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Pythagorean Fuzzy Multigroups

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ABSTRACT: This paper introduces Pythagorean fuzzy multigroups as an extension of intuitionistic fuzzy multigroups, addressing limitations in uncertainty representation. We establish the theoretical framework for these algebraic structures, deriving fundamental properties including closure under group operations and characterizing intersection and union behaviors. Key results include necessary and sufficient conditions for Pythagorean fuzzy multigroup properties and comprehensive analysis of their algebraic structure. The developed theory provides enhanced tools for decision-making under uncertainty with multiple membership degrees

Key Words: Multiset, fuzzy multiset, multigroups, fuzzy multigroups, Pythagorean fuzzy multiset.

Contents

1. Introduction

Since its inception by Zadeh [18], fuzzy set theory has profoundly influenced numerous domains by providing mathematical tools to handle uncertainty and vagueness inherent in real-world phenomena. The limitations of classical crisp sets in modeling imprecise and ambiguous information have motivated researchers to develop various extensions of fuzzy sets. Among these extensions, Yager [16] introduced the concept of fuzzy multisets by generalizing the notion of multisets [13], which themselves extend ordinary sets by allowing repeated occurrences of elements within a collection. Consequently, within the fuzzy multiset framework, an element may possess multiple membership degrees, thereby enabling a more nuanced representation of uncertainty [7].

A significant advancement in fuzzy set theory emerged through the work of Atanassov [1,2], who introduced intuitionistic fuzzy sets (IFS) as a natural extension of classical fuzzy sets. The distinguishing feature of IFS lies in its ability to capture hesitation or indeterminacy in membership assessment by incorporating both a membership function μ and a non-membership function ν, subject to the constraint $\mu + \nu \leq 1$. This dual-function approach provides enhanced modeling capabilities for situations where complete certainty in membership assessment is unattainable. However, despite the theoretical elegance of IFS, practical applications frequently encounter scenarios where $\mu + \nu \geq 1$, thereby violating the fundamental constraint of intuitionistic fuzzy sets.

To address this inherent limitation, Yager [15] proposed Pythagorean fuzzy sets (PFS), which accommodate situations where the sum of membership and non-membership degrees may exceed unity by imposing the relaxed constraint $\mu^2 + \nu^2 \leq 1$. This generalization has significantly expanded the applicability of fuzzy set theory and has stimulated extensive research into algebraic applications of Pythagorean

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fuzzy sets [3,14,4].

The integration of multiset theory with intuitionistic fuzzy sets led to the development of intuitionistic fuzzy multisets (IFMS) by Shinoj and John [11]. This framework extends fuzzy multisets by incorporating count non-membership functions $CN = \{\nu^1, \ldots, \nu^n\}$ alongside the traditional count membership functions $CM = \{\mu^1, \ldots, \mu^n\}$. Subsequent investigations [5,6] have explored various operational aspects and modal operators within the IFMS framework, demonstrating remarkable versatility in addressing complex real-world problems [11].

The systematic study of algebraic structures—encompassing groups, rings, fields, and lattices—constitutes a fundamental area of abstract algebra. These structures consist of sets equipped with one or more finitely defined operations and serve as the mathematical foundation for numerous theoretical and applied investigations. Building upon the pioneering work of Shinoj et al. [10], who established algebraic structures on intuitionistic fuzzy multisets, the present investigation extends this theoretical framework to Pythagorean fuzzy multisets.

The primary motivation for this work stems from the need to develop algebraic structures that can effectively handle the enhanced uncertainty representation capabilities of Pythagorean fuzzy multisets. By introducing Pythagorean fuzzy multigroups, we aim to provide a robust theoretical foundation that combines the advantages of Pythagorean fuzzy sets with the multiplicative capabilities of multiset structures.

The main contributions of this paper are as follows:

1. We formally define Pythagorean fuzzy multigroups and establish their fundamental characterizations;
2. We derive essential algebraic properties including closure and inverse operations;
3. We investigate intersection and union operations on these structures;
4. We provide necessary and sufficient conditions for the multigroup property.

This theoretical development opens new avenues for applications in decision-making systems, pattern recognition, and uncertainty quantification where multiple membership assessments are required.

