investigates the generalized Burgers–KdV (GBKdV) equation to understand nonlinear wave behavior in media exhibiting both dispersion and dissipation. Using the (G'/G)-expansion method within fractional calculus, exact traveling wave solutions are obtained. The GBKdV equation, which merges features of the Burgers and KdV models, effectively models complex nonlinear dynamics such as diffusion, dispersion, and higher-order interactions. The derived solutions include a variety of waveforms bright and dark solitons, kink waves, bell-shaped curves, multi-peak patterns, and singularities. These solutions are studied under different fractional orders and parameter settings to assess their physical significance. Graphical simulations highlight the dynamic behaviors, and comparisons with established results confirm the validity and novelty of several outcomes. This work enhances the understanding of wave propagation in systems like plasmas, shallow water flows, and elastic tubes, offering a solid foundation for future research in nonlinear wave theory.

Key Words: Generalized Burgers- KdV equation, exact solutions, (G'/G)-expansion technique, Caputo's derivative.

## Contents



## 1. Introduction

Many phenomena have been described by linear equations. Exact wave solutions are gaining attention in nonlinear research every day. Researchers can develop and carry out studies to determine these characteristics or functions by setting appropriate parameters and providing reliable responses. Different mathematicians and physicists have successfully developed many contemporary techniques like the bilinear transformation method developed by Hirota [1,2], exp-expansion method [3,4], Enhanced $(G'/G)$-expansion method [5,6,7], the improved F-expansion method [8,9], Kudryashov approach [10], Exp-function method [11,12], simple equation method [13,14], modified simple equation method [15,16,17], etc. Waves are often defined as the sequence of motion through materials. Waves with small amplitudes are almost linear and large amplitude waves might not be linear. Solving the mathematical equations for a soliton wave more than a century ago discovered that the precise balance between the effects of diffusion and nonlinearity allows for the state of a soliton wave. It is due to nonlinearity causing the hill tends to be steep, but dispersion flattens the hill. Although soliton waves can be evaluated using a variety of different nonlinear equations, including the Boussinesq equations, these three equations are especially important for physics applications. They exhibit the most famous solitons and KdV (pulse) solitons, sine-Gordon (topological) solitons and nonlinear Schrödinger equation [18,19].

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Wide range of physical conditions the appearance of the Burgers–Korteweg de Vries (Burgers–KdV) equations has attracted a lot of interest during the past three decades. For example, the propagation of undulating bores in shallow water [20], the flow of liquid with bubbles, the propagation of waves in elastic tubes filled with viscous liquid [21,22], and weak nonlinear plasma waves [23,24,25].

The typical version of the Burgers–KdV equation is,

$$u_t + pu^n u_x + qu_{xx} + ru_{xxx} = 0, \tag{1.1}$$

where $p$, $q$ and $r$ are constants with $n > 1$. These equations provide the generalized sequence versions of Burgers equation:

$$u_t + pu_x + qu_{xx} = 0 \tag{1.2}$$

The equation (1) is regarded as combination of the Burgers equation [26,27] with the KdV equation.

In plasma physics, the Burgers–KdV equation serves as a powerful tool for modeling various nonlinear phenomena, including ion-acoustic waves, dust-ion shock structures, and complex plasma interactions [28, 29,30,31,32]. In the context of fluid dynamics, this equation is often applied to describe undular bores in shallow water and the evolution of solitary waves in elastic tubes [33]. It also finds relevance in nonlinear acoustics, where it helps explain the formation of shock waves and the steepening of sound waves in thermos viscous media.The exact solutions derived from this equation offer meaningful insights into the nature of nonlinear wave propagation across different physical settings. Notably, soliton and periodic wave solutions reveal stable, localized structures that preserve their form during movement and interaction. These types of waves are commonly observed in real-world situations, such as shallow water waves, ion-acoustic waves in plasma, and even in the transmission of signals through nonlinear optical fibers.By deriving and analyzing these solutions analytically, our study enhances the theoretical understanding of how nonlinear effects appear and evolve in practical scenarios. This contributes to a deeper grasp of the physical dynamics within diverse systems, bridging mathematical theory with observable behavior in complex media.

## 2. Description of the $(G'/G)$-Expansion Method

Consider a fractional nonlinear partial differential equation (NLPDE) given by

$$\phi\left(u, D_t^{\alpha}u, u_y, u_{xx}, D_t^{2\alpha}u, D_t^{\alpha}u_y, \ldots\right) = 0, \quad t > 0,\ x \in \mathbb{R},\ 0 \leq \alpha \leq 1, \tag{2.1}$$

where $\phi$ is a polynomial in $u = u(x,t)$, the unknown function.

**Step 1.** Apply the wave transformation

$$u(x,y) = u(\eta), \quad \eta = x \pm V\frac{e}{r_{\Omega+\alpha j}}. \tag{2.2}$$

After applying the transformation, Eq. (2.1) reduces to an ordinary differential equation (ODE) of the form

$$P\left(Y, kY', \omega Y, k^2 Y', k\omega Y', \ldots\right) = 0, \tag{2.3}$$

where the prime denotes derivative with respect to $\eta$.
We assume a solution of the form

$$Y(\xi) = \sum_{i=1}^{m} a_i \left(\frac{G'}{G}\right)^i + \alpha_0, \quad \alpha_m \neq 0, \tag{2.4}$$

where $\alpha_0$ and $\alpha_i$ are constants, and $G(\xi)$ satisfies the second-order linear ODE

$$\frac{d^2G(\xi)}{d\xi^2} + \lambda\frac{dG(\xi)}{d\xi} + \mu G(\xi) = 0, \tag{2.5}$$

with $\lambda$ and $\mu$ being arbitrary constants.

From Eqs. (2.4) and (2.5), we obtain

$$Y' = \sum_{i=1}^{m} i a_i \left[ \left(\frac{\alpha}{g}\right)^{i+1} + \lambda \left(\frac{\alpha}{g}\right)^{i} + \mu \left(\frac{\alpha}{g}\right)^{i-1} \right], \tag{2.6}$$

$$\begin{aligned} Y'' = \sum_{i=1}^{m} i a_i \Big[ &(i+1)\left(\frac{\alpha}{g}\right)^{i+2} + (2i+1)\lambda \left(\frac{\alpha}{g}\right)^{i+1} \\ &+ (i\lambda^2 + 2\mu)\left(\frac{\alpha}{g}\right)^{i} + (2i-1)\lambda\mu \left(\frac{\alpha}{g}\right)^{i-1} \\ &+ (i-1)\mu^2 \left(\frac{\alpha}{g}\right)^{i-2} \Big], \end{aligned} \tag{2.7}$$

and so on.

