Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–9.
ISSN-0037-8712
doi:10.5269/bspm.77208

Some Spectral Properties for Fractional Sturm- Liouville Operator

M. Jasim Mohammed* and Mustafa Akram Saeed

ABSTRACT: This paper investigates the spectral properties of eigenvalues and eigenfunctions associated with the fractional Sturm-Liouville problem of Bessel type. It is demonstrated The eigenvalues is a real number, and the eigenvalues corresponding to the eigenvalues are orthogonal. In addition, the fractional Bessel operator are explored.

Key Words: Fractional differential equations,Fractional Operator, Fractional order model,Fractional calculus.

Contents

1. Introduction

The Sturm-Liouville boundary value formulation involves a linear second-order ordinary differential equation.

$$-\left(p\psi^1\right)^1+9\psi=\sigma w\psi \tag{1.1}$$

Where p, 9, w : $(\kappa,\sigma)\to IR$ is a A real-valued continuous function in the interval $(\kappa,\sigma),\sigma\not\subset$. function $\psi(x)$ satisfies the boundary conditions: [1]

$$\begin{aligned}\kappa,\psi(\kappa)+\kappa_2\psi'(\kappa)=0\\ \sigma,\psi(\sigma)+\sigma_2\psi'(\sigma)=0\end{aligned} \tag{1.2}$$

Where $\kappa_1,\kappa_2,\sigma_1$ and σ_2 are real numbers.
System (1.1)-(1.2) is an example of a Sturm-Liouville system with eigenvalues. The values that make the system non-zero are called eigenvalues. (1.1)-(1.2), and the corresponding solutions are called to be deleted Eigen functions. Sturm-Liouville issues have a significant impact on multiple scientific disciplines, including engineering, mathematics, and science. Their spectra' characteristics include spectra, functions of spectrum, data about scattering, norms, and other associated quantities.

Theory shows that a second-order differential operator with linear terms is the adjoint operator of a differential operator with homogeneous terms. The two operators have L2-orthogonal sequences of eigenfunctions. [2,3,4,5,6,7].

Fractional caucuses are the " doctrine of the derivative and integral of any arbitrary complex or real number, which combines the concept of differentiation with that of integration", [6,7,8,9,10,11,12,13]. Fractional calculus has become more popular recently because of its diverse range of uses in nearly every field of science. It has been employed successfully in fields like viscoelasticity, electrical engineering, electrochemistry, biology, biophysics, and control theory. [14,15,16,17,18].

* Corresponding author
Submitted June 05, 2025. Published July 29, 2025

2. Preliminaries

In this section, we discuss some special functions, including fractional calculus, and the fundamental properties of the definition of fractional differential/integral operators., [19].

The Linear second-order is given as:

$$x^2\frac{d_2\psi}{dx^2}+x\frac{d\psi}{dx}+\left(x^2-v^2\right)\psi=0 \tag{2.1}$$

Is known as Bessel's equation. Where v is real.

Definition 2.1 *[20] The gamma functions are defined as:*

$$\begin{aligned}\lceil(x) &= \int_0^{\propto} t^{x-1}e^{-t}dt\\ \lceil(x) &= \lceil(x+1)/x \quad x>0\end{aligned} \tag{2.2}$$

Definition 2.2 *[22] Let $0<\infty\leq 1$. Left and right.*

The Riemann-Liouville integral of order "proportional to" is defined as follows:

$$\left(I_{\kappa,+}^{\propto}\mathfrak{F}\right)_{(x)}=\frac{1}{\psi(\propto)}\int_{\kappa}^{x}(x-t)^{\propto-1}\mathfrak{F}(t)dt, x>\kappa, \tag{2.3}$$

$$\left(I_{\sigma,-}^{\propto}\mathfrak{F}\right)_{(x)}=\frac{1}{\psi(\propto)}\int_{x}^{\sigma}(t-x)^{\propto-1}\mathfrak{F}(t)dt, x>\sigma$$

Where $\lceil$ Denotes the gamma function.

