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On Congruences of Sixth Order Mock Theta Function

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ABSTRACT: In a recent work, Kaur and Rana [17] obtained several Ramanujan-like congruences and established infinite families of congruences modulo 12 for the coefficients of sixth order mock theta functions $\lambda(q)$ and $\rho(q)$. Inspired by their approach, in this paper, we develop more generalized results. We extend and enrich their findings by deriving additional infinite families of congruences, including new congruences modulo 3, 6 and 9 for the functions $\lambda(q)$ and $\rho(q)$.

Key Words: Mock theta functions, partitions, congruences.

Contents

1. Introduction

In his well-known final letter to Hardy [6], Ramanujan presented 17 mock theta functions but did not provide a clear definition for them. Additionally, Ramanujan introduced the mock theta functions of order six, which are documented in his Lost Notebook. Since then, mock theta functions have been extensively studied. The six primary sixth order mock theta functions, as defined in [2], are:

$$\phi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q;q^2)_n}{(-q;q)_{2n}}, \tag{1.1}$$

$$\psi(q) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n^2} (q;q^2)_{n-1}}{(-q;q)_{2n-1}}, \tag{1.2}$$

$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}} (-q;q)_n}{(q;q^2)_{n+1}}, \tag{1.3}$$

$$\sigma(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{(n+2)(n+1)}{2}} (-q;q)_n}{(q;q^2)_{n+1}}, \tag{1.4}$$

$$\lambda(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q;q^2)_n}{(-q;q)_n}, \tag{1.5}$$

$$\mu(q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n}{(-q;q)_n}, \tag{1.6}$$

where the q-Pochhammer symbol is represented by:

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)(1-aq^3)...(1-aq^{n-1})$$

for a positive integer n, and it's infinite product form is :

$$(a;q)_\infty = \prod_{k=1}^{\infty} (1-aq^{k-1}), \ |q| < 1. \tag{1.7}$$

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Throughout this paper, we use

$$E_k := (q^k; q^k)_\infty,$$

where k is any positive integer.

In recent years, partitions related to these functions have attracted significant interest in the literature. Andrews et al. [3] presented partition functions associated with the Ramanujan–Watson mock theta functions $w(q)$ and $v(q)$, defined as :

$$w(q) = \sum_{n=0}^{\infty} \frac{q^{2(n^2+n)}}{(q;q^2)_{n+1}^2},$$
$$v(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}.$$

The sixth order mock theta functions $\lambda(q)$ and $\rho(q)$, as introduced by Ramanujan, are defined as follows:

$$\lambda(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q;q^2)_n}{(-q;q)_n} = \sum_{n=0}^{\infty} p_\lambda(n) q^n, \tag{1.8}$$
$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}} (-q;q)_n}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} p_\rho(n) q^n. \tag{1.9}$$

The linear relations between the sixth order mock theta functions provided by Ramanujan is given as:

$$2q^{-1}\psi_6(q^2) + \lambda(-q) = (-q;q^2)_\infty^2 f(q;q^5), \tag{1.10}$$
$$q^{-1}\psi_6(q^2) + \rho(q) = (-q;q^2)_\infty^2 f(q;q^5), \tag{1.11}$$

where $\psi_6(q)$ is defined as:

$$\psi_6(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2} (q;q^2)_n}{(-q;q)_{2n+1}}.$$

In [4], Andrews et al. proved several infinite families of congruences for $p_w(n)$ and $p_v(n)$ modulo 2. In work of Fatima and Pore [10], it is shown that a number of infinite families of congruences for $p_w(n)$ and $p_v(n)$ modulo 20. In 2019, Barhua and Begum [5] derived several congruence relations for the same partition functions modulo powers of 5. Recently, Kaur and Rana [17] obtained many infnite family of congruences modulo numbers of the form $2^\alpha 3^\beta$ for $p_\lambda(n)$ and $p_\rho(n)$. Inspired by their work, we have also discovered new infinite families of congruences for these functions, further enriching the understanding of their arithmetic properties. Our results extend the framework established by Kaur and Rana and highlight deeper modular behaviors in the sixth order mock theta functions.

In this study, we derive many infinite families of congruences modulo 2, 3, 4, 6 and 9 for $\lambda(q)$. In second section, we present some preliminary results and lemma to prove main results. In the final section, we establish the proofs of our main results.