The method proceeds with the following three steps:

**Step 2.** Determine the integer $m$. Substituting Eqs. (2.4) and (2.5) into Eq. (2.3) and balancing the highest-order derivative term with the nonlinear terms yields an algebraic equation for $m$.

**Step 3.** Solve the resulting system. Using symbolic computation software such as MAPLE, the explicit values of $a_0$, $a_i$ $(i = 1, 2, \ldots, m)$ and the wave variable $\xi$ are determined.

**Step 4.** Construct the exact solutions. Substituting the obtained coefficients into Eq. (2.4) provides a class of exact traveling wave solutions of Eq. (2.1), depending on the solution $G(\xi)$ of Eq. (2.5).

## 3. Solution Method

**Solution process**

The Generalized Burger-KdV equation is given by

$$D_t^{\alpha} u + u u_x + u_{xx} + u_{xxx} = 0 \tag{3.1}$$

Using the wave transformation

$$\eta = kx \pm \omega \frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

Equation (3.1) reduces to an ODE by using the above transformation:

$$Y'(\xi)\omega + Y(\xi)Y'(\xi)k + Y''(\xi)k^2 + Y'''(\xi)k^3 = 0 \tag{3.2}$$

Integrating the above equation, we obtain:

$$\omega Y(\xi) + \frac{1}{2} k Y^2(\xi) + k^2 Y'(\xi) + k^3 \left( \frac{d^2}{d\xi^2} Y(\xi) \right) = 0 \tag{3.3}$$

Suppose the solution of Eq. (3.1) is of the form

$$u(\xi) = a_m \left(\frac{G'}{G}\right)^m + \cdots \tag{3.4}$$

where

$$G'' + \lambda G' + \mu G = 0 \tag{3.5}$$

Using Eqs. (3.4) and (3.5), we get:

$$u^2 = a_m^2 \left(\frac{G'}{G}\right)^{2m} + \cdots \tag{3.6}$$

$$u' = -ma_m \left(\frac{G'}{G}\right)^{m+1} + \cdots \tag{3.7}$$

$$u'' = m(m+1)a_m \left(\frac{G'}{G}\right)^{m+2} + \cdots \tag{3.8}$$

We obtain $M = 2$ using the homogeneous balancing principle. So we can write Eq. (3.4) as

$$u(\xi) = a_0 + a_1 \left(\frac{G'}{G}\right) + a_2 \left(\frac{G'}{G}\right)^2, \qquad a_0 \neq 0 \tag{3.9}$$

and therefore

$$u^2(\xi) = a_2^2 \left(\frac{G'}{G}\right)^4 + 2a_2a_1 \left(\frac{G'}{G}\right)^3 + \left(a_1^2 + 2a_2a_0\right) \left(\frac{G'}{G}\right)^2 + 2a_1a_0 \left(\frac{G'}{G}\right) + a_0^2 \tag{3.10}$$

By using Eqs. (3.5) and (3.9), it is derived that

$$u''(\xi) = 6a_2 \left(\frac{G'}{G}\right)^4 + (2a_1 + 10a_2\lambda) \left(\frac{G'}{G}\right)^3 + \left(8a_2\mu + 3a_1\lambda + 4a_2\lambda^2\right) \left(\frac{G'}{G}\right)^2 + \left(6a_2\mu + 2a_1\mu + a_1\lambda^2\right) \left(\frac{G'}{G}\right) + 2a_2\mu^2 + a_1\mu \tag{3.11}$$

By substituting Eqs. (15) to (17) into Eq. (12) and collecting all terms with the same powers of $\left(\frac{G'}{G}\right)$ together, we get, and equating each coefficient of the polynomial to zero yields a set of algebraic equations as follows:

$$\begin{aligned}
k^3\lambda^2 a_1 + 6k^3\lambda\mu a_2 + 2k^3\mu a_1 + ka_0a_1 + \omega a_1 &= 0,\\
\omega a_2 + ka_0a_1 + \frac{1}{2}ka_1^2 + 4k^3\lambda^2 a_2 + 3k^3\lambda a_1 + 8k^3\mu a_2 &= 0,\\
10k^3\lambda a_2 + 2k^3 a_1 + ka_1a_2 &= 0,\\
\frac{1}{2}ka_2^2 + 6k^3 &= 0,\\
\omega a_0 + \frac{1}{2}ka_0^2 + k^3\mu\lambda a_1 + 2k^3\mu^2 a_2 &= 0.
\end{aligned}$$

On solving the above algebraic equations using MAPLE, we obtain the solution sets. By choosing a solution, we have:

On solving Eq. (3.5) we deduce after some reduction that

$$\frac{G'}{G} := \left(\frac{\sqrt{(\lambda^2 - 4\mu)}}{2} \left(\frac{A_1 \sinh\left(\frac{1}{2}\text{Trans}\left(\sqrt{(\lambda^2 - 4\mu)}\right)\right) + A_2 \cosh\left(\frac{1}{2}\text{Trans}\left(\sqrt{(\lambda^2 - 4\mu)}\right)\right)}{A_1 \cosh\left(\frac{1}{2}\text{Trans}\left(\sqrt{(\lambda^2 - 4\mu)}\right)\right) + A_2 \sinh\left(\frac{1}{2}\text{Trans}\left(\sqrt{(\lambda^2 - 4\mu)}\right)\right)}\right) - \frac{\lambda}{2}\right) \tag{3.12}$$

where $A_1$ and $A_2$ are arbitrary constants.

Using Eqs. (3.5), (3.9), (3.10), (3.11), and (3.12), we have three types of travelling wave solutions of Eq. (3.1) as follows:

**Case 1:** When $\lambda^2 - 4\mu > 0$, we obtain the solution

$$S_1 = \left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}\left(\frac{A_1\sinh\left(\frac{1}{2}\text{Trans}\left(\sqrt{(\lambda^2-4\mu)}\right)\right)+A_2\cosh\left(\frac{1}{2}\text{Trans}\left(\sqrt{(\lambda^2-4\mu)}\right)\right)}{A_1\cosh\left(\frac{1}{2}\text{Trans}\left(\sqrt{(\lambda^2-4\mu)}\right)\right)+A_2\sinh\left(\frac{1}{2}\text{Trans}\left(\sqrt{(\lambda^2-4\mu)}\right)\right)}\right)-\frac{\lambda}{2}\right) \tag{3.13}$$

**Case 2:** When $\lambda^2 - 4\mu < 0$, we have

$$S_2 = \left(\frac{\sqrt{(-\lambda^2+4\mu)}}{2}\left(\frac{-A_1\sin\left(\frac{1}{2}\text{Trans}\left(\sqrt{(-\lambda^2+4\mu)}\right)\right)+A_2\cos\left(\frac{1}{2}\text{Trans}\left(\sqrt{(-\lambda^2+4\mu)}\right)\right)}{A_1\cos\left(\frac{1}{2}\text{Trans}\left(\sqrt{(-\lambda^2+4\mu)}\right)\right)+A_2\sin\left(\frac{1}{2}\text{Trans}\left(\sqrt{(-\lambda^2+4\mu)}\right)\right)}\right)-\frac{\lambda}{2}\right) \tag{3.14}$$

**Case 3:** When $\lambda^2 - 4\mu = 0$, we get

$$S_3 = \left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}\left(\frac{A_2}{A_1+A_2\text{Trans}}\right)-\frac{\lambda}{2}\right) \tag{3.15}$$

We choose some cases as follows.