Definition 2.3 *[22] Let $\mathrm{o}<\kappa\leq 1$. Both the left-hand and right-hand Riemann-Liouville derivatives of order are given by:*

$$\left(D_{\kappa,+}^{\propto}\mathfrak{F}\right)(x)=D\left(I_{\kappa,+}^{1-\propto}\mathfrak{F}\right)(x), x>\kappa \tag{2.4}$$

$$\left(D_{\sigma,-}^{\propto}\mathfrak{F}\right)(x)=-D\left(I_{\sigma,-}^{1-\propto}\mathfrak{F}\right)(x), x>\sigma$$

Similar formulas give left-hand and right-hand captured derivatives of order

$$\propto: \left(D_{\kappa,+}^{\propto}\mathfrak{F}\right)(x)=D\left(I_{\kappa,+}^{1-\propto}D\mathfrak{F}\right)(x), \quad x>\kappa, o<\propto\leq 1$$

$$\left(D_{\sigma,-}^{\propto}\mathfrak{F}\right)(x)=D\left(I_{\sigma,-}^{1-\propto}D\mathfrak{F}\right)(x), \quad x<\sigma, o<\propto\leq 1 \tag{2.5}$$

Property

The following properties are possessed by the above operators listed in (2.4)-(2.5)

i)

$$\begin{aligned}&\int_{\kappa}^{\sigma}\mathfrak{F}(x)D_{\sigma,-}^{\propto}\mathfrak{G}(x)dx\\ &\quad=\int_{\kappa}^{\sigma}\mathfrak{G}(x)^{c}D_{\kappa}^{\propto}\mathfrak{F}(x)dx-\mathfrak{F}(x)I_{\sigma,-}^{1-\propto}\mathfrak{G}(x)\Big|_{\kappa}^{\sigma}\end{aligned} \tag{2.6}$$

ii)

$$\int_{\kappa}^{\sigma} \mathfrak{F}(x) D_{\sigma,-}^{\propto} \mathfrak{G}(x)^{c} D_{\kappa,+}^{\propto} k(x) dx$$
$$= \int_{\kappa}^{\sigma} \mathfrak{G}(x)^{c} D_{\kappa,+}^{\propto} \mathfrak{F}(x)^{c} D_{\kappa,+}^{\propto} k(x) dx$$
$$- \mathfrak{F}(x) I_{\sigma,-}^{1-\propto} \mathfrak{G}(x)^{c} D_{\kappa,+}^{\propto} k(x) \Big|_{\kappa}^{\sigma} \tag{2.7}$$

iii)

$$\int_{\kappa}^{\sigma} \mathfrak{F}(x) D_{\kappa,+}^{\propto} \mathfrak{G}(x) dx$$
$$= \int_{\kappa}^{\sigma} \mathfrak{G}(x)^{c} D_{\sigma,-}^{\propto} \mathfrak{F}(x) dx + \mathfrak{F}(x) I_{\kappa,+}^{1-\propto} \mathfrak{G}(x) \Big|_{\kappa}^{\sigma} \tag{2.8}$$

Property 2

Assume that $\propto \in (\kappa, \sigma), \beta > \propto$, and $\mathfrak{F} \in [\kappa, \sigma]$ [20].
Then the relation:-

$$D_{\kappa,+}^{\propto} I_{\kappa,+}^{\propto} \mathfrak{F}(x) = \mathfrak{F}(x)$$
$$D_{\sigma,-}^{\propto} I_{\kappa,-}^{\propto} \mathfrak{F}(x) = \mathfrak{F}(x)$$
$$D_{\kappa,+}^{\propto} I_{\kappa,+}^{\beta} \mathfrak{F}(x) = I_{\kappa,+}^{\beta-\kappa} \mathfrak{F}(x) \tag{2.9}$$
$$D_{\sigma,-}^{\propto} I_{\sigma,-}^{\beta} \mathfrak{F}(x) = I_{\sigma,-}^{\beta-\kappa} \mathfrak{F}(x)$$
$${}^{\mathrm{c}}D_{\kappa,+}^{\propto} I_{\kappa,+}^{\propto} \mathfrak{F}(x) = \mathfrak{F}(x)$$
$${}^{\mathrm{c}}D_{\sigma,-}^{\propto} I_{\sigma,-}^{\propto} \mathfrak{F}(x) = \mathfrak{F}(x)$$

Hold for any $x \in [\kappa, \sigma]$. The internal operations define in (2.3) satisfy the following semigroup properties:

$$I_{\kappa,+}^{\propto} I_{\kappa,+}^{\beta} = I_{\kappa,+}^{\propto+\beta} \quad , \quad I_{\sigma,-}^{\propto} I_{\sigma,-}^{\beta} = I_{\sigma,-}^{\propto+\beta}$$

3. Main Result

To formulate approximate To study the properties of eigenfunctions and eigenvalues in the classical Sturm-Liouville theory, we use the integration by parts formulas (2.7)-(2.8) based on first-order derivatives. In the extended theory, left and right fractional derivatives exist simultaneously.