2. Preliminary

In order to prove the main theorems, the following lemmas are required.
Ramanujan's general theta function is given as:

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad where \ |ab| < 1. \tag{2.1}$$

Also, Jacobi's triple product identity is given as:

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.2}$$

Lemma 2.1 *The following* 3*-dissection holds:*

$$\frac{E_2^2}{E_1} = \frac{E_6E_9^2}{E_3E_{18}} + q\frac{E_{18}^2}{E_9}. \tag{2.3}$$

The identity (2.3) can be obtained from [16, p.132].

Lemma 2.2 *The following* 3*-dissection holds:*

$$\frac{E_2^3}{E_1^3} = \frac{E_6}{E_3} + 3q\frac{E_6^4E_9^5}{E_3^8E_{18}} + 6q^2\frac{E_6^3E_9^2E_{18}^2}{E_3^7} + 12q^3\frac{E_6^2E_{18}^5}{E_3^6E_9}. \tag{2.4}$$

The identity (2.2) was proved by Toh [20].

Lemma 2.3 *The following* 3*-dissection holds:*

$$\frac{E_1^3}{E_3} = \frac{E_4^2}{E_{12}} - 3q\frac{E_2^2E_{12}^3}{E_4E_6^2}. \tag{2.5}$$

The equation (2.5) is same as (22.1.13) from [16, p.186].

Lemma 2.4 *The following p-dissection holds from [8, Theorem 2.2]:*
For any prime $p \geq 5$,

$$E_1 = (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} E_{p^2} + \sum_{\substack{k=\frac{-p-1}{2} \\ k\neq \frac{p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{(3k^2+k)}{2}} f(-q^{\frac{3p^2+(6k+1)p}{2}}; -q^{\frac{3p^2-(6k+1)p}{2}}). \tag{2.6}$$

Furthermore, for $-(p-1)/2 \leq k \leq (p-1)/2$ *and* $k \neq (\pm p - 1)/6$,

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}.$$

Lemma 2.5 *From [1, Lemma 2.3] for any prime* $p \geq 3$, *we have*

$$E_1^3 = p(-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} E_{p^2}^3 + \sum_{\substack{k=0 \\ k\neq \frac{\pm p-1}{2}}}^{p-1} (-1)^k q^{\frac{(k^2+k)}{2}} \sum_{n=0}^{\infty} (-1)^n (2pn+2k+1).q^{pn.\frac{pn+2k+1}{2}}. \tag{2.7}$$

Furthermore, for $0 \leq k \leq (p-1)$ *and* $k \neq (p-1)/2$,

$$\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod{p}.$$

From the binomial theorem and for any positive integers k, m, β and prime p, we have

$$E_k^{p^\beta m} \equiv E_{pk}^{p^{\beta-1}m} \pmod{p^\beta}. \tag{2.8}$$

Lemma 2.6 *The following* 3*-dissection holds:*

$$E_1^3 = \frac{E_6E_9^6}{E_3E_{18}^3} - 3qE_9^3 + 4q^3\frac{E_3^2E_{18}^6}{E_6^2E_9^3}. \tag{2.9}$$

The equation (2.9) is the same as (14.8.5) in [16, p.137].

Lemma 2.7 *The following* 3*-dissection holds:*

$$\frac{E_1^2}{E_2} = \frac{E_9^2}{E_{18}} - 2q\frac{E_3E_{18}^2}{E_6E_9}. \tag{2.10}$$

The equation (2.10) is from [12].

Lemma 2.8 *The following 2-dissection holds:*

$$\frac{E_2}{E_1^2} = \frac{E_8^5}{E_2^4 E_{16}^2} + 2q\frac{E_4^2 E_{16}^2}{E_2^4 E_8}. \tag{2.11}$$

Lemma 2.8 was obtained by Hirschhorn and Sellers [14, p.2].

3. Main Results

Theorem 3.1 *For any prime $p \geq 3$, $i \in \{1, 2...p-1\}$ and non-negative integers α and n, we have*

$$\sum_{n=0}^{\infty} p_\lambda\left(2 \cdot p^{2\alpha} n + \frac{p^{2\alpha}-1}{4}\right) q^n \equiv E_1^3 \pmod{2}, \tag{3.1}$$

$$\sum_{n=0}^{\infty} p_\lambda\left(2 \cdot p^{2\alpha+1} n + \frac{p^{2\alpha+2}-1}{4}\right) q^n \equiv E_p^3 \pmod{2}, \tag{3.2}$$

$$p_\lambda\left(2 \cdot p^{2\alpha+1}(pn+i) + \frac{p^{2\alpha+2}-1}{4}\right) \equiv 0 \pmod{2}. \tag{3.3}$$