**Solution 1:**

$$\left[k=k,\ \omega=\omega,\ a_0=-\frac{2\omega}{k},\ a_1=0,\ a_2=0\right] \tag{25}$$

Using Eqs. (3.9), (3.12), and (25), we obtained

$$\begin{aligned} u_1(\xi) =& \frac{1}{2}\Bigg[\sqrt{\lambda^2-4\mu}\cdot\sinh\left(\frac{1}{2}\left(kx+\frac{\omega t^\alpha}{\Gamma(\alpha+1)}\right)\sqrt{\lambda^2-4\mu}\right)A_1 \\ &+\sqrt{\lambda^2-4\mu}\cdot\cosh\left(\frac{1}{2}\left(kx+\frac{\omega t^\alpha}{\Gamma(\alpha+1)}\right)\sqrt{\lambda^2-4\mu}\right)A_2 \\ &-\sinh\left(\frac{1}{2}\left(kx+\frac{\omega t^\alpha}{\Gamma(\alpha+1)}\right)\sqrt{\lambda^2-4\mu}\right)\cdot\lambda A_2 \\ &-\cosh\left(\frac{1}{2}\left(kx+\frac{\omega t^\alpha}{\Gamma(\alpha+1)}\right)\sqrt{\lambda^2-4\mu}\right)\cdot\lambda A_1\Bigg] \\ &\Bigg/\left(A_1\cdot\cosh\left(\frac{1}{2}\left(kx+\frac{\omega t^\alpha}{\Gamma(\alpha+1)}\right)\sqrt{\lambda^2-4\mu}\right)+A_2\cdot\sinh\left(\frac{1}{2}\left(kx+\frac{\omega t^\alpha}{\Gamma(\alpha+1)}\right)\sqrt{\lambda^2-4\mu}\right)\right) \end{aligned}$$

**Solution 2:**

$$\left[k=k,\ \omega=\omega,\ a_0=\frac{2\omega}{k},\ a_1=0,\ a_2=0\right] \tag{26}$$

Using Eqs. (3.9), (3.12), and (26), we get

$$\begin{aligned} u_2(\xi) := \frac{1}{2}\Bigg(&\sin\left(\frac{1}{2}\left(kx+\frac{\omega t}{\Gamma(\alpha+1)}\right)\sqrt{-\lambda^2+4\mu}\right)\sqrt{-\lambda^2+4\mu}A_1+\sin\left(\frac{1}{2}\left(kx+\frac{\omega t}{\Gamma(\alpha+1)}\right)\sqrt{-\lambda^2+4\mu}\right)A_2 \\ &-\cos\left(\frac{1}{2}\left(kx+\frac{\omega t}{\Gamma(\alpha+1)}\right)\sqrt{-\lambda^2+4\mu}\right)A_2-\cos\left(\frac{1}{2}\left(kx+\frac{\omega t}{\Gamma(\alpha+1)}\right)\sqrt{-\lambda^2+4\mu}\right)A_1\Bigg)\Bigg/ \\ &\left(A_1\cos\left(\frac{1}{2}\left(kx+\frac{\omega t}{\Gamma(\alpha+1)}\right)\sqrt{-\lambda^2+4\mu}\right)+A_2\sin\left(\frac{1}{2}\left(kx+\frac{\omega t}{\Gamma(\alpha+1)}\right)\sqrt{-\lambda^2+4\mu}\right)\right) \end{aligned}$$

**Solution 3:**

$$\left[k=-\frac{1}{6}\sqrt{-3a_2},\ \omega=\frac{1}{72}\sqrt{-3a_2}(\lambda^2-4\mu)a_2,\ a_0=\frac{1}{6}(\lambda^2+2\mu)a_2,\ a_1=2\lambda,\ a_2=a_2\right] \tag{27}$$

$$\begin{aligned}
u_3(\xi) := -\tfrac{1}{2}\Bigg[&\sin\Big(\tfrac{1}{144}\sqrt{3}\sqrt{-a_2}\left(\tfrac{a_2t^\alpha\lambda^2}{\Gamma(\alpha+1)}-\tfrac{4a_2t^\alpha\mu}{\Gamma(\alpha+1)}-12x\right)\sqrt{-\lambda^2+4\mu}\Big)\sqrt{-\lambda^2+4\mu}\,A_1\\
&-\cos\Big(\tfrac{1}{144}\sqrt{3}\sqrt{-a_2}\left(\tfrac{a_2t^\alpha\lambda^2}{\Gamma(\alpha+1)}-\tfrac{4a_2t^\alpha\mu}{\Gamma(\alpha+1)}-12x\right)\sqrt{-\lambda^2+4\mu}\Big)\sqrt{-\lambda^2+4\mu}\,A_2\\
&+\sin\Big(\tfrac{1}{144}\sqrt{3}\sqrt{-a_2}\left(\tfrac{a_2t^\alpha\lambda^2}{\Gamma(\alpha+1)}-\tfrac{4a_2t^\alpha\mu}{\Gamma(\alpha+1)}-12x\right)\sqrt{-\lambda^2+4\mu}\Big)\lambda A_2\\
&+\cos\Big(\tfrac{1}{144}\sqrt{3}\sqrt{-a_2}\left(\tfrac{a_2t^\alpha\lambda^2}{\Gamma(\alpha+1)}-\tfrac{4a_2t^\alpha\mu}{\Gamma(\alpha+1)}-12x\right)\sqrt{-\lambda^2+4\mu}\Big)\lambda A_1\Bigg]\\
&\Big/\Big(A_1\cos\Big(\tfrac{1}{144}\sqrt{3}\sqrt{-a_2}\left(\tfrac{a_2t^\alpha\lambda^2}{\Gamma(\alpha+1)}-\tfrac{4a_2t^\alpha\mu}{\Gamma(\alpha+1)}-12x\right)\sqrt{-\lambda^2+4\mu}\Big)\\
&+A_2\sin\Big(\tfrac{1}{144}\sqrt{3}\sqrt{-a_2}\left(\tfrac{a_2t^\alpha\lambda^2}{\Gamma(\alpha+1)}-\tfrac{4a_2t^\alpha\mu}{\Gamma(\alpha+1)}-12x\right)\sqrt{-\lambda^2+4\mu}\Big)\Big)
\end{aligned}$$

