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# Disconnected Captive Domination in Graphs

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ABSTRACT: Presents a novel domination in this paper paradigm on graphs known as disconnected captive domination. A disconnected captive dominating set is the appropriate subset in graph's vertices, if $D$ is a total dominating set, and every vertex of $D$ dominates at least one vertex from $V-D$, and subgraph $G[V-D]$ is disconnected. The disconnected captive domination number in $G$, represented by $\gamma_{dca}(G)$ means least cardinality over all disconnected captive dominating sets of $G$. Limits and characteristics of disconnected captive domination are examined in relation to a graph's order, size, minimum degree, and maximum degree. Lastly, disconnected captive domination in complement graphs is described, and disconnected captive dominating sets for a number of graphs were identified by examining their attributes and using the suggested model.

Key Words: Disconnected captive domination, minimum disconnected captive domination, dominating set, domination number and total domination number

## Contents



## 1. Introduction

Assume that $G = (V, E)$ has size $m = |E|$ and order $n = |V|$. $N(v) = \{uv, uv \in E$ defines $v$ open neighborhood, while $N[v] = N(v) \cup \{v\}\}$ defines its closed neighborhood. Subgraph in $G$ created vertices of $D$ is represented by the symbol $G[D]$ [16,27]. There is no appropriate subset that may be used as a dominating set in the minimal dominating set $D$ of $G$. The domination number $\gamma(G)$ is the cardinality of the minimum dominating set $D$ in $G$ [14,24]. Several types of domination were introduced due to the real life problems. Some types setting conditions on dominating set elements, such as [1,2,3,4,5,6,7,9,10,11,13,26,28,29], or on elements from $V-D$, such as [20] or on both as in [8]. Prior research has looked into the transformation of neighborhood topology obtained from undirected graphs, as well as the creation of topological graphs with numerous properties and new forms of discrete topological graphs [21,22,23].

Captive domination in graphs is a special type of dominating set where $G$ has a total dominating set and every vertex of $D$ dominates at least one vertex from $V-D$ [12,25]. Then the set is considered total dominating set, a set in which an isolated element cannot exist [15]. Total domination number in $G$, represented as $\gamma_t(G)$, cardinality of a minimum total dominating set in $G$. Due to their importance in many applications, numerous types of dominating models were developed based on the goal of domination [17,18,19].

Here, total domination existed accompanied by a new requirement. $DCAD$ is the new type of domination, every vertex in $D$ dominates at least one vertex from $V-D$, and subgraph $G[V-D]$ is disconnected.

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Additionally, a graph $G$ 's $DCAD$ number has many boundaries based on its order. In some graph families and for the complement of some specific graphs, the $DCAD$ number notion is established.

## 2. Definition and Properties

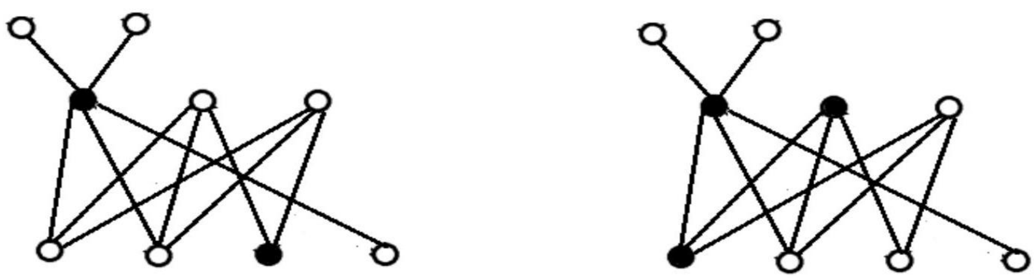
$DCAD$ is described in this section along with its limitations and characteristics. Any graph with this kind of domination has order, minimum degree, maximum degree, along with other features examined.

**Definition 2.1** *Consider $G = (V, E)$ be a simple, undirected, finite, nontrivial graph with no isolated vertices. If $G[D]$ had not an isolated vertices ( $D$ a total dominating set), and every vertex of $D$ is adjacent at least one vertex from $V - D$, and $G[V - D]$ is a disconnected subgraph, then $D \subseteq V(G)$ is a disconnected captive dominating set, and represented by DCADS. For example, see Fig 1.*

**Definition 2.2** *If there is no appropriate $DCAD$ subset, a disconnected captive dominating set $D$ of $G$ is minimum and is represented by MDCADS.*

**Definition 2.3** *The least among all minimal $DCAD$ in $G$ indicates that a minimal DCADS of $G$ is minimum.*

**Definition 2.4** *The $DCAD$ number in $G$, represented as $\gamma_{dc}{}^{-}$set, is the smallest cardinality among all $DCADS$ of $G$.*



(a) Minimum dominating set (b) *MDCADS*

Figure 1: The dominating set and $DCADS$.