Proof: Substituting q by $-q$ in (1.10), we have

$$2(-q)^{-1}\psi_6(q^2) + \lambda(q) = (q;q^2)_\infty^2 f(-q;-q^5). \tag{3.4}$$

Employing Jacobi's triple product identity into (3.4), we get

$$\lambda(q) = \frac{E_1^3 E_6^2}{E_2^3 E_3} + 2q^{-1}\psi_6(q^2). \tag{3.5}$$

Now, using Lemma 2.3 in (3.5) and isolating the terms that involve q^{2n}, we obtain

$$\sum_{n=0}^{\infty} p_\lambda(2n) q^n = \frac{E_2^3 E_3^2}{E_6 E_1^3}. \tag{3.6}$$

Now applying (2.8) for $\beta = 1$ and $p = 2$ in (3.6), we have

$$\sum_{n=0}^{\infty} p_\lambda(2n) q^n \equiv E_1^3 \pmod{2}. \tag{3.7}$$

The congruence (3.7) is the case for $\alpha = 0$ of (3.1). Consider that (3.1) is true for $\alpha \geq 0$. Now utilizing Lemma 2.5 in (3.1) and collecting the terms that involve $q^{pn+\frac{p^2-1}{8}}$, we have

$$\sum_{n=0}^{\infty} p_\lambda\left(2 \cdot p^{2\alpha+1} n + \frac{p^{2\alpha+2}-1}{4}\right) q^n \equiv E_p^3 \pmod{2}. \tag{3.8}$$

Again colleting the terms that involve q^{pn}, we obtain

$$\sum_{n=0}^{\infty} p_\lambda\left(2 \cdot p^{2\alpha+2} n + \frac{p^{2\alpha+2}-1}{4}\right) q^n \equiv E_1^3 \pmod{2}. \tag{3.9}$$

The congruence (3.9) shows that (3.1) is true for $\alpha + 1$. This shows that (3.1) is true for all $\alpha \geq 0$.

Employing Lemma 2.5 in (3.1) and isolating the terms that involve $q^{pn+\frac{p^2-1}{8}}$, we obtain (3.2). Collecting the terms that involve q^{pn+i}, for $i \in \{1, 2....p-1\}$ from (3.2), we get equation (3.3).

□

Theorem 3.2 *For any prime $p \geq 5$, $i \in \{1, 2...p-1\}$ and non-negative integers α and n, we have*

$$\sum_{n=0}^{\infty} p_\lambda\left(6\cdot p^{2\alpha}n + \frac{p^{2\alpha}-1}{4}\right) q^n \equiv E_1 \pmod 3, \tag{3.10}$$

$$\sum_{n=0}^{\infty} p_\lambda\left(6\cdot p^{2\alpha+1}n + \frac{p^{2\alpha+2}-1}{4}\right) q^n \equiv E_p \pmod 3, \tag{3.11}$$

$$p_\lambda\left(6\cdot p^{2\alpha+1}(pn+i) + \frac{p^{2\alpha+2}-1}{4}\right) \equiv 0 \pmod 3. \tag{3.12}$$

Proof: Employing (2.8) for $\beta = 1$ and $p = 3$ in (3.6), we have

$$\sum_{n=0}^{\infty} p_\lambda(2n) q^n \equiv E_1^3 \pmod 3. \tag{3.13}$$

Again isolating the terms that involve q^{3n}, we have

$$\sum_{n=0}^{\infty} p_\lambda(6n) q^n \equiv E_1 \pmod 3. \tag{3.14}$$

The congruence (3.14) is the case for $\alpha = 0$ of (3.10). Consider that (3.10) is true for $\alpha \geq 0$. Now utilizing Lemma 2.4 in (3.10) and collecting the terms that involve $q^{pn+\frac{p^2-1}{24}}$, we have

$$\sum_{n=0}^{\infty} p_\lambda\left(6\cdot p^{2\alpha+1}n + \frac{p^{2\alpha+2}-1}{4}\right) q^n \equiv E_p \pmod 3. \tag{3.15}$$

Again colleting the terms that involve q^{pn}, we obtain

$$\sum_{n=0}^{\infty} p_\lambda\left(6\cdot p^{2\alpha+2}n + \frac{p^{2\alpha+2}-1}{4}\right) q^n \equiv E_1 \pmod 3. \tag{3.16}$$

The congruence (3.16) shows that (3.10) is true for $\alpha + 1$. This shows that (3.10) is true for all $\alpha \geq 0$.

Employing Lemma 2.4 in (3.10) and isolating the terms that involve $q^{pn+\frac{p^2-1}{24}}$, we obtain (3.11). Collecting the terms that involve q^{pn+i}, for $i \in \{1, 2....p-1\}$ from (3.11), we get equation (3.12).