2. Preliminaries

In this section, we establish the fundamental concepts and definitions that form the theoretical foundation for our investigation. We begin with the basic notions of multisets and multigroups, then proceed to fuzzy extensions, ultimately leading to the framework necessary for defining Pythagorean fuzzy multigroups.

Definition 1 *[16] Let S be a set. A multiset (MS) Q drawn from S is represented by a function Count Q or C_Q defined as $C_Q : S \to \{0, 1, 2, 3...\}$.*
For each $s \in S$, $C_Q(s)$ is the characteristic value of s in Q. Here $C_Q(s)$ denotes the number of occurrences of s in Q.

Definition 2 *[8] Let S be a group. A multiset Q over S is a multigroup (MG) over S if the count of Q satisfies the following two conditions.*

1. $C_Q(s.u) \geq C_Q(s) \wedge C_Q(u) \quad \forall s, u \in S$.
2. $C_Q(s^{-1}) \geq C_Q(s) \quad \forall s \in S$.

Definition 3 *[18] If S is a collection of objects, then a Fuzzy Set (FS) Q in S is a set of ordered pairs: $Q = \{(s, \mu(s)) : s \in S, \mu : S \to [0, 1]\}$ where μ is called the membership function of Q, and is defined from S into $[0, 1]$.*

Definition 4 *[16] Assume S is a set of elements. Then, a Fuzzy bag / Multiset (FMS) Q drawn from S can be characterized by a count membership function CM_Q such that $CM_Q : S \to R$.*
where R is the set of all crisp bags or multisets from the unit interval $I = [0, 1]$.
A FMS can also be characterized by a high-order function. In particular, a fuzzy multiset Q can be characterized by a function

$$CM_Q : S \to [0, 1] \to N$$

where of course N is the set of natural numbers.
It follows that $CM_Q(s)$ for $s \in S$ is given as

$$CM_Q(s) = \{\mu_Q^1(s), \mu_Q^2(s), ..., \mu_Q^n(s), ...\}$$

where $\mu_Q^1(s), \mu_Q^2(s), ..., \mu_Q^n(s), ... \in [0, 1]$ such that $\mu_Q^1(s) \geq \mu_Q^2(s) \geq ... \geq \mu_Q^n(s) \geq ...$, whereas in a finite case, we write

$$CM_Q(s) = \{\mu_Q^1(s), \mu_Q^2(s), ..., \mu_Q^n(s)\}$$

for $\mu_Q^1(s) \geq \mu_Q^2(s) \geq ... \geq \mu_Q^n(s)$.
A Fuzzy Multiset Q can be represented in the form

$$Q = \{(s, CM_Q(s))/s \in S\}$$

Definition 5 *[12] Let $(S, \cdot)$ be a group. A fuzzy multiset Q over S is called a* fuzzy multigroup *(FMG) over S if the count membership function CM_Q satisfies the following conditions:*

1. *$CM_Q(s \cdot u) \geq CM_Q(s) \wedge CM_Q(u)$ for all $s, u \in S$;*
2. *$CM_Q(s^{-1}) = CM_Q(s)$ for all $s \in S$,*

3. Basic notions of Pythagorean Fuzzy Multisets

This section establishes the fundamental framework of Pythagorean fuzzy multisets (PFMS), which serves as the theoretical foundation for our subsequent development of Pythagorean fuzzy multigroups. We begin with the basic definition of Pythagorean fuzzy sets and systematically extend this concept to the multiset setting, followed by an analysis of their essential algebraic properties.

Definition 6 *[15] Let S be a crisp set. A Pythagorean Fuzzy Set (PFS) P in S is an object having the form $P = \{(s, \mu_P(s), \nu_P(s)) : s \in S\}$ where $\mu_p(s) \in [0, 1]$ and $\nu_P(s) \in [0, 1]$ are membership degree and non-membership degree of $s \in S$ respectively, which satisfy the condition $0 \leq \mu_P^2(s) + \nu_P^2(s) \leq 1$.*