**Solution 4:**

$$\left[k=-\frac{1}{6}\sqrt{-3a_2},\quad \omega=\frac{1}{72}\sqrt{-3a_2}\left(\lambda^2-4\mu\right)a_2,\quad a_0=-\frac{1}{6}\left(\lambda^2+2\mu\right)a_2,\quad a_1=a_2\lambda,\quad a_2=a_2\right] \tag{28}$$

$$\begin{aligned}
u_4(\xi) := -\frac{1}{2}\Bigg[&\sin\left(\frac{\sqrt{3}\sqrt{-a_2}}{144}\left(\frac{a_2t^\alpha\lambda^2}{\Gamma(\alpha+1)}-\frac{4a_2t^\alpha\mu}{\Gamma(\alpha+1)}-12x\right)\sqrt{-\lambda^2+4\mu}\right)\cdot\sqrt{-\lambda^2+4\mu}\cdot A_1\\
&-\cos\left(\frac{\sqrt{3}\sqrt{-a_2}}{144}\left(\frac{a_2t^\alpha\lambda^2}{\Gamma(\alpha+1)}-\frac{4a_2t^\alpha\mu}{\Gamma(\alpha+1)}-12x\right)\sqrt{-\lambda^2+4\mu}\right)\cdot\sqrt{-\lambda^2+4\mu}\cdot A_2\\
&+\sin\left(\frac{\sqrt{3}\sqrt{-a_2}}{144}\left(\frac{a_2t^\alpha\lambda^2}{\Gamma(\alpha+1)}-\frac{4a_2t^\alpha\mu}{\Gamma(\alpha+1)}-12x\right)\sqrt{-\lambda^2+4\mu}\right)\cdot\lambda A_2\\
&+\cos\left(\frac{\sqrt{3}\sqrt{-a_2}}{144}\left(\frac{a_2t^\alpha\lambda^2}{\Gamma(\alpha+1)}-\frac{4a_2t^\alpha\mu}{\Gamma(\alpha+1)}-12x\right)\sqrt{-\lambda^2+4\mu}\right)\cdot\lambda A_1\Bigg]\\
&\Bigg/\left(A_1\cos\left(\frac{\sqrt{3}\sqrt{-a_2}}{144}\left(\frac{a_2t^\alpha\lambda^2}{\Gamma(\alpha+1)}-\frac{4a_2t^\alpha\mu}{\Gamma(\alpha+1)}-12x\right)\sqrt{-\lambda^2+4\mu}\right)\right.\\
&\left.+A_2\sin\left(\frac{\sqrt{3}\sqrt{-a_2}}{144}\left(\frac{a_2t^\alpha\lambda^2}{\Gamma(\alpha+1)}-\frac{4a_2t^\alpha\mu}{\Gamma(\alpha+1)}-12x\right)\sqrt{-\lambda^2+4\mu}\right)\right)
\end{aligned}$$

**Solution 5:**

$$\left[k=\frac{1}{6}\sqrt{-3a_2},\quad \omega=-\frac{1}{72}\sqrt{-3a_2}\left(\lambda^2-4\mu\right)a_2,\quad a_0=a_2\mu,\quad a_1=a_2\lambda,\quad a_2=a_2\right] \tag{29}$$

$$\begin{aligned} u_5(\xi) = -\frac{1}{2}\Bigg[ & \sin\left(\frac{\sqrt{3}\sqrt{-a_2}}{144}\left(\frac{a_2 t^\alpha \lambda^2}{\Gamma(\alpha+1)} - \frac{4a_2 t^\alpha \mu}{\Gamma(\alpha+1)} - 12x\right)\sqrt{-\lambda^2+4\mu}\right)\cdot\sqrt{-\lambda^2+4\mu}\cdot A_1 \\ & - \cos\left(\frac{\sqrt{3}\sqrt{-a_2}}{144}\left(\frac{a_2 t^\alpha \lambda^2}{\Gamma(\alpha+1)} - \frac{4a_2 t^\alpha \mu}{\Gamma(\alpha+1)} - 12x\right)\sqrt{-\lambda^2+4\mu}\right)\cdot\sqrt{-\lambda^2+4\mu}\cdot A_2 \\ & + \sin\left(\frac{\sqrt{3}\sqrt{-a_2}}{144}\left(\frac{a_2 t^\alpha \lambda^2}{\Gamma(\alpha+1)} - \frac{4a_2 t^\alpha \mu}{\Gamma(\alpha+1)} - 12x\right)\sqrt{-\lambda^2+4\mu}\right)\cdot\lambda A_2 \\ & + \cos\left(\frac{\sqrt{3}\sqrt{-a_2}}{144}\left(\frac{a_2 t^\alpha \lambda^2}{\Gamma(\alpha+1)} - \frac{4a_2 t^\alpha \mu}{\Gamma(\alpha+1)} - 12x\right)\sqrt{-\lambda^2+4\mu}\right)\cdot\lambda A_1\Bigg] \\ \Bigg/\Bigg( & A_1\cos\left(\frac{\sqrt{3}\sqrt{-a_2}}{144}\left(\frac{a_2 t^\alpha \lambda^2}{\Gamma(\alpha+1)} - \frac{4a_2 t^\alpha \mu}{\Gamma(\alpha+1)} - 12x\right)\sqrt{-\lambda^2+4\mu}\right) \\ & + A_2\sin\left(\frac{\sqrt{3}\sqrt{-a_2}}{144}\left(\frac{a_2 t^\alpha \lambda^2}{\Gamma(\alpha+1)} - \frac{4a_2 t^\alpha \mu}{\Gamma(\alpha+1)} - 12x\right)\sqrt{-\lambda^2+4\mu}\right)\Bigg) \end{aligned}$$