*Two vertices in $D$ dominate every vertex of $V - D$ in $G$, in which $G[D]$ contains an isolated vertex, as shown in Fig. 1(a). Three vertices of $D$ dominate all other vertices in $V - D$ of $G$ in Fig. 1 (b), where $G[V - D]$ is a disconnected graph, and $G[D]$ had not an isolated vertex.*

**Observation 2.5** *For any graph $G = (n, m)$, there is the information that follows, with $DCADS$ and $DCAD$ number $\gamma_{dca}(G)$ :*

1. *$G$ has an order $n \geq 4$.*
2. $|D| \geq 2$.
3. $|V - D| \geq 2$.
4. $\delta(G) \geq 1, \Delta(G) \geq 2$.
5. $\deg(u) \geq 2 \quad \forall u \in D$.
6. *If $u \in D$, then $N(u) \cap D \neq \varnothing$ and $N(u) \cap V - D \neq \varnothing$.*
7. *Each support vertex belongs to $D$.*
8. $\gamma(G) \leq \gamma_t(G) \leq \gamma_{dca}(G)$.

**Observation 2.6**

1. *$G$ does not have a DCADS if $G$ it contains a component $P_2$, or $P_3$.*
2. *$\gamma_{dca}(G) \leq n - r$, where $r$ is the number of pendent vertices of graph $G$, indicates DCAD.*

**Theorem 2.7** *$G = (n, m)$ is any graph with DCADS and DCAD number $\gamma_{dca}(G)$ has the following boundaries:*

$$\left\lceil \frac{\gamma_{dca}(G)}{2} \right\rceil + (n - \gamma_{dca}(G)) \leq m \leq \binom{n-1}{2} + \gamma_{dca}(G).$$

**Proof:** The requirements of two situations that rely on the borders are demonstrated as follows, assuming that $D$ is a graph $G$ 's $\gamma_{dca}-$ set:

**Case 1.** In order to demonstrate that $\left\lceil \frac{\gamma_{dca}(G)}{2} \right\rceil + (n - \gamma_{dca}(G)) \leq m$, suppose that $G[V - D]$ is null graph. Since $G[D]$ had not an isolated vertices according to Definition 2.1, assume that $m_1$ represent number of edges of $G[D]$, where $m_1 = \left\lceil \frac{|D|}{2} \right\rceil$. $G$ has as few edges as feasible as a result. According to its definition, a graph with a $DCADS$ has every vertex in $V - D$ has at least one edge incident to it, where $m_2 = |V - D|$. Thus, $m_1 + m_2 = \left\lceil \frac{|D|}{2} \right\rceil + |V - D| = \left\lceil \frac{\gamma_{dca}(G)}{2} \right\rceil + (n - \gamma_{dca}(G))$ equals number of edges. Consequently, $m \geq \left\lceil \frac{\gamma_{dca}(G)}{2} \right\rceil + (n - \gamma_{dca}(G))$ in general.

**Case 2.** Let's assume that $G[D]$ is a complete subgraph with the highest number of edges and that $G[V - D]$ is a union of the complete subgraph and isolated vertex. When $G[D] = K_{t-1} \cup K_1$ so that $|E(G[V - D])| = m_2$, where $m_1$ and $m_2$ are the number of edges of $G[D]$ and $G[V - D]$, respectively. Thus,
$m_1 = \frac{|D||D-1|}{2} = \frac{\gamma_{dca}(\gamma_{dca}-1)}{2}$
$m_2 = \frac{|V-D-1||V-D-2|}{2} = \frac{(n-\gamma_{dca}-1)(n-\gamma_{dca}-2)}{2}$
Based on Definition $DCAD$ a maximum of $|V - D|$ edges connecting every vertex of $D$ to $V - D$, so that each vertex of $D$ dominates all vertices in $V - D$. After that, $|D||V - D| = \gamma_{dca}(n- \ \gamma_{dca}) = m_3$ is number of the edges connecting $D$ to $V - D$. Then, $m \leq m_1 + m_2 + m_3$ equals the number of edges in $G$.
$\leq \frac{\gamma_{dca}(\gamma_{dca}-1)}{2} + \frac{(n-\gamma_{dca}-1)(n-\gamma_{dca}-2)}{2} + n\gamma_{dca} - \gamma_{dca}^2$
$\leq \binom{n-1}{2} + \gamma_{dca}$.

This is the upper limit in general.

For $G = P_4$ where $\gamma_{dca}(P_4) = 2$ and $m = 3$ see Fig 3 (a), the lower limit is acute. While $G = F_3$ where $\gamma_{dca}(F_3) = 2$ and $m = 5$ see Fig 2, the upper limit is acute. □

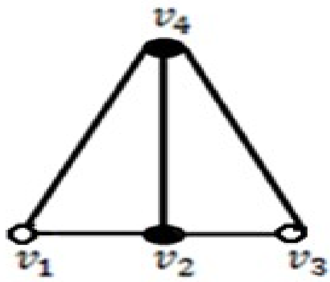



Figure 2: $\gamma_{dca}(F_3) = 2$

**Theorem 2.8** *DCAD number of any G be $\gamma_{dca}(G)$, we have* $2 \leq \gamma_{dca}(G) \leq \lceil \frac{2n}{3} \rceil$.