Definition 3.1 *Let $\propto \in (\kappa, \sigma)$, the fractional Bessel operator is expressed as follows:* $\int_{\propto[B]} =$

$$D_{\sigma,-}^{\propto} p(x)^{c} D_{\kappa,+}^{\propto} + \left(q(x) - \frac{v^2 - 1/4}{w^2} \right) \tag{3.1}$$

Considering the fractional Resell equation

$$\int_{\propto[B]} \psi(x) + w_{\propto}(x) \psi(x) = o \tag{3.2}$$

The boundary conditions of the operator are as follows:

$$d_{11} \psi(\kappa) + d_{12} I_{b,-}^{1-\propto} p(\kappa)^{c} D_{\kappa,+}^{\propto} \psi(\kappa) = o \tag{3.3}$$
$$d_{21} \psi(\sigma) + d_{22} I_{\sigma,-}^{1-\kappa} p(\sigma)^{c} D_{\kappa,+}^{\kappa} \psi(\sigma) = o$$

Where $d_{11}^2 + d_{12}^2 \neq c, d_{21}^2 + d_{22}^2 \neq c$

The fractional boundary-value problem (3.2)-(3.3) constitutes a fractional SturmLiouville problem for the Bessel differential operator, generalizing the classical Bessel eigenvalue problem to fractional calculus

Theorem 3.2 *Fractional Bessel operator* $\int_{\propto[B]}$ *is self-adjoint on* $[\kappa,\sigma]$

Proof: Let us consider the following equation:-

$$\begin{aligned}
<\int_{\propto[B]}\mathfrak{F},\mathfrak{G}> &= \int_{\kappa}^{\sigma}\int_{\propto[B]}\mathfrak{F}(x)\mathfrak{G}(x)dx \\
&= \int_{\kappa}^{\sigma}\mathfrak{G}(x)\left[D_{,-}^{\propto}p(x)^{c}D_{a,+}^{\propto}\mathfrak{F}(x)\sigma\right. \\
&+ \left.\left(q(x)-\frac{v^{2}-1/4}{x^{2}}\right)\mathfrak{F}(x)\right]dx \\
&= \int_{\kappa}^{\sigma}\mathfrak{G}(x)D_{\sigma,-}^{\propto}P(x)^{c}D_{a,+}^{\propto}\mathfrak{F}(x)dx \\
&+ \int_{\kappa}^{\sigma}\left(q(x)-\frac{v^{2}-1/4}{x^{2}}\mathfrak{F}(x,\mathfrak{G},\propto,dx)\right.
\end{aligned} \tag{3.4}$$

Utilizing equality (2.7) and boundary condition (3.3), we obtain the identity.

$$\begin{aligned}
<\int_{\propto[B]}\mathfrak{F},\mathfrak{G}> &= \int_{\kappa}^{\sigma}p(x)^{c}D_{\kappa,+}^{\propto}\mathfrak{G}(x)^{c}D_{\kappa,+}^{\propto}\mathfrak{F}(x)dx \\
&- g(x)I_{\sigma,-}^{1-\propto}p(x)^{c}D_{\kappa,+}^{\propto}\mathfrak{F}(x)\Big|_{\kappa}^{\sigma} \\
&+ \int_{\kappa}^{\sigma}\left(q(x)-\frac{v^{2}-1/4}{x^{2}}\right)\mathfrak{F}(x)\mathfrak{G}(x)dx \\
&\int_{\kappa}^{\sigma}p(x)^{c}D_{\kappa,+}^{\propto}\mathfrak{G}(x)^{c}D_{\kappa,+}^{\propto}\mathfrak{F}(x),dx \\
&- \mathfrak{G}(b)I_{\sigma,-}^{1-\propto}p(\sigma)^{c}D_{\kappa,+}^{\propto}\mathfrak{F}(\sigma) \\
&+ \mathfrak{G}(\kappa)I_{\sigma,-}^{1-\propto}p(\kappa)^{c}D_{\kappa,+}^{\propto}\mathfrak{F}(\kappa) \\
&+ \int_{\kappa}^{\sigma}\left(q(x)-\frac{v^{2}-1/4}{x^{2}}\right)\mathfrak{F}(x)\mathfrak{G}(x)dx
\end{aligned} \tag{3.5}$$