Definition 7 *[4] Let X be a nonempty set. Then, an PFMS A drawn from X is of the form*

$$A = \left\{ \left\langle \frac{CM_A(x)}{x}, \frac{CN_A(x)}{x} \right\rangle \middle| \, x \in X \right\}$$

or

$$A = \{\langle x, CM_A(x), CN_A(x)\rangle \mid x \in X\}$$

where

$$CM_A(x) = \mu_A^1(x), \ldots, \mu_A^n(x)$$

and

$$CN_A(x) = \nu_A^1(x), \ldots, \nu_A^n(x)$$

are the count membership and count non-membership degrees defined by the functions

$$CM_A : X \to N^{[0,1]} \text{ and } CN_A : X \to N^{[0,1]}$$

such that $0 \leq [CM_A(x)]^2 + [CN_A(x)]^2 \leq 1$, *where* $N = \mathbb{N} \cup \{0\}$. *For each PFMS A of X,*

$$CH_A(x) = \sqrt{1 - [CM_A(x)]^2 - [CN_A(x)]^2}$$

is the count hesitation margin of x in A, where

$$CH_A(x) = \pi_A^1(x), \ldots, \pi_A^n$$

The count hesitation margin $CH_A(x)$ *is the degree of non-determinacy of* $x \in X$ *to A and* $CH_A(x) \in [0,1]$. *The count hesitation margin is the function that expresses lack of knowledge of whether* $x \in A$ *or* $x \notin A$. *Thus,*

$$[CM_A(x)]^2 + [CN_A(x)]^2 + [CH_A(x)]^2 = 1.$$

We denote the set of all PFMS over X by PFMS(X).

Example 1 *Let A be an PFMS of* $S = \{a_1, a_2, a_3\}$ *such that*

$$\begin{aligned}
&CM_A(a_1) = \{0.27, 0.12\}, \quad CN_A(a_1) = \{0.88, 0.56\}, \\
&CM_A(a_2) = \{0.53, 0.24, 0.11\}, \quad CN_A(a_2) = \{0.62, 0.56, 0.67\}, \\
&CM_A(a_3) = \{0.91, 0.57\}, \quad CN_A(a_3) = \{0.17, 0.67\}.
\end{aligned}$$

That is

$$A = \left\{ \frac{a_1}{(0.27, 0.12), (0.88, 0.56)}, \frac{a_2}{(0.53, 0.24, 0.11), (0.62, 0.56, 0.67)}, \frac{a_3}{(0.91, 0.57), (0.17, 0.67)} \right\}$$

Then

$$\begin{aligned}
CH_A(a_1) &= 0.391,\ 0.820 \\
CH_A(a_2) &= 0.578,\ 0.793,\ 0.734 \\
CH_A(a_3) &= 0.378,\ 0.475
\end{aligned}$$

Definition 8 *The cardinality of* $CM_A(a)$ *(or that of* $CN_A(a)$ *) in a PFM-set A is called length of the element* $a \in A$ *and is designated as* $L(a : A)$ *i.e.*

$$L(a : A) = \mid CM_A(a) \mid = \mid CN_A(a) \mid$$

where $\mid CM_A(a) \mid$ *denotes the cardinality of the membership sequence* $CM_A(a)$ *and* $\mid CN_A(a) \mid$ *that of non-membership sequence* $CN_A(a)$.

For example, in Example 1

$$\begin{aligned}
&L(a_1 : P) = 2, \\
&L(a_2 : P) = 3, \text{ and} \\
&L(a_3 : P) = 2
\end{aligned}$$

If A_1 and A_2 are two PFM-sets extracted from X, then

$$L(a : A_1, A_2) = \max \{L(a : A_1), L(a : A_2)\}.$$

For the sake of transience, we use $L(a)$ to mean $L(a : A_1, A_2)$.

Definition 9 *Two PFM-sets* A_1 *and* A_2 *drawn from a non-empty set X are said to be equivalent, written* $A_1 \sim A_2$, *if and only if* $L(a, A_1) = L(a, A_2)$.

The following fundamental operations on PFMS are adapted from [4] and provide the algebraic framework for subsequent developments.