**Proof:** The conditions of two cases that rely on borders are demonstrated as below, assuming $D$ is of $G$.

**Case 1.** By $DCADS$ definition, the lower bound is true.

**Case 2.** When there are $k$ components in $G$, every which is triangle, the upper bound remains valid. Each component's $DCAD$ number is equal to two. This means that, $\gamma_{dca}(G) \leq \lceil \frac{2n}{3} \rceil$.

Where $\gamma_{dca}(B_{n,n}) = 2$, there will be a steep lower bound when $G = B_{n,n}$, see Fig 6. Where $\gamma_{dca}(C_6) = 4$, there will be a steep lower bound when $G = C_6$, see Fig 4 (c). □

## 3. Disconnected Captive Domination of Some Graphs

Some graphs, including the path, cycle, star, complete, barbell and double fan graphs, are investigated using the $DCAD$ model.

**Proposition 3.1** *Only when $n = 3, 5, 6,$ and 9 does $P_n$ have no DCAD.*

**Proof:** Assuming that $V(P_n) = \{v_1, v_2, \ldots, v_n\}$,

1. If $n = 3$, based on Observation 2.5 (5 and 6), then $P_3$ has no $DCAD$.
2. When $n = 5, 6$, according to Definition 2.1, and Observation 2.5, support vertices (apart from the pendant vertex) belong to $DCADS$, in addition their neighboring vertices. Although, a set is a total dominating set, it has no a $DCADS$.
3. If $n = 9$, the dominating set includes all support vertices, in addition their neighboring vertices, with the exception of the pendent vertices. This set does not dominate the $v_5$ vertex, as per the $DCAD$ definition. It is not possible to add the two vertices $\{v_4, v_6\}$ to $D$ since they need to be in $V - D$.

□

**Theorem 3.2** *Given path graph $P_n, (n \geq 4)$ and $n \neq 5, 6, 9, \gamma_{dca}(P_n) = 2\lceil \frac{n}{4} \rceil$.*

**Proof:** Assume that $v_1, v_2, \ldots, v_n$ the vertices in $P_n$, and assume that $D \subseteq V(P_n)$ so that:

$$D = \begin{cases} \{v_{2+4i}, v_{3+4i}, i = 0, 1, \ldots, \frac{n}{4} - 1\} & if\ n \equiv 0(\text{mod}4) \\ \{v_{2+4i}, v_{3+4i}, i = 0, 1, \ldots, \lceil \frac{n}{4} \rceil - 2\} \cup \{v_{n-1}, v_{n-2}\} & if\ n \equiv 3(\text{mod}4) \\ \{v_{2+4i}, v_{3+4i}, i = 0, 1, \ldots, \lceil \frac{n}{4} \rceil - 3\} \cup \{v_{n-1}, v_{n-2}, v_{n-4}, v_{n-5}\} & if\ n \equiv 2(\text{mod}4) \\ \{v_{2+4i}, v_{3+4i}, i = 0, 1, \ldots, \lceil \frac{n}{4} \rceil - 4\} \cup \{v_{n-1}, v_{n-2}, v_{n-4}, v_{n-5}, v_{n-7}, v_{n-8}\} & if\ n \equiv 1(\text{mod}4) \end{cases}$$

Four vertices are the most that might be disconnected captive dominated by two vertices. Therefore, from any four consecutive vertices, then can select the midpoint of the vertices. Thus, the following four cases exist.

**Case 1.** The $DCADS$ is clearly represented by vertices of the $D = \{v_{2+4i}, v_{3+4i}, i = 0, 1, \ldots, \frac{n}{4} - 1\}$ where $n \equiv 0(\text{mod}\,4)$. In this instance, set $D$ 's neighboring vertices all have maximum neighborhood. So, there isn't a $DCADS$ whose cardinality is smaller than $|D|$. Thus, $D$ is the $MDCADS$ and $G[V - D]$ is a disconnected graph. Consequently, $\gamma_{dca}(P_n) = \frac{n}{2}$.

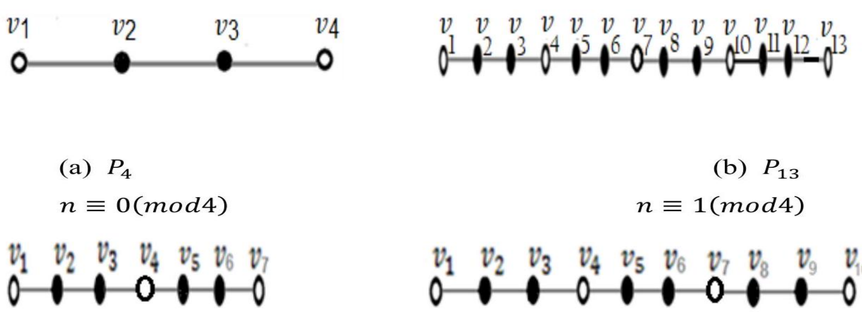
**Case 2.** Assume that $D_1 = \{v_{2+4i}, v_{3+4i}, i = 0, 1, \ldots, \lceil \frac{n}{4} \rceil - 2\}$, if $n \equiv 3(\text{mod}4)$. It's obvious that