From the boundary contention (3.3) we get:

$$I_{\sigma,-}^{1-\propto}p(\kappa)^{c}D_{\kappa,+}^{\propto}\mathfrak{F}(a) = -\frac{d_{11}}{d_{12}}\mathfrak{F}(\kappa) \tag{3.6}$$

$$I_{\sigma,-}^{1-\propto}p(\kappa)^{c}D_{\kappa,+}^{\propto}\mathfrak{F}(\kappa) = -\frac{d_{21}}{d_{22}}\mathfrak{F}(\kappa) \tag{3.7}$$

We make up the two equations (3.6),(3.7) in equation (3.5), we get

$$\begin{aligned}
<\int_{\propto[B]}\mathfrak{F},\mathfrak{G}> =& \int_{\kappa}^{\sigma}p(x)^{c}D_{\kappa,+}^{\propto}\mathfrak{G}(x)^{c}D_{\kappa,+}^{\propto}\mathfrak{F}(x)dx \\
&+ \frac{d_{21}}{d_{22}}\mathfrak{F},(\sigma)\mathfrak{G}(\sigma) \\
&+ \frac{d_{11}}{d_{12}}\mathfrak{F},(\kappa)\mathfrak{G}(\kappa) \\
&+ \int_{\kappa}^{\sigma}q(x)-\frac{v^{2}-1/4}{x^{2}}\mathfrak{F}(x)\mathfrak{G}(x)dx
\end{aligned} \tag{3.8}$$

Similar operations applied to the dual system lead to

$$
\begin{aligned}
< \mathfrak{F}, \int_{\propto[B]}, \mathfrak{G} >= & \int_{\kappa}^{\sigma} p(x)^c D_{\kappa,+}^{\propto} \mathfrak{F}(x)^c D_{\kappa,+}^{\propto} \mathfrak{G}(x) dx \\
& + \frac{d_{21}}{d_{22}} \mathfrak{F}, (\sigma) \mathfrak{G}(\sigma) \\
& + \frac{d_{11}}{d_{12}} \mathfrak{F}, (\kappa) \mathfrak{G}(\kappa) \\
& + \int_{\kappa}^{\sigma} q(x) - \frac{v^2 - 1/4}{x^2} \mathfrak{F}(x) \mathfrak{G}(x) dx
\end{aligned} \tag{3.9}
$$

The identities on the right sides of (3.8) and (3.9) require that the left sides also be equal.

$$
< \int_{\propto[B]} \mathfrak{F}, \mathfrak{G} >=< \mathfrak{F}, \int_{\propto[B]} \mathfrak{G} >
$$

□

Theorem 3.3 *The eigenvalues of the fractional Bessel operators (3.2)-(3.3) are all real numbers.*

Proof: From formula (2.7), we can see that the following relationship exists:

$$
\begin{aligned}
\int_{\kappa}^{\sigma} \mathfrak{F}(x) \int_{\propto[B]} \mathfrak{G}(x) dx = \int_{\kappa}^{\sigma} p(x) = & \int_{c}^{\sigma} p(x)^c D_{\kappa,+}^{\propto} \mathfrak{F}(x)^c D_{\kappa,+}^{\propto} \mathfrak{G}(c) dx \\
& - \mathfrak{F}(x) I_{\sigma,-}^{1-\propto} p(x) D_{\kappa,+}^{\propto} \mathfrak{G}(x) \Big|_{\kappa}^{\sigma} \\
& + \int_{\kappa}^{\sigma} q(x) - \frac{v^2 - 1/4}{x^2} \mathfrak{G}(x) \mathfrak{F}(x) dx
\end{aligned} \tag{3.10}
$$

