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Investigation of Torsional Waves Propagation in a Sandwiched and Initially Stressed Dry Sandy Gibson Poroelastic Dissipative Isotropic Solid

Ch. Balu[1]*, M. Ramesh[2], B. Radhika[3], and P. Malla Reddy[4]

ABSTRACT: In this paper, torsional waves in sandwiched solids are investigated in the framework of Biot's incremental theory. The solid consists of dry sandy Gibson poroelastic dissipative isotropic cylindrical solid sandwiched between two heterogeneous isotropic poroelastic cylindrical solids, all are initially stressed. The solutions for the problems of torsional waves in upper heterogeneous poroelastic cylindrical solid, dry sandy Gibson poroelastic cylindrical middle solid, and lower heterogeneous poroelastic cylindrical solids are presented. The solution of the problem reduced to that of Whittaker's differential equation. Frequency equation is obtained from the boundary conditions of displacement components and stresses which are assumed to be continuous at the interfaces between upper solid and middle, and middle solid and lower solid. The solid under consideration is dissipative, the frequency equation is implicit and complex valued. Employing the values in the frequency equation, the frequency and attenuation coefficient against heterogeneous parameter at fixed sandy parameter, gravity parameter and initial stress are computed. The values are computed using the bisection method implemented in MATLAB.

Key Words: Poroelasticity, initial stress, torsional wave, frequency equation, attenuation coefficient.

Contents

1. Introduction

Seismic wave research has been used to pinpoint the epicenter of earthquakes and provides valuable insights into the layered structure of the Earth. Large-scale waves are produced by earthquakes, and one of the surface waves that may be seen after multiple global trips is a torsional wave. The systematic study of these waves has significant consequences for both human safety and scientific interest in the composition and history of the Earth. In addition, torsional waves produced artificially also yield information about the arrangement of rock layers for oil exploration and, on a smaller scale, about the stiffness of superficial layers for Engineering applications. In general, the Earth contains heterogeneous layers and has significant effect on propagation of elastic waves. In particular, the Earth crust is made of diversity of igneous, metamorphic and sedimentary rocks. These rocks are capable to generate magnetic field due to presence of iron, nickel, and cobalt, etc in them, and are magneto poroelastic in nature. In the frame work of Biot's theory [1] , the effect of boundaries on torsional vibrations in a poroelastic composite cylinder was reported in the papers [2]- [3]. Axially symmetric vibrations of composite poroelastic cylinder were investigated by Reddy and Tajuddin [4]. Poromechanic analysis of a fully saturated transversely isotropic hollow cylinder was made by Kanj et. al. [5]. Radial vibrations in thick-walled transversely isotropic poroelastic cylindrical bone in presence of dissipative were studied by Reddy and Sandhya [6]. Torsional vibrations

* Corresponding author.
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in thick-walled cylindrical bone in the framework of transversely isotropic Poroelasticity were investigated by Sandhya and Reddy [7]. Torsional vibrations in composite transversely isotropic poroelastic cylinders were studied by Sandhya et. al. [8]. In the paper [8], poroelastic composite cylinder consists of two concentric cylindrical layers of different materials. It is established that Berea sandstone and Shale rock exhibit transversely isotropic behavior at low effective stress [9]-[10]. Kundu et. al. [11] investigated propagation of a torsional surface wave in a non-homogeneous anisotropic layer over a heterogeneous half-space, under assumption that homogeneity varies exponentially with depth. Shear waves in magneto-elastic transversely isotropic layer bonded between two heterogeneous elastic media was studied by Kundu et. al. [12]. Torsional surface wave propagation in anisotropic layer sandwiched between heterogeneous half-space was studied by Vaishnav et. al. [13]. In the paper [13], the dispersion relation of torsional surface waves was obtained for heterogeneous, initially stressed solid. Studies of torsional vibrations in isotropic poroelastic cylinder were reported in the papers [14] [15] [16]. However, torsional waves in shell sandwiched between two heterogeneous transversely poroelastic cylindrical shell is not yet considered. Therefore, in the present paper, the same is investigated. The rest of the paper is organized as follows: In section 2, geometry, formulation and solution of the problem are presented. Boundary conditions and frequency equation are presented in section 3. In section 4, numerical results are discussed. Finally, conclusion is given in section 5.