$D_1$ is the $MDCADS$ to vertices $\{v_1, v_2, v_3, \ldots, v_{n-3}\}$ in the exact same way as in proof of Case 1. Thus, $\{v_{n-2}, v_{n-1}, v_n\}$ are the leftover vertices of $P_n$ that are not dominated by set $D_1$. The $DCADS$ condition is not met if the two vertices in $D_1$ are selected in the same way, which indicates that $\{v_{n-1}, v_n\}$ will dominate the three remaining vertices. Since $D = D_1 \cup \{v_{n-1}, v_n\}$, no vertex of $V - D$ is dominated by the $v_n$ vertex. Hence, $D = D_1 \cup \{v_{n-1}, v_{n-2}\}$, and $G[V - D]$ is a disconnected graph. Consequently, $\gamma_{dca}(P_n) = |D_1 \cup \{v_{n-2}, v_{n-1}\}| = 2\left\lceil \frac{n}{4} \right\rceil$.

**Case 3.** Assume that $D_2 = \left\{v_{2+4i}, v_{3+4i}, i = 0, 1, \ldots, \left\lceil \frac{n}{4} \right\rceil - 3\right\}$ when $n \equiv 2(\bmod 4)$. Once more, it is evident that is $D_2$ the $MDCADS$ to vertices $\{v_1, v_2, v_3, \ldots, v_{n-6}\}$ in the exact same way as we did of proof in Case 1. Set $D_2$ does not dominate the leftover vertices of $P_n$, which are $\{v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n\}$. The two vertices $\{v_{n-4}, v_{n-3}\}$ cannot be selected to dominate the vertices $\{v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}\}$. In Case 2 in the same way. The vertices $\{v_{n-1}, v_n\}$ must be part of the dominating set if they are added to $D_2$. Based on Observation 2.5, the vertex $v_n$ cannot be included to dominant set in Case 2 for the same reason. Thus, $D = D_1 \cup \{v_{n-1}, v_{n-2}\}$ and $G[V - D]$ is a disconnected graph, and $\gamma_{dca}(P_n) = |D_2 \cup \{v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}\}| = 2\left\lceil \frac{n}{4} \right\rceil$.

**Case 4.** Let $D_3 = \left\{v_{2+4i}, v_{3+4i}, i = 0, 1, \ldots, \left\lceil \frac{n}{4} \right\rceil - 4\right\}$, where $n \equiv 1(\text{mod}4)$. Once more, it is evident that set $D_3$ is the $MDCADS$ to vertices $\{v_1, v_2, v_3, \ldots, v_{n-9}\}$ in the exact same way as we did in demonstration of Case 1. Thus, $\{v_{n-8}, v_{n-7}, v_{n-6}, \ldots, v_n\}$ are the leftover vertices of $P_n$ that are not dominated by $D_3$. The vertex $v_n$ cannot be included to dominant set in Case 2 for the same reason. Consequently, $\gamma_{dca}(P_n) = |D_3 \cup \{v_{n-1}, v_{n-2}, v_{n-4}, v_{n-5}, v_{n-7}, v_{n-8}\}| = 2\left\lceil \frac{n}{4} \right\rceil$ and $G[V - D]$ is disconnected graph. From every instance mentioned above $\gamma_{dca}(P_n) = 2\left\lceil \frac{n}{4} \right\rceil$.

To demonstrate that $D$ is an $MDCADS$ in each of the earlier cases. Assume that $D'$ is a $DCADS$ in $G$ such that $|D'| < |D|$. This means that either there at least one vertex of $V - D$ does not dominated by any vertex from $D$ or $G[D']$ has an isolated vertex. Combining this contraction with the $DCADS$ notion. Thus, $D$ is the $MDCADS$ and $D'$ is not $DCADS$. For example, see Fig 3. □



(a) $P_4$ $n \equiv 0(mod4)$

(b) $P_{13}$ $n \equiv 1(mod4)$

(c) $P_7$ $n \equiv 3(mod4)$

(d) $P_{10}$ $n \equiv 2(mod4)$

Figure 3: $DCADS$ of $P_n$.

**Proposition 3.3** *Only when $n = 3, 4,$ and $5$ does $C_n$ have no $DCAD$.*

**Proof:** Suppose that $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ then,

1. When $n = 3, 4$, since there exist two vertices in $D$, then each vertex of $D$ dominates one vertex, but $G[D]$ is connected graph. Thus, $C_3$ and $C_4$ have no $DCAD$.
2. When $n = 5$, suppose that $V(C_5) = \{v_1, v_2, v_3, v_4, v_5\}$. Assume that $v_1$ and $v_2$ belong to dominating set $D$. Thus, $v_5$ and $v_3$ should belong to $V - D$ based on Definition 2.1. So, $G[D]$ has an isolated vertex if vertex $v_4$ belongs to set $D$. Based on Observation 2.5 (5) $D$ has no total set.