□

Theorem 3.3 *For any prime $p \geq 5$, $i \in \{1, 2...p-1\}$ and non-negative integers α and n, we have*

$$\sum_{n=0}^{\infty} p_\lambda\left(6\cdot p^{2\alpha}n + \frac{p^{2\alpha}-1}{4}\right) q^n \equiv E_1 \pmod 4, \tag{3.17}$$

$$\sum_{n=0}^{\infty} p_\lambda\left(6\cdot p^{2\alpha+1}n + \frac{p^{2\alpha+2}-1}{4}\right) q^n \equiv E_p \pmod 4, \tag{3.18}$$

$$p_\lambda\left(6\cdot p^{2\alpha+1}(pn+i) + \frac{p^{2\alpha+2}-1}{4}\right) \equiv 0 \pmod 4. \tag{3.19}$$

Proof: Employing Lemma 2.2 in (3.6), we have

$$\sum_{n=0}^{\infty} p_\lambda(2n) q^n = \left(\frac{E_6}{E_3} + 3q\frac{E_6^4 E_9^5}{E_3^8 E_{18}} + 6q^2\frac{E_6^3 E_9^2 E_{18}^2}{E_3^7} + 12q^3\frac{E_6^2 E_{18}^5}{E_3^6 E_9}\right)\frac{E_3^2}{E_6}. \tag{3.20}$$

Isolating the terms that involve q^{3n}, q^{3n+1} and q^{3n+2} from (3.20), we get

$$\sum_{n=0}^{\infty} p_\lambda(6n)q^n = E_1 + 12q\frac{E_2E_6^5}{E_1^4E_3}, \tag{3.21}$$

$$\sum_{n=0}^{\infty} p_\lambda(6n+2)q^n = 3\frac{E_2^3E_3^5}{E_1^6E_6}, \tag{3.22}$$

$$\sum_{n=0}^{\infty} p_\lambda(6n+4)q^n = 6\frac{E_2^2E_3^2E_6^2}{E_1^5}, \tag{3.23}$$

respectively.
From (3.21), we have

$$\sum_{n=0}^{\infty} p_\lambda(6n)q^n \equiv E_1 \pmod 4. \tag{3.24}$$

The congruence (3.24) is the case for $\alpha = 0$ of (3.17). The proof of identities (3.17)-(3.19) follows a similar approach to those of identities (3.10)-(3.12) of Theorem 3.2. □

Theorem 3.4 *For any prime $p \geq 3$, $i \in \{1, 2...p-1\}$ and non-negative integers α and n, we have*

$$\sum_{n=0}^{\infty} p_\lambda\left(2\cdot p^{2\alpha}n + \frac{p^{2\alpha}-1}{4}\right) q^n \equiv E_1^3 \pmod 6, \tag{3.25}$$

$$\sum_{n=0}^{\infty} p_\lambda\left(2\cdot p^{2\alpha+1}n + \frac{p^{2\alpha+2}-1}{4}\right) q^n \equiv E_p^3 \pmod 6, \tag{3.26}$$

$$p_\lambda\left(2\cdot p^{2\alpha+1}(pn+i) + \frac{p^{2\alpha+2}-1}{4}\right) \equiv 0 \pmod 6. \tag{3.27}$$

Proof: Combining equations (3.7) and (3.13), we obtain

$$\sum_{n=0}^{\infty} p_\lambda(2n)q^n \equiv E_1^3 \pmod 6. \tag{3.28}$$

The congruence (3.28) is the case for $\alpha = 0$ of (3.25). The proof of identities (3.25)-(3.27) follows a similar approach to those of identities (3.1)-(3.3) of Theorem 3.1. □

Theorem 3.5 *For any prime $p \geq 5$, $i \in \{1, 2...p-1\}$ and non-negative integers α and n, we have*

$$\sum_{n=0}^{\infty} p_\lambda\left(54\cdot p^{2\alpha}n + \frac{9p^{2\alpha}-1}{4}\right) q^n \equiv 3E_1 \pmod 9, \tag{3.29}$$

$$\sum_{n=0}^{\infty} p_\lambda\left(54\cdot p^{2\alpha+1}n + \frac{9p^{2\alpha+2}-1}{4}\right) q^n \equiv 3E_p \pmod 9, \tag{3.30}$$

$$p_\lambda\left(54\cdot p^{2\alpha+1}(pn+i) + \frac{9p^{2\alpha+2}-1}{4}\right) \equiv 0 \pmod 9. \tag{3.31}$$