2. Geometry, Formulation and Solution of the Problem

Consider the isotropic poroelastic cylindrical solid sandwiched between two inhomogeneous poroelastic solid, all under gravity field and initial stress. The equations of motion in this case are as follows:

$$\begin{aligned}
\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta}+\frac{\partial \sigma_{rz}}{\partial r}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}-p\frac{\partial \omega_z}{\partial \theta}+p\frac{\partial \omega_z}{\partial z} &= \frac{\partial^2}{\partial t^2}(\rho_{11}u_i+\rho_{12}U_i)+b\frac{\partial}{\partial t}(u_i-U_i)\\
\frac{\partial \sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}+\frac{\partial \sigma_{\theta z}}{\partial z}+\frac{2\sigma_{r\theta}}{r}+p\frac{\partial \omega_z}{\partial r} &= \frac{\partial^2}{\partial t^2}(\rho_{11}v_i+\rho_{12}V_i)+b\frac{\partial}{\partial t}(v_i-V_i)\\
\frac{\partial \sigma_{rz}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta}+\frac{\partial \sigma_{zz}}{\partial z}+\frac{\sigma_{rz}}{r}-p'\frac{\partial \omega_\theta}{\partial r} &= \frac{\partial^2}{\partial t^2}(\rho_{11}w_i+\rho_{12}W_i)+b\frac{\partial}{\partial t}(w_i-W_i) \qquad (2.1)\\
\frac{\partial s}{\partial r} &= \frac{\partial^2}{\partial t^2}(\rho_{12}u_i+\rho_{22}U_i)-b\frac{\partial}{\partial t}(u_i-U_i)\\
\frac{\partial s}{\partial \theta} &= \frac{\partial^2}{\partial t^2}(\rho_{12}v_i+\rho_{22}V_i)-b\frac{\partial}{\partial t}(v_i-V_i)\\
\frac{\partial s}{\partial z} &= \frac{\partial^2}{\partial t^2}(\rho_{12}w_i+\rho_{22}W_i)-b\frac{\partial}{\partial t}(w_i-W_i)
\end{aligned}$$

where (u_i, v_i, w_i) and $(U_i, V_i, W_i)(i = 1, 2, 3)$ are the displacement components of solid and fluid, respectively, $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{rz}, \sigma_{r\theta}$ and $\sigma_{\theta z}$ are the stress components, ρ_{ij} are mass coefficients, b is disspative co-efficient, t is time, s is fluid pressure, p is initial stress. For torsional waves in cylindrical solids, it is convenient to consider the cylindrical polar coordinate system (r, θ, z). The strain-displacement relations in the cylindrical system are

$$\begin{aligned}
e_{rr} &= \frac{\partial u_i}{\partial r}, e_{\theta\theta} = \frac{\partial v_i}{\partial \theta}, e_{r\theta} = \frac{1}{2}\left(\frac{\partial u_i}{\partial \theta}+\frac{\partial v_i}{\partial r}-\frac{v_i}{r}\right),\\
e_{rz} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial z}+\frac{\partial w_i}{\partial r}\right), e_{z\theta} = \frac{1}{2}\left(\frac{\partial v_i}{\partial z}+\frac{1}{r}\frac{\partial w_i}{\partial \theta}\right). \qquad (2.2)
\end{aligned}$$

The rotational components are given by

$$\omega_\theta = \frac{1}{2}\left(\frac{\partial u_i}{\partial z}-\frac{\partial w_i}{\partial r}\right), \omega_z = \frac{1}{2}\left(\frac{\partial v_i}{\partial r}+\frac{\partial u_i}{\partial \theta}\right). \qquad (2.3)$$

The problem of torsional waves in the upper heterogeneous poroelastic cylindrical solid, the dry sandy gibson poroelastic cylindrical middle solid, and the lower heterogeneous poroelastic cylindrical solid are presented in the following three subsections.