□

**Theorem 3.4** *Any cycle graph $C_n, (n \geq 6)$ and $n \neq 3,4,5, \gamma_{dca}(C_n) = 2\left\lceil \frac{n}{4} \right\rceil$.*

**Proof:** Given $C_n$ with vertices $v_1, v_2, \ldots, v_n$, let $D \subseteq V(C_n)$ so that:

$$D = \begin{cases} \left\{v_{2+4i}, v_{3+4i}, i = 0,1,\ldots, \left|\frac{n}{4}\right| - 1\right\} & if\ n \equiv 0,3(\text{mod}4) \\ \left\{v_{2+4i}, v_{3+4i}, i = 0,1,\ldots, \left|\frac{n}{4}\right| - 2\right\} \cup \{v_{n-1}, v_n\} & if\ n \equiv 2(\text{mod}4) \\ \left\{v_{2+4i}, v_{3+4i}, i = 0,1,\ldots, \left|\frac{n}{4}\right| - 3\right\} \cup \{v_{n-4}, v_{n-3}, v_{n-1}, v_n\} & if\ n \equiv 1(\text{mod}4) \end{cases}$$

The most number of vertices that can, as stated in Theorem 3.2. Therefore, from any four successive vertices, we can select the middle vertices. Thus, the following four situations exist.

**Case 1.** Let $D = \left\{v_{2+4i}, v_{3+4i}, i = 0,1,\ldots, \frac{n}{4} - 1\right\}$ when $n \equiv 0,3(\text{mod}4)$. In the exact same way of proof Theorem 3.2 (Case 1). $D$ is the $MDCADS$. Consequently, $\gamma_{dca}(C_n) = |D| = 2\left\lceil \frac{n}{4} \right\rceil$.

**Case 2.** Assume that $D_1 = \left\{v_{2+4i}, v_{3+4i}, i = 0,1,\ldots, \left\lceil \frac{n}{4} \right\rceil - 2\right\}$ if $n \equiv 2(mod4)$. It's obvious that set $D_1$ is $MDCADS$ to vertices $\{v_1, v_2, v_3, \ldots, v_{n-2}\}$ in the exact same way in proof of Case 1. Thus, $\{v_{n-1}, v_n\}$ are the leftover vertices in $C_n$ that are not dominated by $D_1$. In event that the two vertices are selected in the exact same way as of $D_1$, then $D = D_1 \cup \{v_{n-1}, v_n\}$ and $G[V - D]$ is disconnected graph. At that point, the $DCADS$ a requirement is met. Consequently, $\gamma_{dca}(C_n) = |D_1 \cup \{v_{n-1}, v_n\}| = 2\left\lceil \frac{n}{4} \right\rceil$.

**Case 3.** Assume that $D_2 = \left\{v_{2+4i}, v_{3+4i}, i = 0,1,\ldots, \left\lceil \frac{n}{4} \right\rceil - 3\right\}$ where $n \equiv 1(\text{mod}4)$. Once more, it is evident that $D_2$ is the $MDCADS$ to vertices $\{v_1, v_2, v_3, \ldots, v_{n-5}\}$ in the exact same way we did in demonstration of Case 1. Set $D_1$ does not dominate the leftover vertices from $C_n$, which are $\{v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n\}$. The vertex $v_n$ has no dominated by any vertex of this set if two vertices $\{v_{n-3}, v_{n-2}\}$ are selected to dominate the remaining vertices. Hence, $\gamma_{dca}(C_n) = | D_2 \cup \{v_{n-4}, v_{n-3}, v_{n-1}, v_n\} | = 2\left\lceil \frac{n}{4} \right\rceil$, and subgraph $G[V - D]$ is disconnected.

In three instances mentioned above, the set $D$ is an $MDCADS$, and the proof of this is comparable to that Theorem 3.2. For example, see Fig 4. □