1. **Inclusion:** $Q \subseteq G$ if and only if
$$CM_Q^p(s) \leq CM_G^p(s) \quad \text{and} \quad CN_Q^p(s) \geq CN_G^p(s) \quad \forall s \in S.$$
2. **Equality:** $Q = G$ if and only if
$$CM_Q(s) = CM_G(s) \quad \text{and} \quad CN_Q(s) = CN_G(s) \quad \forall s \in S.$$
3. **Complement:**
$$\overline{Q} = \{(s, CN_Q^p(s), CM_Q^p(s)) \mid s \in S\}.$$
4. **Intersection:**
$$Q \cap G = \{(s, CM_Q^p(s) \wedge CM_G^p(s), CN_Q^p(s) \vee CN_G^p(s)) \mid s \in S\}.$$
5. **Union:**
$$Q \cup G = \{(s, CM_Q^p(s) \vee CM_G^p(s), CN_Q^p(s) \wedge CN_G^p(s)) \mid s \in S\}.$$

Definition 10 *[4] Let $Q \in PFMS(S)$. Then Q^{-1} is defined as*
$$CM_{Q^{-1}}^p(s) = CM_Q^p(s^{-1}) \quad \text{and} \quad CN_{Q^{-1}}^p(s) = CN_Q^p(s^{-1})$$

Definition 11 *[4] Let $Q, G \in PFMS(S)$. Then define $Q \circ G$ as*
$$CM_{Q\circ G}^p(s) = \vee\{CM_Q^p(u) \wedge CM_G^p(t); u, t \in S \quad \text{and} \quad ut = s\}$$
$$CN_{Q\circ G}^p(s) = \wedge\{CN_Q^p(u) \vee CN_G^p(t); u, t \in S \quad \text{and} \quad ut = s\}$$

Proposition 1 *Let $Q, G, Q_i \in PFMS(S)$. Then the following properties hold:*

1. $(Q^{-1})^{-1} = Q$;
2. $Q \subseteq G \Rightarrow Q^{-1} \subseteq G^{-1}$;
3. $\left(\bigcup_{i=1}^n Q_i\right)^{-1} = \bigcup_{i=1}^n (Q_i^{-1})$;
4. $\left(\bigcap_{i=1}^n Q_i\right)^{-1} = \bigcap_{i=1}^n (Q_i^{-1})$;
5. $(Q \circ G)^{-1} = G^{-1} \circ Q^{-1}$.

Proof: We provide detailed proofs for each property:

1. For any $s \in S$:
$CM_{(Q^{-1})^{-1}}^p(s) = CM_{(Q^{-1})}^p(s^{-1}) = CM_Q^p((s^{-1})^{-1}) = CM_Q^p(s) \quad \forall s \in S.$

   and $CN_{(Q^{-1})^{-1}}^p(s) = CN_{(Q^{-1})}^p(s^{-1}) = CN_Q^p((s^{-1})^{-1}) = CN_Q^p(s) \quad \forall s \in S,$
   Therefore, $(Q^{-1})^{-1} = Q$.
2. Suppose $Q \subseteq G$. Then for any $s \in S$: $Q \subseteq G$ then $CM_{(Q^{-1})}^p(s) = CM_Q^p(s^{-1}) \leq CM_G^p(s^{-1}) = CM_{(G^{-1})}^p(s) \quad \forall s \in S$
   and $CN_{(Q^{-1})}^p(s) = CN_Q^p(s^{-1}) \geq CN_G^p(s^{-1}) = CN_{(G^{-1})}^p(s) \quad \forall s \in S,$
   Hence $Q^{-1} \subseteq G^{-1}$.

3. For the union property:

$$\begin{aligned} CM^p_{\left(\bigcup_{i=1}^n Q_i\right)^{-1}}(s) &= CM^p_{\left(\bigcup_{i=1}^n Q_i\right)}(s^{-1}) \\ &= \vee\{CM^p_{Q_i}(s^{-1}); i=1,...,n\} \\ &= \vee\{CM^p_{Q_i^{-1}}(s); i=1,...,n\} \\ &= CM^p_{\bigcup_{i=1}^n (Q_i^{-1})}(s). \end{aligned}$$

Similar reasoning applies to the non-membership function.

4. The intersection property follows by analogous arguments, replacing $\bigvee$ with $\bigwedge$ and vice versa for membership and non-membership functions.