2.1. Upper heterogeneous poroelastic cylindrical solid

Let (u_1, v_1, w_1) and (U_1, V_1, W_1) be the displacement vectors of the solid and fluid parts, respectively, in the homogeneous cylindrical solid initially stressed poroelastic M_1 (say). For torsional vibrations,$u_1 = w_1 = 0, v_1 = v_1(r, z, t)$ and $U_1 = W_1 = 0, V_1 = V_1(r, z, t)$. In this case, the equations of motion (2.1) are reduced to the following:

$$\begin{aligned} \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + p_1 \frac{\partial \omega_z}{\partial r} &= \frac{\partial^2}{\partial t^2}(\rho_{11} v_1 + \rho_{12} V_1) + b\frac{\partial}{\partial t}(v_1 - V_1), \\ 0 &= \frac{\partial^2}{\partial t^2}(\rho_{12} v_1 + \rho_{22} V_1) - b\frac{\partial}{\partial t}(v_1 - V_1). \end{aligned} \tag{2.4}$$

The upper cylindrical solid is assumed to be heterogeneous along axial direction with respect to mass coefficients and initial stress. Therefore, one can have

$$\begin{aligned} \rho_{11} = \rho_{110}(1 + \alpha_1 z)e^{az}, \rho_{12} = \rho_{120}(1 + \alpha_1 z)e^{az}, \\ \rho_{22} = \rho_{220}(1 + \alpha_1 z)e^{az}, p_1 = p_{100}(1 + \alpha_1 z)e^{az}, \end{aligned} \tag{2.5}$$

where $\rho_{110}, \rho_{120}, \rho_{220}$ are initial values of mass coefficients, α_1 is heterogeneity parameter and p_{100} is initial value of initial stress. Using stress- strain relations, Eq. (2.5) in Eq. (2.4) gives the following equations:

$$\begin{aligned} & N\frac{\partial^2 v_1}{\partial r^2} + (N - \frac{p_{100}(1 + \alpha_1 z)e^{az}}{2})\frac{\partial^2 v_1}{\partial z^2} + \frac{N}{r}\frac{\partial v_1}{\partial r} - \frac{N}{r^2}v_1 \\ &= \left(\rho_{110}\frac{\partial^2 v_1}{\partial t^2} + \rho_{120}\frac{\partial^2 V_1}{\partial t^2}\right)(1 + \alpha_1 z)e^{az} + b\left(\frac{\partial v_1}{\partial t} - \frac{\partial V_1}{\partial t}\right), \\ 0 &= \left(\rho_{120}\frac{\partial^2 v_1}{\partial t^2} + \rho_{220}\frac{\partial^2 V_1}{\partial t^2}\right)(1 + \alpha_1 z)e^{az} - b\left(\frac{\partial v_1}{\partial t} - \frac{\partial V_1}{\partial t}\right). \end{aligned} \tag{2.6}$$

For harmonic torsional waves, the displacement components can be taken as follows:

$$v_1(r, z, t) = h_1(z)J_1(kr)e^{i\omega t}, V_1(r, z, t) = H_1(z)J_1(kr)e^{i\omega t}. \tag{2.7}$$

In Eq. (2.7), J_1 is Bessel's function of the first order and the first kind, k is wave number, ω is wave frequency, and i is complex unity. Substitution of Eq. (2.7) in Eq. (2.6) gives

$$\frac{d^2 h_1}{dz^2} + q_1^2 h_1 = 0. \tag{2.8}$$

Solution of Eq. (2.8) is

$$h_1(z) = (c_1 e^{q_1 z} + c_2 e^{-q_1 z}). \tag{2.9}$$

In Eq. (2.9), c_1 and c_2 are arbitrary constants, and

$$\begin{aligned} q_1 &= \frac{\sqrt{L^2 - 4q^1}}{2}, L = 2N((2N - p_{110}(1 + \alpha_1 z)e^{az^{-1}}, \\ q^1 &= 2(2N - p_{100}(1 + \alpha_1 z)e^{az)^{-1}}(\rho_{100}\omega^2(1 + \alpha_1 z)e^{az)^{-1}} - ib\omega - \frac{2N}{r^2} - \\ & \left(\left(\rho_{120}\omega^2(1 + \alpha_1 z)e^{az^2} + ib\omega\right)^2 \left(\rho_{220}\omega^2(1 + \alpha_1 z)e^{az^2} - ib\omega\right)^{-1}\right). \end{aligned}$$