Proof: Utilizing (2.8) for $\beta = 1$ and $p = 3$ in (3.22), we obtain

$$\sum_{n=0}^{\infty} p_\lambda(6n+2)q^n \equiv 3E_3^3 \pmod 9, \tag{3.32}$$

Extracting terms involving q^{9n}, we have

$$\sum_{n=0}^{\infty} p_\lambda(54n+2)q^n \equiv 3E_1 \pmod 9, \tag{3.33}$$

The equation (3.32) shows that (3.29) holds for $\alpha_1 = 0$. Consider that (3.29) is true for all $\alpha_1 \geq 0$. Now employing Lemma 2.4 in (3.29) and isolating the terms involving $q^{pn+\frac{p^2-1}{24}}$, we have

$$\sum_{n=0}^{\infty} p_\lambda\left(54\cdot p^{2\alpha+1}n + \frac{9p^{2\alpha+2}-1}{4}\right) q^n \equiv 3E_p \pmod 9. \tag{3.34}$$

Again extracting coefficients of $q^{pn+\frac{p^2-1}{24}}$ from (3.34), we have

$$\sum_{n=0}^{\infty} p_\lambda\left(54\cdot p^{2\alpha+2}n + \frac{9p^{2\alpha+2}-1}{4}\right) q^n \equiv 3E_1 \pmod 9. \tag{3.35}$$

The equation (3.35) shows that (3.29) is true for $\alpha+1$. This shows that (3.29) holds for all integer $\alpha_1 \geq 0$.

Employing Lemma 2.4 in (3.29) and extracting terms involving $q^{pn+\frac{p^2-1}{24}}$, we obtain (3.30). Extracting the terms that involve q^{pn+i}, for $i \in \{1, 2....p-1\}$ from (3.30), we get equation (3.31).

□

Theorem 3.6 *For any prime $p \geq 3$, $i \in \{1, 2...p-1\}$ and non-negative integers α and n, we have*

$$\sum_{n=0}^{\infty} p_\lambda\left(18\cdot p^{2\alpha}n + \frac{9p^{2\alpha}-1}{4}\right) q^n \equiv 3E_1^3 \pmod 6, \tag{3.36}$$

$$\sum_{n=0}^{\infty} p_\lambda\left(18\cdot p^{2\alpha+1}n + \frac{9p^{2\alpha+2}-1}{4}\right) q^n \equiv 3E_p^3 \pmod 6, \tag{3.37}$$

$$p_\lambda\left(18\cdot p^{2\alpha+1}(pn+i) + \frac{9p^{2\alpha+2}-1}{4}\right) \equiv 0 \pmod 6. \tag{3.38}$$

Proof: Employing (2.8) for $\beta = 1$ and $p = 2$ in (3.22), we obtain

$$\sum_{n=0}^{\infty} p_\lambda(6n+2)q^n \equiv 3E_3^3 \pmod 6, \tag{3.39}$$

Extracting terms involving q^{3n}, we have

$$\sum_{n=0}^{\infty} p_\lambda(18n+2)q^n \equiv 3E_1^3 \pmod 6, \tag{3.40}$$

The equation (3.40) shows that (3.36) holds for $\alpha_1 = 0$. Consider that (3.36) is true for all $\alpha_1 \geq 0$. Now employing Lemma 2.5 in (3.36) and isolating the terms involving $q^{pn+\frac{p^2-1}{8}}$, we have

$$\sum_{n=0}^{\infty} p_\lambda\left(18\cdot p^{2\alpha+1}n + \frac{9p^{2\alpha+2}-1}{4}\right) q^n \equiv 3E_p^3 \pmod 6. \tag{3.41}$$

Again extracting coefficients of $q^{pn+\frac{p^2-1}{8}}$ from (3.41), we have

$$\sum_{n=0}^{\infty} p_\lambda\left(18\cdot p^{2\alpha+2}n + \frac{9p^{2\alpha+2}-1}{4}\right) q^n \equiv 3E_1^3 \pmod 6. \tag{3.42}$$

The equation (3.42) shows that (3.36) is true for $\alpha+1$. This shows that (3.36) holds for all integer $\alpha_1 \geq 0$.

Employing Lemma 2.5 in (3.36) and extracting terms involving $q^{pn+\frac{p^2-1}{8}}$, we obtain (3.37). Extracting the terms that involve q^{pn+i}, for $i \in \{1, 2....p-1\}$ from (3.37), we get equation (3.38).

□

Author Contributions

Both the authors equally contributed for prepairing this manuscript.

Declaration of competing interest

The authors declared that they have no conflicts of interest to this work.

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