Let σ be the eigenvalue corresponding to the eigenfunction ψ in (3.2)-(3.3).

$$
\int_{\propto[B]} \psi(x) + \sigma w_{\propto}(x) \psi(x) = o \tag{3.11}
$$

$$
d_{11} \psi(\kappa) + d_{12} I_{\sigma,-}^{1-\propto} p(\kappa) D_{\kappa,+}^{\propto} \psi(\kappa) = o
$$

$$
d_{21} \psi(\sigma) + d_{22} I_{\sigma,-}^{1-\propto} p(\sigma) D_{\kappa,+}^{\propto} \psi(\sigma) = o \tag{3.12}
$$

$$
\int_{\propto[B]} \bar{\psi}(x) + \bar{\sigma} w_{\propto} \bar{\psi}(x) = o \tag{3.13}
$$

$$
d_{11} \bar{\psi}(\kappa) + d_{22} I_{\sigma,-}^{1-\propto} p(\kappa) D_{\kappa,+}^{\propto} \bar{\psi}(x) = o \tag{3.14}
$$

$d_{22} \bar{\psi}(\sigma) + d_{22} I_{\sigma,-}^{1-\propto} p(\sigma) D_{\kappa,+}^{\propto} \bar{\psi}(\sigma) = o$
where $d_{11}^2 + d_{21}^2 \neq d_{21}^2 + d_{22}^2 \neq 0$.

$$
(\sigma - \bar{\sigma}) w_{\propto}(x) \psi(x) \bar{\psi}(x) = \psi(x) \int_{\propto[B]} \bar{\psi}(x) - \bar{\psi}(x) \int_{\propto[B]} \psi(x) = o \tag{3.15}
$$

Integration across $[\kappa, \sigma]$ under identity (3.10) yields a right-hand side comprising only boundary terms:

$$
\begin{aligned}
&(\sigma-\bar{\sigma}) \int_{\kappa}^{\sigma} w_{\propto}(x) \psi(x) \bar{\psi}(x) d x \\
&\quad=\int_{\kappa}^{\sigma} \psi(x) \int_{\propto[B]} \bar{\psi}(x) d x-\int_{\kappa}^{\sigma} \bar{\psi}(x) \int_{\propto[B]} \psi(x) d x \\
&=\int_{\kappa}^{\sigma} \psi(x)\left[D_{\sigma,-}^{\propto} p(x)^{c} D_{\kappa,+}^{\propto} \bar{\psi}(x)\right. \\
&\quad\left.+\left(q(x)-\frac{v^{2}-1/4}{x^{2}}\right) \bar{\psi}(x)\right] d x \\
&\quad-\int_{\kappa}^{\sigma} \bar{\psi}(x)\left[D_{\sigma,-}^{\propto} p(x)^{c} D_{\kappa,+}^{\propto} \psi(x)\right. \\
&\quad\left.+\left(q(x)-\frac{v^{2}-1/4}{x^{2}}\right) \bar{\psi}(x)\right] d x \\
&=-\psi(b) I_{\sigma,-}^{1-\propto} p(1)^{c} D_{\kappa,+}^{\propto} \bar{\psi}(\sigma) \\
&+\psi(\kappa) I_{\sigma,-}^{1-\propto} p(\kappa)^{c} D_{\kappa,+}^{\propto} \bar{\psi}(\kappa) \\
&+\bar{\psi}(\sigma) I_{\sigma,-}^{1-\propto} p(\sigma)^{c} D_{\kappa,+}^{\propto} \psi(\sigma) \\
&-\bar{\psi}(\kappa) I_{\sigma,-}^{1-\propto} p(\kappa)^{c} D_{\kappa,+}^{\propto} \psi(\kappa)
\end{aligned} \tag{3.16}
$$

From the boundary conditions (3.12), (3.14) we get
$I_{\sigma,-}^{1-\propto} p(\kappa)^{c} D_{\kappa,+}^{\propto} \psi(\kappa)=-\frac{d_{11}}{d_{12}} \psi(\kappa)$

$$
I_{\sigma,-}^{1-\propto} p(\sigma)^{c} D_{\kappa,+}^{\propto} \psi(\sigma)=-\frac{d_{21}}{d_{22}} \psi(\sigma) \tag{3.17}
$$