Substitution of Eq. (2.9) in Eq. (2.8) gives

$$H_1(z) = D_1 D_2^{-1} h_1(z), \tag{2.10}$$

where $D_1 = \rho_{120}(1+\alpha_1 z)e^{az} + ib\omega^{-1}$ and $D_2 = \rho_{220}(1+\alpha_1 z)e^{az} - ib\omega^{-1}$.
Substituting Eq. (2.9) in Eq. (2.7), and then using stress- displacement relations, the following stress components are obtained

$$\begin{aligned}(\sigma_{r\theta})_{M_1} &= -N(c_1 e^{q_1 z} + c_2 e^{-q_1 z})J_1(kr)e^{i\omega t},\\ (\sigma_{\theta z})_{M_1} &= N(c_1 e^{q_1 z} - c_2 e^{-q_1 z})J_1(kr)e^{i\omega t}.\end{aligned} \tag{2.11}$$

2.2. Dry sandy Gibson poroelastic cylindrical middle solid

Let (u_2, v_2, w_2) and (U_2, V_2, W_2) be the solid and fluid displacement components, respectively, in the sandwiched dry sandy gibson poroelastic isotropic cylindrical solid (M_2) (say). For torsional waves, Eq. (2.1) can be written as

$$\begin{aligned}&\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + p_2\frac{\partial \omega_r}{\partial z} +\\ &\frac{\partial}{\partial z}(p_2 - (\frac{\rho_{110}-\rho_{120}^2}{\rho_{220}})gz)e_z\theta) - (\frac{\rho_{110}-\rho_{120}^2}{\rho_{220}})gz\frac{\partial}{\partial r}\frac{1}{2}(\frac{\partial v_2}{\partial r} + \frac{v_2}{r}) = \frac{\partial^2}{\partial t^2}(\rho_{110}v_2 + \rho_{120}V_2) + b\frac{\partial}{\partial t}(v_2 - V_2)\\ &0 = \frac{\partial^2}{\partial t^2}(\rho_{120}v_2 + \rho_{220}V_2) - b\frac{\partial}{\partial t}(v_2 - V_2)\end{aligned} \tag{2.12}$$

In the above equation, g is acceleration due to gravity, p_2 is compressive initial stress. Also, $N = \eta\mu$, where η is sandy parameter and μ is the modulus of rigidity.

$$\begin{aligned}\eta\mu\frac{\partial^2 v_2}{\partial r^2} + \frac{\eta\mu}{r}\frac{\partial v_2}{\partial r} - \frac{\eta\mu}{r^2}v_2 + \eta\mu - \frac{p_2}{2} - \frac{1}{2}((\frac{\rho_{110}-\rho_{120}^2}{\rho_{220}})gz))\frac{\partial^2 v_2}{\partial z^2} - ((\frac{\rho_{110}-\rho_{120}^2}{\rho_{220}})gz)\\ = \left(\rho_{110}\frac{\partial^2 v_2}{\partial t^2} + \rho_{120}\frac{\partial^2 V_2}{\partial t^2}\right) + b\left(\frac{\partial v_2}{\partial t} - \frac{\partial V_2}{\partial t}\right)\\ 0 = \left(\rho_{120}\frac{\partial^2 v_2}{\partial t^2} + \rho_{220}\frac{\partial^2 V_2}{\partial t^2}\right) - b\left(\frac{\partial v_2}{\partial t} - \frac{\partial V_2}{\partial t}\right)\end{aligned} \tag{2.13}$$

For torsional harmonic waves, the displacement components can be expressed as

$$v_2(r,z,t) = h_2(z)J_1(kr)e^{i\omega t}, V_2(r,z,t) = H_2(z)J_1(kr)e^{i\omega t}. \tag{2.14}$$

Substitution of Eq. (2.14) in Eq. (2.13) gives

$$\frac{d^2 h_2}{dz^2} - q_2^2 h_2 = 0, \tag{2.15}$$

where,

$$\begin{aligned}q_2^2 = &\frac{2\rho_2 20}{(2(\eta\mu)^2 - (\rho_{220}-\rho_{220}^2))gzJ_1(kr)}(\frac{\eta\mu - (\rho_{220}-\rho_{220}^2)gz}{2\rho_{220}})k^2 J_1(kr)\\ &+\eta\mu - (\rho_{110}-\rho_{120}^2)gz2\rho_{220})k^2 J_1(kr) + (\rho_{110}^2\omega^2 - ib\omega - \frac{\eta\mu}{r^2} - \frac{\rho_{120}\omega^2 + ib\omega}{\rho_{220}\omega^2 - ib\omega})J_1(kr).\end{aligned}$$