**Solution 6:**

$$\left[k = \frac{1}{6}\sqrt{-3a_2}, \quad \omega = \frac{1}{72}\sqrt{-3a_2}\left(\lambda^2 - 4\mu\right)a_2, \quad a_0 = a_2\mu, \quad a_1 = a_2\lambda, \quad a_2 = a_2\right] \tag{30}$$

$$\begin{aligned} u_6(\xi) = -\frac{1}{2}\Bigg[ & \sin\left(\frac{\sqrt{3}\sqrt{-a_2}}{144}\left(\frac{a_2 t^\alpha \lambda^2}{\Gamma(\alpha+1)} - \frac{4a_2 t^\alpha \mu}{\Gamma(\alpha+1)} - 12x\right)\sqrt{-\lambda^2+4\mu}\right)\cdot\sqrt{-\lambda^2+4\mu}\cdot A_1 \\ & + \sin\left(\frac{\sqrt{3}\sqrt{-a_2}}{144}\left(\frac{a_2 t^\alpha \lambda^2}{\Gamma(\alpha+1)} - \frac{4a_2 t^\alpha \mu}{\Gamma(\alpha+1)} - 12x\right)\sqrt{-\lambda^2+4\mu}\right)\cdot\sqrt{-\lambda^2+4\mu}\cdot A_2 \\ & + \cos\left(\frac{\sqrt{3}\sqrt{-a_2}}{144}\left(\frac{a_2 t^\alpha \lambda^2}{\Gamma(\alpha+1)} - \frac{4a_2 t^\alpha \mu}{\Gamma(\alpha+1)} - 12x\right)\sqrt{-\lambda^2+4\mu}\right)\cdot\lambda A_2 \\ & + \cos\left(\frac{\sqrt{3}\sqrt{-a_2}}{144}\left(\frac{a_2 t^\alpha \lambda^2}{\Gamma(\alpha+1)} - \frac{4a_2 t^\alpha \mu}{\Gamma(\alpha+1)} - 12x\right)\sqrt{-\lambda^2+4\mu}\right)\cdot\lambda A_1\Bigg] \\ \Bigg/\Bigg( & A_2\sin\left(\frac{\sqrt{3}\sqrt{-a_2}}{144}\left(\frac{a_2 t^\alpha \lambda^2}{\Gamma(\alpha+1)} - \frac{4a_2 t^\alpha \mu}{\Gamma(\alpha+1)} - 12x\right)\sqrt{-\lambda^2+4\mu}\right) \\ & + A_1\cos\left(\frac{\sqrt{3}\sqrt{-a_2}}{144}\left(\frac{a_2 t^\alpha \lambda^2}{\Gamma(\alpha+1)} - \frac{4a_2 t^\alpha \mu}{\Gamma(\alpha+1)} - 12x\right)\sqrt{-\lambda^2+4\mu}\right)\Bigg) \end{aligned}$$

**Solution 7:**

$$\left[k = -\frac{1}{6}\sqrt{-3a_2}, \quad \omega = -\frac{1}{72}\sqrt{-3a_2}\left(\lambda^2 - 4\mu\right)a_2, \quad a_0 = a_2\mu, \quad a_1 = a_2\lambda, \quad a_2 = a_2\right] \tag{31}$$

$$u_7(\xi) = \frac{-\frac{1}{2}\left(\frac{\sqrt{3}\lambda^3 t^\alpha(-a_2)^{3/2}A_2}{\Gamma(\alpha+1)} - \frac{4\sqrt{3}\lambda\mu t^\alpha(-a_2)^{3/2}A_2}{\Gamma(\alpha+1)} + 12\sqrt{3}\lambda x\sqrt{-a_2}A_2 - 72\lambda A_1 + 72\sqrt{\lambda^2-4\mu}A_2\right)}{\frac{A_2\sqrt{3}(-a_2)^{3/2}t^\alpha\lambda^2}{\Gamma(\alpha+1)} - \frac{4A_2\sqrt{3}(-a_2)^{3/2}t^\alpha\mu}{\Gamma(\alpha+1)} + 12A_2\sqrt{3}\sqrt{-a_2}x - 72A_1}$$

**solution 8:**

$$\left[k=-\frac{1}{6}\sqrt{-3a_2},\quad \omega=\frac{1}{72}\sqrt{-3a_2}\left(\lambda^2+4\mu\right)a_2,\quad a_0=a_2\mu,\quad a_1=a_2\lambda,\quad a_2=a_2\right] \tag{32}$$

$$u_8(\xi)=\frac{-\frac{1}{2}\left(\frac{\sqrt{3}\lambda^3t^\alpha(-a_2)^{3/2}A_2}{\Gamma(\alpha+1)}-\frac{4\sqrt{3}\lambda\mu t^\alpha(-a_2)^{3/2}A_2}{\Gamma(\alpha+1)}+12\sqrt{3}\lambda x\sqrt{-a_2}A_2-72\lambda A_1+72\sqrt{\lambda^2-4\mu}A_2\right)}{\frac{A_2\sqrt{3}(-a_2)^{3/2}t^\alpha\lambda^2}{\Gamma(\alpha+1)}-\frac{4A_2\sqrt{3}(-a_2)^{3/2}t^\alpha\mu}{\Gamma(\alpha+1)}+12A_2\sqrt{3}\sqrt{-a_2}x-72A_1}$$

## 4. Graphical Representation

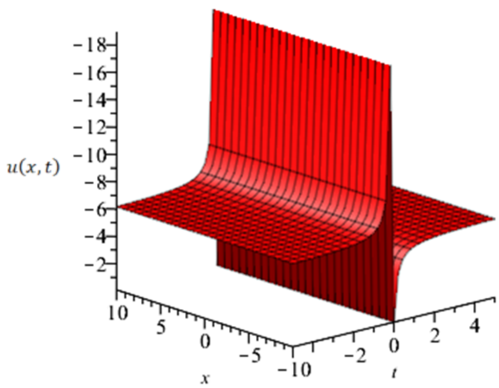


Figure 1:
Dark-bright soliton solution of $u_1$
$\alpha=0.7$, $\lambda=4$, $\mu=1$, $\omega=2$, $k=0.4$, $A_1=0.4$, and $A_2=0.23$.

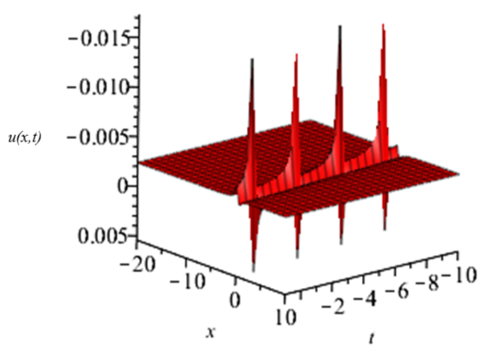


Figure 2:
Multi-peak soliton solution of $u_2$
$\alpha=0.3$, $\lambda=1$, $\mu=2$, $k=0.8$, $\omega=6$, $A_1=0.5$, and $A_2=0.30$.