5. For the composition property, detailed algebraic manipulation shows that:

$$\begin{aligned} CM^p_{(Q\circ G)^{-1}}(s) &= CM^p_{Q\circ G}(s^{-1}) \\ &= \vee\{CM^p_Q(u)\wedge CM^p_G(t); u,t\in S \quad \textit{and} \quad u.t=s^{-1}\} \\ &= \vee\{CM^p_G(t)\wedge CM^p_Q(u); u,t\in S \quad \textit{and} \quad (u.t)^{-1}=s\} \\ &= \vee\{CM^p_G(t^{-1})^{-1}\wedge CM^p_Q(u^{-1})^{-1}; u^{-1},t^{-1}\in S \quad \text{and} \quad t^{-1}.u^{-1}=s\} \\ &= \vee\{CM^p_{G^{-1}}(t^{-1})\wedge CM^p_{Q^{-1}}(u^{-1}); u^{-1},t^{-1}\in S \quad \text{and} \quad t^{-1}.u^{-1}=s\} \\ &= CM^p_{G^{-1}\circ Q^{-1}}(s) \quad \forall s\in S \end{aligned}$$

and

$$\begin{aligned} CN^p_{(Q\circ G)^{-1}}(s) &= CN^p_{Q\circ G}(s^{-1}) \\ &= \wedge\{CN^p_Q(u)\vee CN^p_G(t); u,t\in S \quad \text{and} \quad u.t=s^{-1}\} \\ &= \wedge\{CN^p_G(t)\vee CN^p_Q(u); u,t\in S \quad \text{and} \quad (u.t)^{-1}=s\} \\ &= \wedge\{CN^p_G(t^{-1})^{-1}\vee CN^p_Q(u^{-1})^{-1}; u^{-1},t^{-1}\in S \quad \text{and} \quad t^{-1}.u^{-1}=s\} \\ &= \wedge\{CN^p_{G^{-1}}(t^{-1})\vee CN^p_{Q^{-1}}(u^{-1}); u^{-1},t^{-1}\in S \quad \text{and} \quad t^{-1}.u^{-1}=s\} \\ &= CN^p_{G^{-1}\circ Q^{-1}}(s) \quad \forall s\in S. \end{aligned}$$

Hence $(Q\circ G)^{-1} = G^{-1}\circ Q^{-1}$.

□

4. Pythagorean Fuzzy Multigroups

In this section, we introduce the central concept of Pythagorean fuzzy multigroups (PFMG) and establish their fundamental algebraic properties. Throughout this section, we assume that $(S,\cdot)$ is a group with identity element e and binary operation $\cdot$.

Definition 12 *[17] Let $(S,\cdot)$ be a group and $Q=\{(s,\mu_p(s),\nu_p(s))\mid s\in S\}$ be a Pythagorean fuzzy set over S. Then Q is called a* Pythagorean fuzzy subgroup *(PFG) of S if the following conditions are satisfied:*

1. $\mu_p(s.u)\geq\mu_p(s)\wedge\mu_p(u)$ *and* $\nu_p(s.u)\leq\nu_p(s)\vee\mu_p(u)$ *for all* $s,u\in S$.
2. $\mu_p(s^{-1})\geq\mu_p(s)$ *and* $\nu_p(s^{-1})\leq\nu_p(s)$ *for all* $s\in S$.

Definition 13 *Let $(S,\cdot)$ be a group. A PFMS Q over S is called a* Pythagorean fuzzy multigroup *(PFMG) over S if the count membership and count non-membership functions satisfy the following conditions:*

1. $CM_Q^p(s.u) \geq CM_Q^p(s) \wedge CM_Q^p(u), \forall s, u \in S$ *and* $CN_Q^p(s.u) \leq CN_Q^p(s) \vee CN_Q^p(u), \forall s, u \in S.$
2. $CM_Q^p(s^{-1}) \geq CM_Q^p(s), \forall s \in S$ *and* $CN_Q^p(s^{-1}) \leq CN_Q^p(s), \forall s \in S.$

Proposition 2 *Let* $Q \in PFMS(S)$ *and* $CM_Q^p(s^{-1}) \geq CM_Q^p(s)$ *and* $CN_Q^p(s^{-1}) \leq CN_Q^p(s)$. *Then* $CM_Q^p(s^{-1}) = CM_Q^p(s)$ *and* $CN_Q^p(s^{-1}) = CN_Q^p(s)$.

Proof: From the given condition, we have $CM_Q^p(s^{-1}) \geq CM_Q^p(s)$. Additionally, since $CM_Q^p(s) = CM_Q^p((s^{-1})^{-1}) \geq CM_Q^p(s^{-1})$ by applying the condition to s^{-1}, we obtain $CM_Q^p(s^{-1}) = CM_Q^p(s)$.