Solution of Eq. (2.15) is

$$h_2(z) = (c_3 e^{q_2 z} + c_4 e^{-q_2 z}). \tag{2.16}$$

In Eq. (2.16), c_3 and c_4 are arbitrary constants. Substitution of Eq. (2.16) in Eq. (2.14) gives

$$H_2(z) = -D_3 D_4^{-1} h_2(z), \tag{2.17}$$

where $D_3 = \rho_{120} + ib\omega^{-1}$ and $D_4 = \rho_{220} - ib\omega^{-1}$.
Substituting Eq. (2.16) in Eq. (2.14), and then using stress- displacement relations, the following stress components are obtained

$$\begin{aligned}(\sigma_{r\theta})_{M_2} &= -\eta\mu(c_3 e^{q_2 z} + c_4 e^{-q_2 z})J_1(kr)e^{i\omega t},\\ (\sigma_{\theta z})_{M_2} &= \eta\mu q_2(c_3 e^{q_2 z} - c_4 e^{-q_2 z})J_1(kr)e^{i\omega t}.\end{aligned} \tag{2.18}$$

2.3. Lower heterogeneous poroelastic cylindrical solid

For the lower heterogeneous poroelastic cylindrical solid M_3 (say), let $(u_3, v_3, w3)$ and (U_3, V_3, W_3) be the displacement components of solid and fluid parts, respectively. For the torsional vibrations, the equations of motion are reduced to the following:

$$\begin{aligned}\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} + \frac{\partial\sigma_{\theta z}}{\partial z} + p_3\frac{\partial\omega_z}{\partial r} &= \frac{\partial^2}{\partial t^2}(\rho_{11}v_3 + \rho_{12}V_3) + b\frac{\partial}{\partial t}(v_3 - V_3),\\ 0 &= \frac{\partial^2}{\partial t^2}(\rho_{12}v_3 + \rho_{22}V_3) - b\frac{\partial}{\partial t}(v_3 - V_3).\end{aligned} \tag{2.19}$$

As in the earlier cases, the lower solid is assumed to be heterogeneous with respect to the mass coefficients and initial stress, then one can have

$$\begin{aligned}\rho_{11} = \rho_7(1+\alpha_3 z)e^{az}, \rho_{12} = \rho_8(1+\alpha_3 z)e^{az},\\ \rho_{22} = \rho_9(1+\alpha_3 z)e^{az}, p_3 = p_3(1+\alpha_3 z)e^{az},\end{aligned} \tag{2.20}$$

where ρ_7, ρ_8, ρ_9 are initial values of mass coefficients, and p_3 is initial value of initial stress. Using stress-strain relation and Eq. (2.20) in Eq. (2.19) gives,

$$\begin{aligned}&N\frac{\partial^2 v_3}{\partial r^2} + \frac{N}{r}\frac{\partial v_3}{\partial r} - 2\frac{N}{r^2}v_3 + (N - \frac{p_3(1+\alpha_3 z)e^{az}}{2})\frac{\partial^2 v_3}{\partial z^2}\\ &= \left(\rho_{10}\frac{\partial^2 v_3}{\partial t^2} + \rho_{20}\frac{\partial^2 V_3}{\partial t^2}\right)(1+\alpha_3 z)e^{az} + b\left(\frac{\partial v_3}{\partial t} - \frac{\partial V_3}{\partial t}\right),\\ 0 &= \left(\rho_{20}\frac{\partial^2 v_3}{\partial t^2} + \rho_{30}\frac{\partial^2 V_3}{\partial t^2}\right)(1+\alpha_3 z)e^{az} - b\left(\frac{\partial v_3}{\partial t} - \frac{\partial V_3}{\partial t}\right).\end{aligned} \tag{2.21}$$

For harmonic torsional waves, the displacement components can be taken as follows:

$$v_3(r,z,t) = h_3(z)J_1(kr)e^{i\omega t}, V_3(r,z,t) = H_3(z)J_1(kr)e^{i\omega t}. \tag{2.22}$$