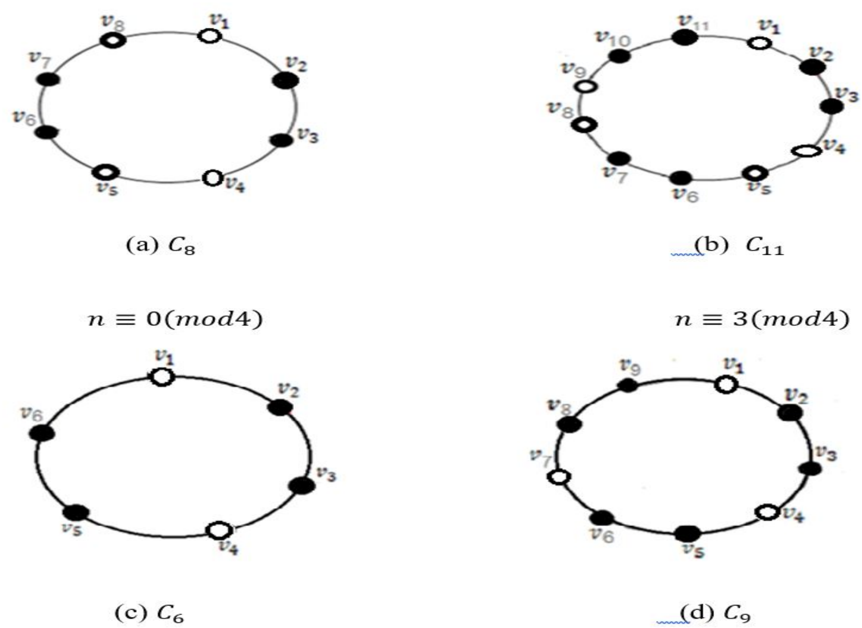



Figure 4: $MDCADS$ of $C_n$.

**Proposition 3.5** *There is no $DCAD$ in a star graph $S_n (n \geq 3)$.*

**Proof:** Based on definition bipartite graph known as star graph $K_{1,n}$. If $v_1 \in D$, it dominates $n \geq 3$ end vertices, where $v_1 \in S_n$ support vertex adjacent pendant vertices $u_1, u_2, \ldots, u_n$. However, the vertex of $G[D]$ is isolated. If $v_1 \notin D$, each vertex of $n \geq 3$ end vertices dominates only $v_1$. However, the graph in $G[D]$ has an isolated vertices, if $u_1, v_1 \in D$, since $u_1$ does not dominate any vertex in $V - D$. It is contradictory, $S_n$ does not have a $DCADS$ as a result. □

**Proposition 3.6** *There is no $DCADS$ in any complete graph $K_n$.*

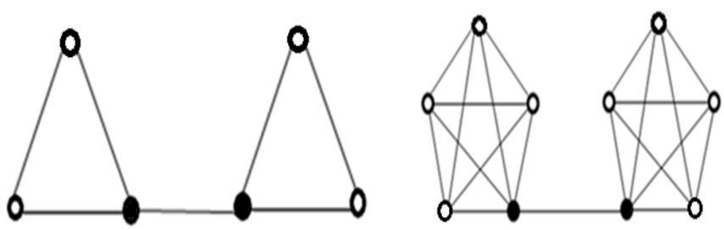
**Proof:** Every vertex in $DCADS$ can dominates one or more vertices, and since every vertex in $K_n$ is connected all other vertices. Accordingly, there is two vertices of $D$ dominates all vertices in $V - D$ of $K_n$. Then, $G[D]$ had not an isolated vertex, however $G[V - D]$ is connected graph. $K_n$ does not have a $DCADS$. For example, see Fig 5. □

$K_4$ $K_5$

Figure 5: $K_n$ has no $DCAD$.

**Proposition 3.7** *Any graph of the barbell $B_{n,n} (n \geq 3)$, has $DCAD$ and $\gamma_{dca}(B_{n,n}) = 2$.*

**Proof:** Considering that $K_n$ has no $DCAD$ according to Proposition 3.6 and that $B_{n,n}$ has two copies from $K_n$ connected by a bridge. Then, two adjacent vertices of $B_{n,n}$ must be present in $D$. So that each one vertex in $D$ dominates $n - 1$ of vertices in $K_n$ in each copy of the complete graph, and the graph $G[V - D]$ is disconnected. When two vertices that belong to $D$ must be the bridge's location. For example, see Fig 6. □



$B_{3,3}$ $B_{5,5}$

Figure 6: $MDCADS$ of $B_{n,n}$.

**Proposition 3.8** *Assuming that $G$ is the double fan graph $\left(P_n + \bar{k}_2\right)$, we have*
$$\gamma_{dca}\left(P_n + \bar{K}_2\right) = \begin{cases} 2 & \text{if } n = 2 \\ 3 & \text{if } n \geq 3 \end{cases}$$

**Proof:** If $n = 2$, let $D = \{v_1, v_2\} = V\left(P_2\right)$, because all of the vertices of $\bar{K}_2$ adjacent every vertex of $P_2$. After that, every vertex in $D$ dominates all vertices of $\bar{K}_2$ and $G[V - D]$ is disconnected graph. So $\gamma_{dca}\left(P_n + \bar{K}_2\right) = 2$.

If $n \geq 3$, assume that $D = V\left(\bar{K}_2\right)$ and one vertex of $\{v_2, v_3, \ldots, v_{n-1}\}$. After that, $D$ has three vertices that dominate all vertices of $V - D$, and subgraph $G[V - D]$ is disconnected. Thus, $\gamma_{dca}\left(P_n + \bar{K}_2\right) = 3$. For example, see Fig 7.