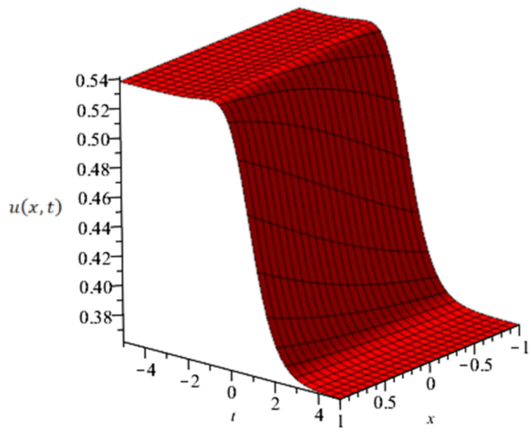


Figure 3:
Kink-type soliton solution of $u_3$
$\alpha = 0.6$, $\lambda = 1$, $\mu = 3$, $a_2 = -3$, $A_1 = 0.40$, and $A_2 = 0.60$.

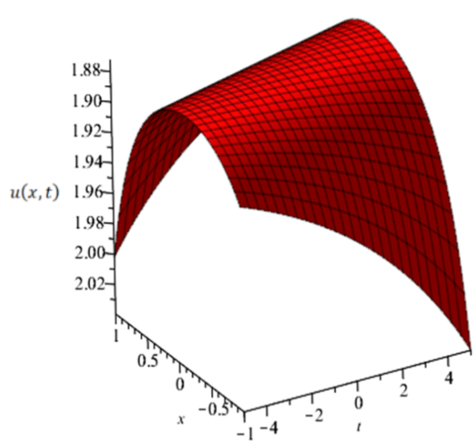


Figure 4:
Bell-type soliton solution of $u_4$
$\alpha = 0.3$, $\lambda = 3$, $\mu = 8$, $a_2 = -2$, $A_1 = 0.2$, and $A_2 = 0.3$.

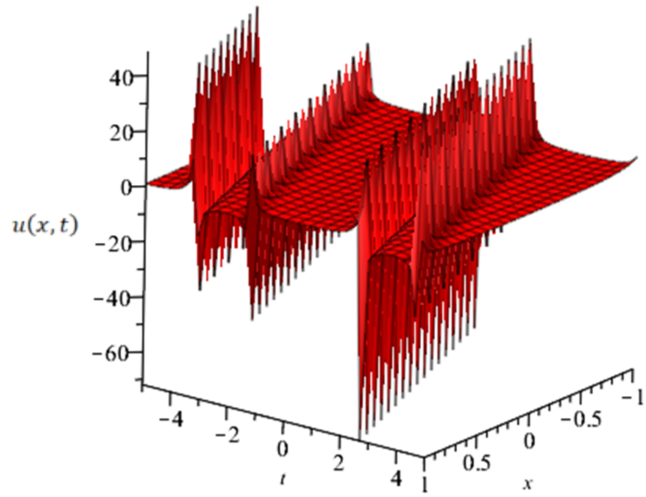


Figure 5:
Multi-peak wave solution of $u_5$
$\alpha = 0.7$, $\lambda = 2$, $\mu = 4$, $a_2 = -6$, $A_1 = 0.50$, and $A_2 = 0.30$.

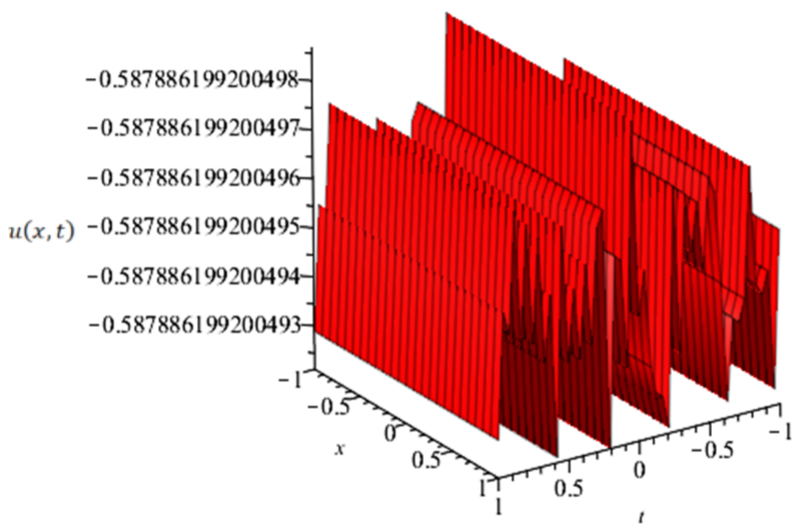


Figure 6:
Chaotic wave solution of $u_6$
$\alpha = 0.4$, $\lambda = 3$, $\mu = 1$, $a_2 = -9$, $A_1 = 0.9$, and $A_2 = 0.4$.

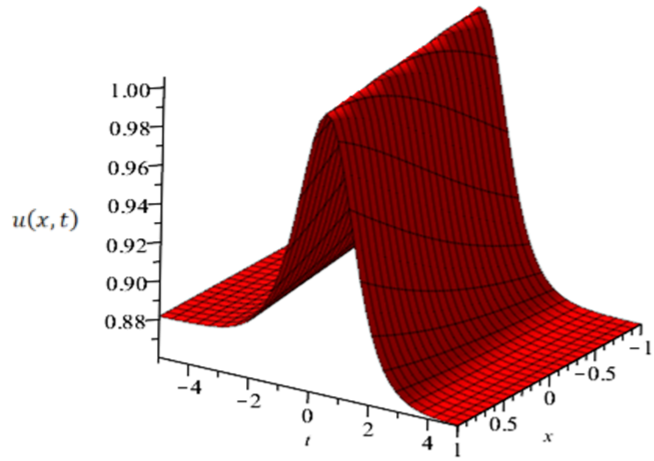


Figure 7:
Bell-shaped solitary wave solution of $u_7$
$\alpha = 0.3$, $\lambda = 8$, $\mu = 0.2$, $a_2 = -7$, $A_1 = 0.80$, and $A_2 = 0.70$.