Substitution of Eq. (2.22) in Eq. (2.21) gives

$$\frac{d^2 h_3}{dz^2} - q_3^2 h_3 = 0. \tag{2.23}$$

Substitution of $h_3(z) = \phi(z)e^{\frac{L_1 z}{2}}$ in Eq. (2.23) gives

$$\phi^{//}(z) + (\frac{R}{4z} - \frac{1}{4})\phi(z) = 0, \tag{2.24}$$

where, $R = z(-L_1^2 - 4q_3^2 + 1)$. The Eq. (2.24) is the standard Whittaker's equation, and its solution is given by

$$\phi(z) = C_5 W_{-R,0}(-z) + C_6 W_{R,0}(z). \tag{2.25}$$

In the above, C_5 and C_6 are arbitrary constants. Substituting Eq. (2.25) and Eq. (2.22), and using stress-displacement relations, the following stress components are obtained.

$$\begin{aligned}(\sigma_{r\theta})_{M_3} &= -N(c_5 W_{-R,0} + c_6 W_{R,0})J_1(kr)e^{i\omega t},\\ (\sigma_{\theta z})_{M_3} &= Nq_3(c_5 W_{-R,0} - c_6 W_{R,0})J_1(kr)e^{i\omega t}.\end{aligned} \tag{2.26}$$

3. Boundary Conditions and Frequency Equation

The displacement components and stresses are assumed to be continuous at $z = a_1$, the interface between upper cylindrical solid M_1 and middle cylindrical solid M_2 . That is,

$$\begin{aligned}(v_1)_{M_1} &= (v_2)_{M_2}\\ (\sigma_{r\theta})_{M_1} &= (\sigma_{r\theta})_{M_2}\\ (\sigma_{\theta z})_{M_1} &= (\sigma_{\theta z})_{M_2}\end{aligned} \tag{3.1}$$

The displacement components and stresses are assumed to be continuous at $z = a_2$, the interface middle cylindrical solid M_2 and lower cylindrical solid M_3. That is,

$$\begin{aligned}(v_2)_{M_2} &= (v_3)_{M_3}\\ (\sigma_{r\theta})_{M_2} &= (\sigma_{r\theta})_{M_3}\\ (\sigma_{\theta z})_{M_2} &= (\sigma_{\theta z})_{M_3}\end{aligned} \tag{3.2}$$

The above boundary conditions lead to the system of homogeneous equations in C_i's. For non-trivial solution,

$$|B_{\mathrm{ij}}| = 0, \qquad (i,\ j = 1,\ 2,\ 3,\ 4,\ 5,\ 6). \tag{3.3}$$

4. Numerical Results

In the presence of dissipation(b), frequency equations (3.3) is implicit and complex valued. If the identifies $\cos(x + \mathrm{i}y) = \cos x\ \cosh y - i\sin x\ \sinh y$ and $\sin(x + \mathrm{i}y) = \sin x\cosh y + i\cos x\ \sinh y$ are used, the following explicit complex valued frequency equation for Eq. (3.3) is obtained:

$$|b_{\mathrm{ij}}| + i\left|b_{\mathrm{ij}}\right| = 0. \quad (i,\ j = 1,\ 2,\ 3,\ 4,\ 5,\ 6). \tag{4.1}$$

These real and imaginary parts together give frequency and attenuation coefficient. For numerical process, the following materials are used. Both upper and lower cylindrical solids are assumed to be of Berea sandstone saturated with water [9] (say Mat-II). The Berea sandstone is a building material, and is a host of oil and natural gas. The parameter values of the said material are as follows:
$\rho_{11} = 2407.64kg/m^3, \rho_{12} = -266kg/m^3, \rho_{22} = 456kg/m^3$.
Middle cylindrical solid is assumed to be of Shale Rock [10] (say Mat-I). The Shale rock is formed from mud consist of thin layers, and is most common Shale rock for hydrocarbons (natural gas and petroleum). This rock is a building material, particularly in elevations work. The parameter values of the said material are as follows:
$\rho_{11} = 1398.72kg/m^3, \rho_{12} = -257.28kg/m^3, \rho_{22} = 771.84kg/m^3$.
Employing the values in Eq. (4.1), the implicit relation between frequency and attenuation coefficient against heterogeneous parameter at fixed sandy parameter, gravity parameter and initial stress at Mat-I and Mat-II, respectively are computed. The values are computed using the bisection method implemented in MATLAB. The attenuation coefficient $Q^{-1} = \frac{2\mathrm{Im}(\omega)}{\mathrm{Re}(\omega)}$. In the above formula, $\mathrm{Im}(\omega)$is frequency of imaginary part in Eq.(4.1), and $\mathrm{Re}(\omega)$is frequency of real part in Eq. (4.1). The results are depicted in Figures 1-4. Figures 1-4 depict the variation of frequency and attenuation coefficient against heterogeneous parameter when sandy parameter, gravity parameter and initial stress are constants for Mat-I and Mat-II, respectively. From Figures 1, 3 and 4, it is clear that the frequency values are, in general, greater than that of attenuation coefficient. From Figure 2, it is clear that, in general, the frequency and attenuation coefficient values are independent.