In three instances mentioned above, the set $D$ is an $MDCADS$, and the proof of this is comparable to that of Theorem 3.2. □

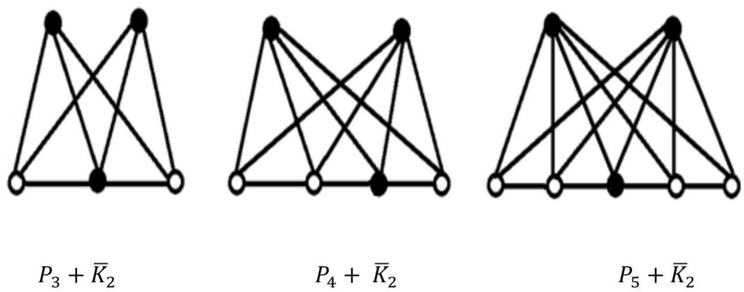



Figure 7: $MDCADS$ in $P_2 + \bar{K}_2$.

## 4. Disconnected Captive Domination for Complement Graphs

This section defines the $DCAD$ for several complement graphs include the complement path, cycle, wheel, complete, bipartite complete, fan, and double fan graphs.

**Observation 4.1** *Consider a graph $G$ where $\Delta(G) = n - 1$. After that, $\bar{G}$ has no DCADS.*

**Theorem 4.2** *Assume that $P_n$ is a path graph, after that $\bar{P}_n$ has $DCAD$ if and only if $n \geq 4$, then*
$$\gamma_{dca}\left(\bar{P}_n\right) = \begin{cases} 2 & \text{if } n = 4 \\ n - 3 & \text{if } n \geq 5 \end{cases}$$

**Proof:** If $n = 4$, let $D$ contains the two support vertices in $\bar{P}_4$, then each vertex of $D$ dominates one vertex of $V - D$ and $G[V - D]$ is disconnected graph. Thus, $\gamma_{dca}\left(\bar{P}_n\right) = 2$.

If $n \geq 5$, suppose that $D$ include all vertices except three consecutive vertices, where $v_1$ and $v_n$ belongs to $D$. Each vertex of $V - D$ adjacent one or more vertices from $D$ in $\bar{P}_n$, thus, each of them is dominated by at least one vertex from $D$. However, each vertex in $D$ adjacent at least two from three vertices in $V - D$.

In three instances mentioned above, the set $D$ is an $MDCADS$, and the proof of this is comparable to that of Theorem 3.2. For example, see Fig 8. □

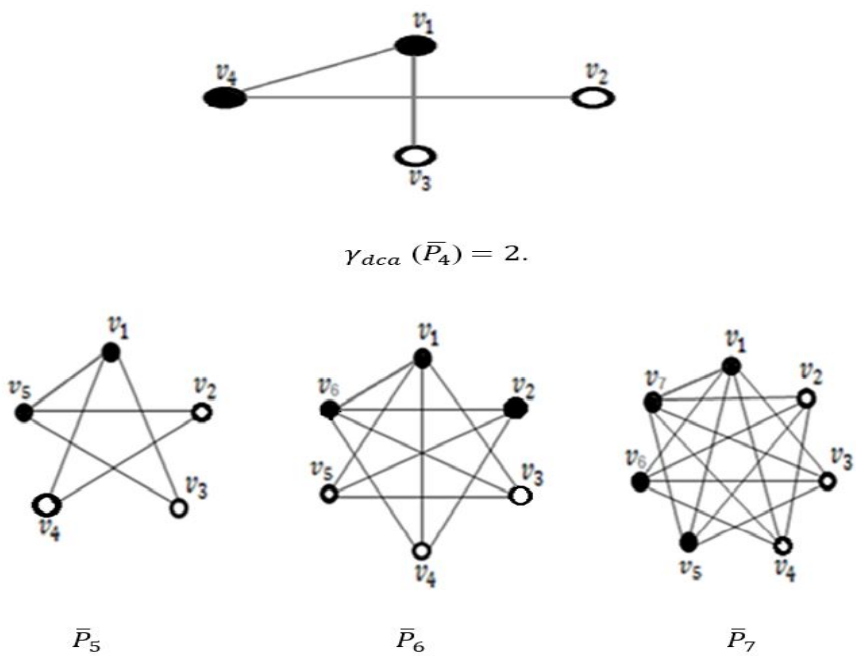



Figure 8: $DCAD$ in $\bar{P}_n$.

**Proposition 4.3** *Only when $n = 3, 4, 5$, then $\bar{C}_n$ has no $DCAD$.*

**Proof:** When $\bar{C}_3$ is null graph and $\bar{C}_4 = P_2 \cup P_2$, then $\bar{C}_3$ and $\bar{C}_4$ have no $DCAD$.