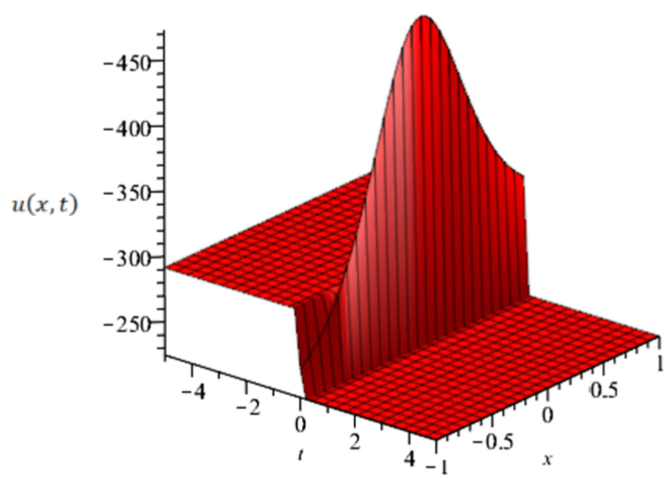


Figure 8:
Singular soliton solution of $u_8$
$\alpha = 0.4$, $\lambda = 4$, $\mu = 0.1$, $a_2 = -3$, $A_1 = 0.2$, and $A_2 = 0.3$.

## 5. Physical interpretation

The solutions obtained in this study offer valuable insights into how waves behave in nonlinear environments such as fluids, plasmas, and elastic materials. By using the fractional form of the generalized Burgers–KdV equation, we capture physical behaviors more effectively especially those involving memory and hereditary effects, which are common in materials like viscoelastic media and in plasma physics. The use of Caputo fractional derivatives enhances the model's ability to reflect real-world phenomena more accurately than traditional integer-order models. Each type of solution presented corresponds to a distinct physical behavior. For example, Solutions 1 and 2 describe stable, solitary waves that keep their shape as they travel, showing a balance between nonlinearity and dispersion. Solutions 3 and 6, on the other hand, show periodic or oscillatory patterns, modeled using trigonometric functions similar to wave behaviors seen in elastic tubes filled with fluid or in weakly nonlinear plasma waves [34,35,36,37]. Solutions 7 and 8 capture more extreme wave behaviors, where the waveforms become highly localized or even singular, meaning the amplitude spikes sharply in a small region. These cases are especially important for understanding critical or intense phenomena in nonlinear systems. To achieve these diverse waveforms, we carefully selected key parameters $a_0$, $a_1$ and $a_2$, the wave number $k$ and the fractional order $\alpha \in (0, 1]$. The fractional order $\alpha$ plays a crucial role by introducing memory effects: smaller values of $\alpha$ lead to faster amplitude decay, which indicates stronger energy loss over time. The constants $a_0$, $a_1$ and $a_2$ were fine-tuned to produce realistic waveforms, including both bright and dark solitons, as seen in Figures 1 and 2. These solitons form when there is a precise balance between dispersive spreading and nonlinear steepening. The wave number $k$ affects how sharp and fast the waves are, increasing results in narrower waves that move more quickly. We also observed that more complex wave patterns, such as multi-peaked or chaotic structures (see Figures 5 and 6), arise under specific parameter combinations. These patterns reflect the kind of complexity found in real-world nonlinear systems, especially in plasma physics and acoustics.

## 6. Results and Discussions

The chosen values are representative of common conditions found in real-world physical systems, such as those in fluid dynamics and plasma physics. These selections help ensure that the resulting solutions display important behaviors, including soliton formation, periodic patterns, and wave breaking. Furthermore, we've included a short discussion on how changes in key parameters (like $\lambda$, $\mu$, $k$, and $\omega$) affect the wave's amplitude, width and stability. This analysis highlights the physical significance of the mathematical solutions and underscores their practical importance.The $(G'/G)$-expansion method provides some new exact solutions that are not found in other literature. By comparing our results, we discovered that some of them are similar to the current literature, while others solutions $u_6$, $u_7$ and $u_8$ are newly discovered and have not been explored elsewhere. As a result, we have taken specific values of the physical parameters, and some of our obtained solutions, $u_1$, $u_2$, $u_3$, $u_4$ and $u_5$ coincide with some of the particular solutions obtained by other methods mentioned in the tables and the references [38,39,40,41].

Table 1: Comparison of our results with A. Borhanifar and R. Abazari [38].

| **Our Results** | **Borhanifar and Abazari[38]** |
|---|---|
| (i) If $\lambda = 0$, $\sqrt{\lambda^2 - 4\mu} = 1$, $(kx + \omega t) = \sqrt{\Delta}$, $A_1 = A_2$, then the solution $u_1$ is $u_1 = \frac{1}{2}$ | (i) If $n = 1$, $\lambda = 0$, $\alpha_1\sqrt{\Delta} = 1$, $k = 0$, $p = 0$, $\xi = 1$, $C_1 = C_2$, then the solution $u_1$ is $u_1 = \frac{1}{2}$ |
| (ii) If $\lambda = 0$, $\sqrt{\lambda^2 - 4\mu} = 1$, $(kx + \omega t) = \sqrt{\Delta}$, $A_1 = A_2 = 1$, then the solution $u_2$ is $u_2 = -\frac{1}{2}\left[\frac{\sin\left(\frac{1}{2}\sqrt{\Delta}\right)\cos\left(\frac{1}{2}\sqrt{\Delta}\right)}{\sin\left(\frac{1}{2}\sqrt{\Delta}\right)+\cos\left(\frac{1}{2}\sqrt{\Delta}\right)}\right]$ | (ii) If $n = 1$, $i = 1$, $\lambda = 0$, $\alpha_1\sqrt{\Delta} = 1$, $k = 0$, $p = 0$, $\xi = 1$, $C_1 = C_2 = 1$, then the solution $u_2$ is $u_2 = -\frac{1}{2}\left[\frac{\sin\left(\frac{1}{2}\sqrt{\Delta}\right)-\cos\left(\frac{1}{2}\sqrt{\Delta}\right)}{\sin\left(\frac{1}{2}\sqrt{\Delta}\right)+\cos\left(\frac{1}{2}\sqrt{\Delta}\right)}\right]$ |
| (iii) If $\lambda = 0$, $\sqrt{\lambda^2 - 4\mu} = 1$, $\left(\frac{1}{144}\sqrt{3}\sqrt{-\alpha_2(\lambda^2 t\alpha_2 - 4\mu t\alpha_2 - 12x)}\right) = \frac{1}{2}\xi$, $A_2 = 0$, then the solution $u_2$ is $u_2 = -\frac{1}{2}\tan\left(\frac{1}{2}\xi\right)$ | (iii) If $n = 1$, $i = 1$, $\lambda = 0$, $\alpha_1 = 1$, $k = 0$, $p = 0$, $\frac{q^2n^2}{q^2n^2 + (n+4)^2} = 1$, then the solution $u_2$ is $u_2 = -\frac{1}{2}\tan\left(\frac{1}{2}\xi\right)$ |
| (iv) If $\lambda = 0$, $\sqrt{\lambda^2 - 4\mu} = 1$, $\left(\frac{1}{144}\sqrt{3}\sqrt{-\alpha_2(2\lambda^2 t\alpha_2 - 4\mu t\alpha_2 - 12x)}\right) = \frac{1}{2}\xi$, $A_1 = 0$, then the solution $u_5$ is $u_5 = -\frac{1}{2}\coth\left(\frac{1}{2}\xi\right)$ | (iv) If $n = 1$, $\lambda = 0$, $\alpha_1 = -1$, $k = 0$, $p = 0$, $\frac{q^2n^2}{q^2n^2 + (n+4)^2} = 1$, then the solution $u_5$ is $u_5 = -\frac{1}{2}\coth\left(\frac{1}{2}\xi\right)$ |