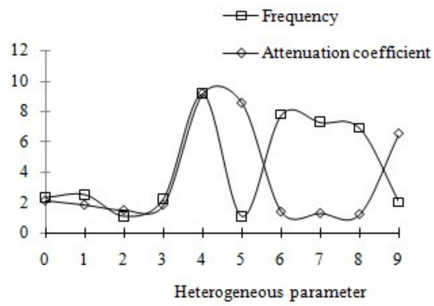

Figure 1: Variation of frequency and attenuation coefficient against heterogeneous parameter at fixed sandy parameter = 2.5 ar

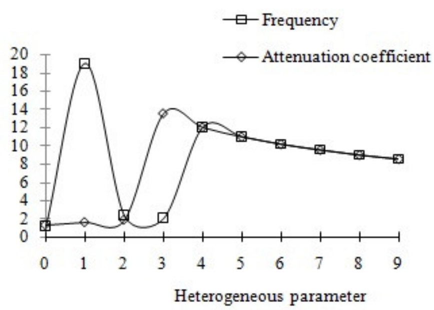

Figure 2: Variation of frequency and attenuation coefficient against heterogeneous parameter at fixed sandy parameter= 2.5 and gravity parameter= $0.5, p = 2$ at Mat-II.

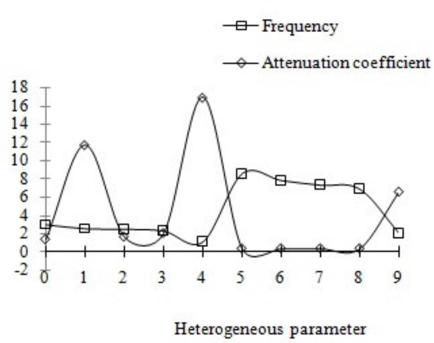

Figure 3: Variation of frequency and attenuation coefficient against heterogeneous parameter at fixed sandy parameter= 3.5 and gravity parameter$1.5, p = 2$ at Mat-I.

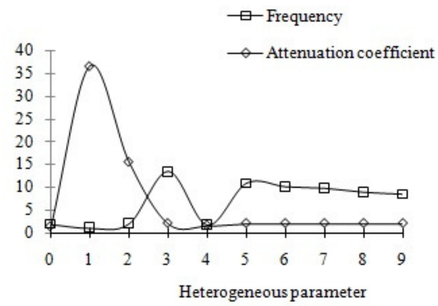

Figure 4: Variation of frequency and attenuation coefficient against heterogeneous parameter at fixed sandy parameter= 3.5 and gravity parameter= 1.5, $p = 2$ at Mat-II.

5. Conclusion

Torsional waves in poroelastic sandwiched between two heterogeneous isotropic poroelastic solid, all under initial stress are investigated in the frame work of Biot's incremental theory. Employing the boundary conditions at the interfaces, frequency equation is obtained. Frequency and attenuation coefficient are computed against heterogeneous parameter at fixed initial stress, sandy parameter, and gravity parameter at Mat-I and Mat-II. From numerical results, it is clear, in general, the frequency values are greater than that of attenuation coefficient.

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[1,4]*Department of Mathematics, Kakatiya University, Warangal-506009, T.G., India.*

[2]*University College of Technology, Osmania University, Hyderabad-500007, T.G., India.*

[3]*Rishi MS Institute of Engineering and Technology for Women, Hyderabad, T.G., India.*

E-mail address: [1]balu.ac.987@gmail.com,

[2]mramesh.ac@gmail.com,

[3] brgoud74@gmail.com,

[4]mperati@yahoo.com