$I_{b,-}^{1-\propto} p(a)^{c} D_{a,+}^{\propto} \bar{\psi}(a)=-\frac{d_{11}}{d_{12}} \bar{\psi}(a)$
$I_{\sigma,-}^{1-\propto} p(\sigma)^{c} D_{\kappa,+}^{\propto} \bar{\psi}(\sigma)=-\frac{d_{21}}{d_{22}} \bar{\psi}(\sigma)$
We make up the equation (3.17) in equation (3.16) we find

$$
(\sigma-\bar{\sigma}) \int_{\kappa}^{\sigma} w_{\propto}(x)|\psi(x)|^{2} d x=o \tag{3.18}
$$

□

Theorem 3.4 *For the fractional Bessel Sturm-Liouville problem* (3.2)-(3.3)*, eigenfunctions associated with different eigenvalues exhibit orthogonality with respect to the weight function* w(x)*. On* (κ, σ) *: that is:*

$$
\int_{\kappa}^{\sigma} w_{\propto}(x) \psi_{\sigma_{1}}(x) \psi_{\sigma_{2}}(x) d x=o, \sigma_{1} \neq \sigma_{2} \tag{3.19}
$$

Proof: By assumption, we have a Bessel-type fractional-order Sturm-Liouville operator with two distinct eigenvalues (σ_1, σ_2) and corresponding eigenfunctions satisfying $(\psi_{\sigma_1}, \psi_{\sigma_2})$:

$$
\int_{\propto[B]} \psi_{\sigma_{1}}(x)+\sigma_{1} w_{\propto}(x) \psi_{\sigma_{1}}=o \tag{3.20}
$$

$$
d_{11} \psi_{\sigma_{1}}(\kappa)+d_{12} I_{b,-}^{1-\propto} p(\kappa) D_{\kappa,+}^{\propto} \psi_{\sigma_{1}}(\kappa)=o \tag{3.21}
$$

$$
d_{21} \psi_{\sigma_{1}}(\sigma)+d_{22} I_{\sigma,-}^{1-\propto} p(\sigma) D_{\kappa,+}^{\propto} \psi_{\sigma_{1}}(\sigma)=o
$$

$$
\int_{\propto[B]} \psi_{\sigma_{2}}(x)+\sigma_{2} w_{\propto}(x) \psi_{\sigma_{2}}=o \tag{3.22}
$$

$$d_{11}\psi_{\sigma_2}(\kappa) + d_{12}I^{1-\propto}_{\sigma,-}p(\kappa)D^{\propto}_{\kappa,+}\psi_{\sigma_2}(\kappa) = o \tag{3.23}$$

$d_{21}\psi_{\sigma_2}(\sigma) + d_{22}I^{1-\propto}_{\sigma,-}p(\sigma)D^{\propto}_{\kappa,+}\psi_{\sigma_2}(\sigma) = o$
We multiply (3.20) by the function ψ_{σ_2} and (3.22) by ψ_{σ_1}, respectively, and subtract:

$$(\sigma_1 - \sigma_2)\, w_{\propto}(x)\psi_{\sigma_1}\psi_{\sigma_2} = \psi_{\sigma_1}\int_{\propto[B]}\psi_{\sigma_2} - \psi_{\sigma_2}\int_{\propto[B]}\psi_{\sigma_1} \tag{3.24}$$

After integration over $[\kappa, 1][\kappa, 1]$ and application of identity (3.10), the right-hand side simplifies to boundary contributions at x = ax = a and x = 1x = 1 :