Similarly, from $CN_Q^p(s^{-1}) \leq CN_Q^p(s)$ and $CN_Q^p(s) = CN_Q^p((s^{-1})^{-1}) \leq CN_Q^p(s^{-1})$, we conclude that $CN_Q^p(s^{-1}) = CN_Q^p(s)$. □

Example 2 *Consider the group* $G = \mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$ *under addition modulo 3. We define a PFMS A over G with the following count membership and count non-membership functions:*

$$CM_A(\overline{0}) = \{0.9, 0.7, 0.3\}, \quad CN_A(\overline{0}) = \{0.1, 0.3, 0.7\}, \tag{4.1}$$
$$CM_A(\overline{1}) = \{0.9, 0.6, 0.4\}, \quad CN_A(\overline{1}) = \{0.1, 0.4, 0.6\}, \tag{4.2}$$
$$CM_A(\overline{2}) = \{0.9, 0.6, 0.4\}, \quad CN_A(\overline{2}) = \{0.1, 0.4, 0.6\}. \tag{4.3}$$

We verify that A forms a PFMG by checking the required conditions:

For all $a, b \in G$,

$$CM_A(a+b) \geq CM_A(a) \wedge CM_A(b),$$
$$CN_A(a+b) \leq CN_A(a) \vee CN_A(b).$$

- ***Case 1:*** $\overline{0} + \overline{0} = \overline{0}$
  $CM_A(\overline{0}) = 0.9, 0.7, 0.3, \quad CM_A(\overline{0}) \wedge CM_A(\overline{0}) = 0.9, 0.7, 0.3.$
  Verified: $CM_A(\overline{0}) \geq 0.9, 0.7, 0.3.$
  Similarly, $CN_A(\overline{0}) = 0.1, 0.3, 0.7, \quad CN_A(\overline{0}) \vee CN_A(\overline{0}) = 0.1, 0.3, 0.7.$
  Verified: $CN_A(\overline{0}) \leq 0.1, 0.3, 0.7.$

- ***Case 2:*** $\overline{0} + \overline{1} = \overline{1}$
  $CM_A(\overline{1}) = 0.9, 0.6, 0.4, \quad CM_A(\overline{0}) \wedge CM_A(\overline{1}) = 0.9, 0.6, 0.3.$
  Verified: $CM_A(\overline{1}) \geq 0.9, 0.6, 0.3.$
  Similarly, $CN_A(\overline{1}) = 0.1, 0.4, 0.6, \quad CN_A(\overline{0}) \vee CN_A(\overline{1}) = 0.1, 0.4, 0.7.$
  Verified: $CN_A(\overline{1}) \leq 0.1, 0.4, 0.7.$

- ***Case 3:*** $\overline{1} + \overline{2} = \overline{0}$
  $CM_A(\overline{0}) = 0.9, 0.7, 0.3, \quad CM_A(\overline{1}) \wedge CM_A(\overline{2}) = 0.9, 0.6, 0.4.$
  Verified: $CM_A(\overline{0}) \geq 0.9, 0.6, 0.4.$
  Similarly, $CN_A(\overline{0}) = 0.1, 0.3, 0.7, \quad CN_A(\overline{1}) \vee CN_A(\overline{2}) = 0.1, 0.4, 0.6.$
  Verified: $CN_A(\overline{0}) \leq 0.1, 0.4, 0.6.$

For all $a \in G$, *verify:*

$$CM_A(-a) \geq CM_A(a), \quad CN_A(-a) \leq CN_A(a).$$

- ***Case 1:*** $-\overline{0} = \overline{0}$
  $CM_A(\overline{0}) = 0.9, 0.7, 0.3, \quad CN_A(\overline{0}) = 0.1, 0.3, 0.7.$
  Trivially satisfied.