If $n = 5$, there exists three vertices of $D$ dominates two vertices in $V - D$, but the subgraph $G[V-D]$ is connected. Then, $\bar{C}_5$ has no $DCAD$. □

**Theorem 4.4** *Suppose that $C_n$ is a cycle graph in $n \geq 3$, after that $\bar{C}_n$ has DCAD if and only if $n \geq 6$, then* $\gamma_{dca}\left(\bar{C}_n\right) = \begin{cases} 4 & \text{if } n = 6 \\ n-3 & \text{if } n \geq 7 \end{cases}$

**Proof:** If $n = 6$, let $D$ contains all vertices except two non-adjacent vertices of $\bar{C}_6$. Each vertex of $V - D$ adjacent three vertices of $D$ in $\bar{C}_6$, thus each of them is dominated by three vertices of $D$. However, each vertex in $D$ adjacent one or two vertices from two vertices in $V - D$, and $G[V - D]$ is disconnected graph. Thus, $\gamma_{dca}\left(\bar{C}_6\right) = 4$.

If $n \geq 7$, let $D$ contains all vertices except three consecutive vertices. Every vertex in $V - D$ is adjacent three or more vertices from $D$ in $\bar{C}_n$, thus each of them is dominated by at least three vertices from $D$. However, each vertex in $D$ adjacent at least two from three vertices of $V - D$, and $G[V- D]$ is disconnected graph.

In three instances mentioned above, the set $D$ is an $MDCADS$, and the proof of this is comparable to that of Theorem 3.2. For example, see Fig 9. □

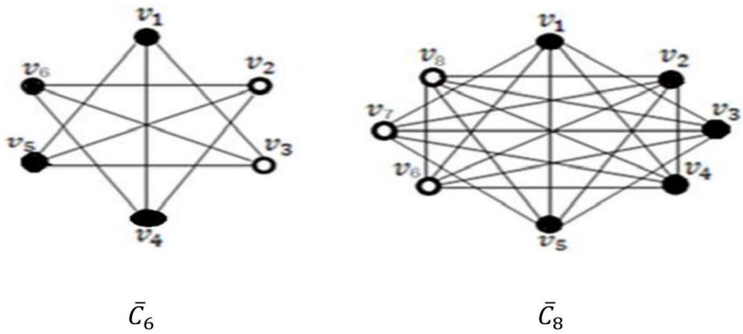



Figure 9: $MCADS$ in $\bar{C}_n$.

**Theorem 4.5** *For any $K_{n,m}$ a complete bipartite graph then $\bar{K}_{n,m}$ has $DCAD$ if and only if $n \geq 3$ and $m \geq 3$, we have $\gamma_{dca}\left(\bar{K}_{n,m}\right) = 4$.*

**Proof:** For $n, m \geq 3$, there are two graphs ($K_n$ and $K_m$) with no $DCAD$ by Proposition 3.6. However, $\bar{K}_{n,m}$ has two components. Then, two vertices of $K_n$ dominate all other vertices of $K_n$, and two vertices of $K_m$ dominate all other vertices of $K_m$. Therefore, $D$ contains four vertices from $\bar{K}_{n,m}$ and $G[V - D]$ is disconnected graph. So, $\gamma_{dca}\left(\bar{K}_{n,m}\right) = 4$.

To demonstrate that $D$ is a $MDCADS$. If $D'$ is a $DCADS$ of $G$ where $|D'| < |D|$, after that either one copy of $K_n$ or $K_m$ is not dominated by any vertex from $D'$, or $G\left[D'\right]$ has an isolated vertex. Thus, $D'$ is not a $DCADS$, but $D$ is $MDCADS$. For example, see Fig 10. □

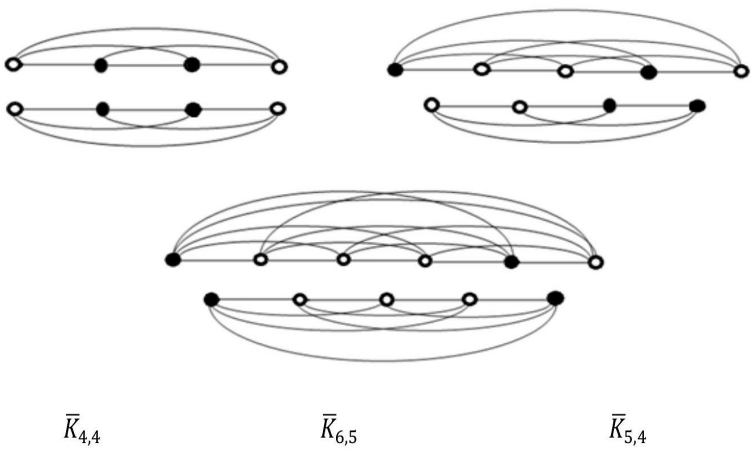



Figure 10: $MDCAD$ in $\bar{K}_{n,m}$.

**Observation 4.6**

1. *$\bar{K}_n$ has no $DCAD$ and $\bar{W}_n$ has no $DCAD$.*
2. *$\bar{F}_n$ has no $DCAD$ and $\bar{P}_n + \bar{K}_2$ has no $DCAD$.*

## 5. Conclusion

The term " $DCAD$ " refers to a new type of control. The $DCAD$ number is correlated with the graph's order, size, minimum degree, and maximum degree. This piece of labor generates a range from conventional and modified graphs that allow the domination number to be computed.

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