Table 2: Comparison of Attained Outcomes with Z.H. Kheiri, M.R. Moghaddam, and V. Vafaeiayed et al [39].

| **Attained Outcomes** | **Z.H. Kheiri, M.R. Moghaddam and V. Vafaeiayed et al. Results** |
|---|---|
| (i) If $\lambda = 0$, $\sqrt{\lambda^2 - 4\mu A_1} = A_1$, $\sqrt{\lambda^2 - 4\mu A_2} = A_2$, $A_1 = A_2 = 1$, $(kx + \omega t) = 1$, then the solution $u_1$ is $u_1 = \frac{1}{2}$ | (i) If $\xi = 1$, $\sqrt{\lambda^2 - 4\mu + 2D} - \sqrt{\lambda^2 - 4\mu} = \frac{1}{2}$, $C_1 = C_2 = 1$, then the solution $u_1$ is $u_1 = \frac{1}{2}$ |
| (ii) If $\lambda = 0$, $\sqrt{\lambda^2 - 4\mu A_1} = A_1$, $\sqrt{\lambda^2 - 4\mu A_2} = -A_2$, $A_1 = A_2 = 1$, $(kx + \omega t) = 1$, then the solution $u_2$ is $u_2 = -\frac{1}{2}$ | (ii) If $\xi = 1$, $\sqrt{\lambda^2 - 4\mu + 2D} - \sqrt{\lambda^2 - 4\mu} = -\frac{1}{2}$, $-C_1 = C_1 = C_2 = 1$, then the solution $u_2$ is $u_2 = -\frac{1}{2}$ |

Table 3: Comparison of our outcomes with the results of A.-M. Wazwaz et al[40]

| **Attained Outcomes** | **A.-M. Wazwaz et al. Results** |
|---|---|
| **(i) If** $\lambda = -1$, $\sqrt{\lambda^2 - 4\mu} = 1$, $\sqrt{\lambda^2 - 4\mu}A_1 = A_1$, $A_2 = 0$, $k = 1$, then the solution $u_1$ is: $u_1 = \frac{1}{2}\left[1 + \tanh(x + \omega t)\right]$ | **(i) If** $n = 2$, $a = 1$, $b = -1$, $a^2 b = \omega$, then the solution $u_1$ is: $u_1 = \frac{1}{2}\left[1 + \tanh(x + \omega t)\right]$ |
| **(ii) If** $\lambda = -1$, $\sqrt{\lambda^2 - 4\mu} = 1$, $\sqrt{\lambda^2 - 4\mu}A_2 = A_2$, $A_1 = 0$, $k = 1$, then the solution $u_2$ is: $u_2 = \frac{1}{2}\left[1 + \coth(x + \omega t)\right]$ | **(ii) If** $n = 2$, $a = 1$, $b = -1$, $a^2 b = \omega$, then the solution $u_2$ is: $u_2 = \frac{1}{2}\left[1 + \coth(x + \omega t)\right]$ |

If we set $b = -\beta$, $c = \gamma$ and $\eta = \xi$, then the obtained solutions $v_2$ and $v_4$ in the referenced article are equivalent to $u_2$ and $u_2$ for $k > 0$, $\beta > 0$, and $v_1$ and $v_3$ are equivalent to $u_4$ and $u_4$ for $k < 0$, $\beta < 0$ respectively, as found using the tanh method (see Table 4).

Table 4: Comparison of our results with the results of Sierra et al.[41]

| **Attained Outcomes** | **Sierra et al. Results** |
|---|---|
| If $b=-\beta$, $c=\gamma$, $\mu=k$, and $\eta=\xi$, then the solution $u_2$ becomes: $$u_2=\sqrt{\frac{\beta}{\gamma}}\tan(\sqrt{k}\,\xi)$$ | (i) The solution $u_2$ is given by: $$u_2=\pm\sqrt{\frac{\beta}{\gamma}}\tan(\sqrt{k}\,\xi)$$ |
| (ii) If $b=-\beta$, $c=\gamma$, $\mu=k$, and $\eta=\xi$, then the solution $u_4$ becomes: $$u_4=-\sqrt{\frac{\beta}{\gamma}}\tan(\sqrt{k}\,\xi)$$ | (ii) The solution $u_2$ is given by: $$u_2=\sqrt{\frac{\beta}{\gamma}}\tan(\sqrt{k}\,\xi)$$ |
| (iii) If $b=-\beta$, $c=\gamma$, $\mu=-k$, and $\eta=\xi$, then the solution $u_1$ becomes: $$u_1=-\sqrt{\frac{\beta}{\gamma}}\tanh(\sqrt{-k}\,\xi)$$ | (iii) The solution $u_4$ is given by: $$u_4=\pm\sqrt{\frac{\beta}{\gamma}}\tanh(\sqrt{-k}\,\xi)$$ |
| (iv) If $b=\beta$, $c=\gamma$, $\mu=-k$, and $\eta=\xi$, then the solution $u_4$ becomes: $$u_4=\sqrt{\frac{\beta}{\gamma}}\tanh(\sqrt{-k}\,\xi)$$ | (iv) The solution $u_4$ is given by: $$u_4=\sqrt{\frac{\beta}{\gamma}}\tanh(\sqrt{-k}\,\xi)$$ |

## 7. Conclusion

In this article, we have effectively utilized the suggested method to obtain the generalized solitary solutions of the generalized Burger's-KdV equation. Periodical, trigonometric, hyperbolic, and rational functions are the only solutions. Various new wave features may be expressed by the obtained data. After comparing our solutions, we came to the conclusion that while some of them are well-established and haven't been studied before while the others are new. Initiating fresh outcomes is a dependable strategy, and we have chosen a new class of exact solutions. The major modification of the wave dynamics by the physical factors is examined. The solutions developed in this work may prove useful for wave breaking research. In plasma physics and atmospheric gravity waves, wave breaking is employed. Additionally, it is used in the analysis to analyze global existence in non-peaked solutions as well as local well-poses. The recommended process completely validated our computational work's reliability and may be used to investigate further physical issues.

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