$$\begin{aligned}
&(\sigma_1 - \sigma_2)\int_{\kappa}^{\sigma} w_{\propto}(x)\psi_{\sigma_1}(x)\psi_{\sigma_2}(x)\\
&= \int_{\kappa}^{\sigma}\psi_{\sigma_1}(x)\int_{\propto[B]}\psi_{\sigma_2}(x)dx - \int_{\kappa}^{\sigma}\psi_{\sigma_2}(x)\int_{\propto[B]}\psi_{\sigma_1}(x)dx\\
&= \int_{\kappa}^{\sigma}\psi_{\sigma_1}(x)\left[D^{\propto}_{\sigma,-}p(x)^{c}D^{\propto}_{\kappa,+}\psi_{\sigma_2}(x)\right.\\
&+\left.\left(q(x) - \frac{v^2 - 1/4}{x^2}\right)\psi_{\sigma_2}(x)\right]dx\\
&- \int_{\kappa}^{\sigma}\psi_{\sigma_2}(x)\left[D^{\propto}_{\sigma,-}p(x)D^{\propto}_{\kappa,+}\psi_{\sigma_1}(x)\right.\\
&\quad+\left.\left(q(x) - \frac{v^2 - 1/4}{x^2}\right)\psi_{\sigma_1}(x)\right]dx\\
&= -\psi_{\sigma_1}(\sigma)I^{1-\propto}_{\sigma,-}p(\sigma)^{c}D^{\propto}_{\kappa,+}\psi_{\sigma_1}(\sigma)\\
&= -\psi_{\sigma_1}(\kappa)I^{1-\propto}_{\sigma,-}p(\kappa)^{c}D^{\propto}_{\kappa,+}\psi_{\sigma_1}(\kappa)\\
&= -\psi_{\sigma_2}(\sigma)I^{1-\propto}_{\sigma,-}p(\sigma)^{c}D^{\propto}_{\kappa,+}\psi_{\sigma_2}(\sigma)\\
&= -\psi_{\sigma_2}(\kappa)I^{1-\propto}_{\sigma,-}p(\kappa)^{c}D^{\propto}_{\kappa,+}\psi_{\sigma_2}(\kappa)
\end{aligned} \tag{3.25}$$

From the boundary conditions (3.21), (3.23) we get:

$$\begin{aligned}
I^{1-\propto}_{\sigma,-}p(\kappa)D^{\propto}_{\kappa,+}\psi_{\sigma_1}(\kappa) &= -\frac{d_{11}}{d_{12}}\psi_{\sigma_1}(\kappa)\\
I^{1-\propto}_{\sigma,-}p(\sigma)^{c}D^{\propto}_{\kappa,+}\psi_{\sigma_1}(\sigma) &= -\frac{d_{21}}{d_{22}}\psi_{\sigma_1}(\sigma)\\
I^{1-\propto}_{\sigma,-}p(\kappa)^{c}D^{\propto}_{\kappa,+}\psi_{\sigma_2}(\kappa) &= -\frac{d_{11}}{d_{12}}\psi_{\sigma_1}(\kappa)\\
I^{1-\propto}_{\sigma,-}p(\sigma)^{c}D^{\propto}_{\kappa,+}\psi_{\sigma_2}(\sigma) &= -\frac{d_{21}}{d_{22}}\psi_{\sigma_2}(\sigma)
\end{aligned} \tag{3.26}$$

We make up the equation (3.26) in the equation (3.25) we find:

$$(\sigma_1 - \sigma_2)\int_{\kappa}^{\sigma} w_{\propto}(x)\psi_{\sigma_1}(x)\psi_{\sigma_2}(x)dx = o \tag{3.27}$$

Because $\sigma_1 \neq \sigma_2$, it is easily seen that

$$\int_{\kappa}^{\sigma} w_{\propto}(x)\psi_{\sigma_1}(x)\psi_{\sigma_2}(x)dx = o$$

The eigenfunction is orthogonal. □

4. Conclusion

The research presented the fractional Bessel operator with general boundary conditions. Some spectral properties have been studied which related to eigenvalues and Eigen functions. And also proved that the fractional Bessel operator is self-adjoint.