- ***Case 2:*** $-\overline{1} = \overline{2}$
  $CM_A(\overline{2}) = 0.9, 0.6, 0.4, \quad CM_A(\overline{1}) = 0.9, 0.6, 0.4.$
  Verified: $CM_A(\overline{2}) \geq CM_A(\overline{1}).$
  Similarly, $CN_A(\overline{2}) = 0.1, 0.4, 0.6, \quad CN_A(\overline{1}) = 0.1, 0.4, 0.6.$
  Verified: $CN_A(\overline{2}) \leq CN_A(\overline{1}).$

- ***Case 3:*** $-\overline{2} = \overline{1}$
  $CM_A(\overline{1}) = 0.9, 0.6, 0.4, \quad CM_A(\overline{2}) = 0.9, 0.6, 0.4.$
  Verified: $CM_A(\overline{1}) \geq CM_A(\overline{2})$.
  Similarly, $CN_A(\overline{1}) = 0.1, 0.4, 0.6, \quad CN_A(\overline{2}) = 0.1, 0.4, 0.6.$
  Verified: $CN_A(\overline{1}) \leq CN_A(\overline{2})$.

Proposition 3 *Let $Q \in PFMG(S)$. Then the following properties hold:*

1. $CM_Q^p(e) \geq CM_Q^p(s) \quad \forall s \in S$
2. $CN_Q^p(e) \leq CN_Q^p(s) \quad \forall s \in S$
3. $CM_Q^p(s^n) \geq CM_Q^p(s) \quad \forall s \in S$
4. $CN_Q^p(s^n) \leq CN_Q^p(s) \quad \forall s \in S$
5. $Q = Q^{-1}$

Proof: Let $s \in S$. We establish each property:

1. Since $e = s \cdot s^{-1}$, we have

$$\begin{aligned} CM_Q^p(e) &= CM_Q^p(ss^{-1}) \\ &\geq CM_Q^p(s) \wedge CM_Q^p(s^{-1}) \\ &= CM_Q^p(s) \wedge CM_Q^p(s) \\ &= CM_Q^p(s) \end{aligned}$$

2. Similarly, $CN_Q^p(e) = CN_Q^p(s \cdot s^{-1}) \leq CN_Q^p(s) \vee CN_Q^p(s^{-1}) = CN_Q^p(s)$.

3. By induction on n:

$$\begin{aligned} CM_Q^p(s^n) &\geq CM_Q^p(s^{n-1}) \wedge CM_Q^p(s) \\ &\geq CM_Q^p(s) \wedge CM_Q^p(s) \wedge ... \wedge CM_Q^p(s) \quad \text{by recursion} \\ &= CM_Q^p(s) \end{aligned}$$

4. The proof for the non-membership function follows analogously.

5. From Definition 13 and Proposition 2, we have

$$CM_{Q^{-1}}^p(s) = CM_Q^p(s^{-1}) = CM_Q^p(s), \tag{4.4}$$
$$CN_{Q^{-1}}^p(s) = CN_Q^p(s^{-1}) = CN_Q^p(s). \tag{4.5}$$

   Therefore, $Q = Q^{-1}$.

□

Theorem 1 *Let $Q \in PFMS(S)$. Then $Q \in PFMG(S)$ if and only if*

$$CM_Q^p(s \cdot u^{-1}) \geq CM_Q^p(s) \wedge CM_Q^p(u) \quad \text{and} \quad CN_Q^p(s \cdot u^{-1}) \leq CN_Q^p(s) \vee CN_Q^p(u)$$

for all $s, u \in S$.

Proof: ($\Rightarrow$) Suppose $Q \in \text{PFMG}(S)$. Then for any $s, u \in S$:

$$\begin{aligned} CM_Q^p(s \cdot u^{-1}) &\geq CM_Q^p(s) \wedge CM_Q^p(u^{-1}) \\ &\geq CM_Q^p(s) \wedge CM_Q^p(u), \end{aligned}$$

where the last inequality follows from the inverse condition in Definition 13.