References

1. R.S. Johnson, *An Introduction To sturm- Liouville Theory*, University of Newcastle, 2006.
2. A. Zettl, *Sturm- Liouville Theory, vol 121 of Mathematical surveys and Monographs* , American Mathematical Society, Providence RI, USA , 2005.
3. W.O. Amrein, A. M.Hinz, and D. B. Pearson, *Eds, Sturm- Liouville Theory*, past and present, Birkhauser, Basel, Switzerland, 2005.
4. E. S. Panakhov and R. Yilmazer, "*A Hochstadt-Lieberman theorem for the hydrogen atom equation*," Applied and Com putational Mathematics, vol. 11, no. 1, pp. 74-80, 2012.
5. B. M. Levitan and I. S. Sargsjan, *Introduction to Spectral Theory: Self adjoint Ordinary Differential Operators*, American Mathematical Society, Providence, RI, USA, 1975.
6. J. Qi and S. Chen, "*Eigenvalue problems of the model from non local continuum mechanics*," Journal of Mathematical Physics, vol. 52, no. 7, Article ID 073516, 2011.
7. Ibrahim, Rabha W., M. Z. Ahmad, and M. Jasim Mohammed. "*Periodicity and positivity of a class of fractional differential equations.*" SpringerPlus 5 (2016): 1-10.
8. E. S. Panakhov and M. Sat, "*Reconstruction of potential function for Sturm-Liouville operator with Coulomb potential*," Boundary Value Problems, vol. 2013, article 49, 2013.
9. A. Carpinteri and F. Mainardi, *Eds., Fractals and Fractional Calculus in Continum Mechanics*, Telos: Springer, 1998.
10. Ibrahim, Rabha W., M. Z. Ahmad, and M. Jasim Mohammed. "*Symmetric-periodic solutions for some types of generalized neutral equations.*" Mathematical Sciences 10 (2016): 219-226.
11. - B. J. West, M. Bologna, and P. Grigolini, *Physics of Fractal Operators*, Springer, New York, NY, USA, 2003.
12. I. Podlubny, *Fractional Differential Equations*, vol. 198, Aca demic Press, San Diego, Calif, USA, 1999.
13. R. Hilfer, Ed., *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
14. S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach*, Philadelphia, Pa, USA, 1993.
15. K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, NY, USA, 1993.
16. A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204, Elsevier, Amsterdam, The Netherlands, 2006.
17. R. Yilmazer and E. Bas, "*Fractional solutions of confluent hypergeometric equation*," Journal of the Chungcheong Mathe matical Society, vol. 25, no. 2, pp. 149-157, 2012.
18. X. Jiang and H. Qi, "*Thermal wave model of bioheat transfer with modified Riemann-Liouville fractional derivative*," Journal of Physics A, vol. 45, no. 48, Article ID 485101, 2012.
19. - X. Jiang and M. Xu, "*The time fractional heat conduction equation in the general orthogonal curvilinear coordinate and the cylindrical coordinate systems*," Physica A, vol. 389, no. 17, pp. 3368-3374, 2010.
20. E. Nakai and G. Sadasue, "*Martingale Morrey-Campanato spaces and fractional integrals*," Journal of Function Spaces and Applications, vol. 2012, Article ID 673929, 29 pages, 2012.
21. Y. Wang, L. Liu, and Y. Wu, "*Existence and uniqueness of a positive solution to singular fractional differential equations*," Boundary Value Problems, vol. 2012, article 81, 2012.
22. D. Baleanu and O. G. Mustafa, "*On the existence interval for the initial value problem of a fractional differential equation*," Hacettepe Journal of Mathematics and Statistics, vol. 40, no. 4, pp. 581-587, 2011.
23. M. Klimek, *On Solutions of Linear Fractional Differential Equations of a Variational Type*, The Publishing Office of Czestochowa, University of Technology, Czestochowa, Poland, 2009.
24. Q. M. Al-Mdallal, "*An efficient method for solving fractional SturmLiouville problems*," Chaas, Solitons and Fractals, vol. 40, no. 1, pp. 183189, 2009.
25. Luke, y. *Lintegrals of Bessel functions*, Me Graw-Hill, New yourk, 1962.
26. Mujeeb ur R. *Boundary value problem for fractional Differential Equations : Existence Theory and Numerical solutions*, Center for Advanced Math and physics, National unv. PhD , Thesis 2011.

M. Jasim Mohammed,
Department of Mathematics,
College of science, University Of Anbar, Ramadi,
Iraq.
E-mail address: mohadmath87@uoanbar.edu.iq

and

Mustafa Akram Saeed,
Department of Mathematics,
College of science, University Of Anbar, Ramadi,
Iraq.
E-mail address: mustafa.akram@uoanbar.edu.iq