($\Leftarrow$) Conversely, assume the given condition holds. We need to verify the PFMG conditions:

For the inverse property, setting $u = e$ gives:

$$CM_Q^p(s^{-1}) = CM_Q^p(s \cdot e^{-1}) \geq CM_Q^p(s) \wedge CM_Q^p(e) = CM_Q^p(s).$$

For the closure property, we have:

$$CM_Q^p(s \cdot u) = CM_Q^p(s \cdot (u^{-1})^{-1}) \geq CM_Q^p(s) \wedge CM_Q^p(u^{-1}) \geq CM_Q^p(s) \wedge CM_Q^p(u).$$

The non-membership functions follow by analogous reasoning. □

Proposition 4 *Let $Q \in PFMS(S)$. Then $Q \in PFMG(S)$ if $Q \circ Q^{-1} \subseteq Q$.*

Proof:

Assume $Q \circ Q^{-1} \subseteq Q$. By Theorem 1, it suffices to show that $CM_Q^p(s.u^{-1}) \geq CM_Q^p(s) \wedge CM_Q^p(u)$ and $CN_Q^p(s.u^{-1}) \leq CN_Q^p(s) \vee CM_Q^p(u)$, for all $s, u \in S$.

From the definition of composition and the inclusion assumption:

$$\begin{aligned} CM_Q^p(s.u^{-1}) &\geq CM_{Q\circ Q^{-1}}^p(s.u^{-1}) \\ &= \vee_{t\in S}\{CM_Q^p(t) \wedge CM_{Q^{-1}}^p(t^{-1}.s.u^{-1})\} \\ &\geq \{CM_Q^p(s) \wedge CM_{Q^{-1}}^p(u^{-1})\}; t = s \\ &= CM_Q^p(s) \wedge CM_Q^p(u).and \end{aligned}$$

$$\begin{aligned} CN_Q^p(s.u^{-1}) &\leq CN_{Q\circ Q^{-1}}^p(s.u^{-1}) \\ &= \wedge_{t\in S}\{CN_Q^p(t) \vee CN_{Q^{-1}}^p(t^{-1}.s.u^{-1})\} \\ &\leq \{CN_Q^p(s) \vee CN_{Q^{-1}}^p(u^{-1})\}; t = s \\ &= CN_Q^p(s) \vee CN_Q^p(u) \end{aligned}$$

□

Theorem 2 *Let $Q, G \in PFMG(S)$. Then $Q \cap G \in PFMG(S)$.*

Proof: Let $s, u \in Q \cap G \in PFMS(S)$.
$\Rightarrow s, u \in Q$ and $s, u \in G$
$\Rightarrow CM_Q^p(s.u^{-1}) \geq CM_Q^p(s) \wedge CM_Q^p(u^{-1}), CM_G^p(s.u^{-1}) \geq CM_G^p(s) \wedge CM_G^p(u^{-1})$ and
$CN_Q^p(s.u^{-1}) \leq CN_Q^p(s) \vee CN_Q^p(u^{-1}), CN_G^p(s.u^{-1}) \leq CN_G^p(s) \vee CN_G^p(u^{-1})$
Now

$$\begin{aligned} CM_{Q\cap G}^p(s.u^{-1}) &= CM_Q^p(s.u^{-1}) \wedge CM_G^p(s.u^{-1}) \\ &\geq \{CM_Q^p(s) \wedge CM_Q^p(u^{-1})\} \wedge \{CM_G^p(s) \wedge CM_G^p(u^{-1})\} \\ &= \{CM_Q^p(s) \wedge CM_G^p(s)\} \wedge \{CM_Q^p(u^{-1}) \wedge CM_G^p(u^{-1})\} \\ &\geq \{CM_Q^p(s) \wedge CM_G^p(s)\} \wedge \{CM_Q^p(u) \wedge CM_G^p(u)\} \\ &= CM_{Q\cap G}^p(s) \wedge CM_{Q\cap G}^p(u) \end{aligned}$$

Then $CM^p_{Q\cap G}(s.u^{-1}) \geq CM^p_{Q\cap G}(s) \wedge CM^p_{Q\cap G}(u)$

and

$$\begin{aligned} CN^p_{Q\cap G}(s.u^{-1}) &= CN^p_Q(s.u^{-1}) \vee CN^p_G(s.u^{-1}) \\ &\leq \{CN^p_Q(s) \vee CN^p_Q(u^{-1})\} \vee \{CN^p_G(s) \vee CN^p_G(u^{-1})\} \\ &= \{CN^p_Q(s) \vee CN^p_G(s)\} \vee \{CN^p_Q(u^{-1}) \vee CN^p_G(u^{-1})\} \\ &\leq \{CN^p_Q(s) \vee CN^p_G(s)\} \vee \{CN^p_Q(u) \vee CN^p_G(u)\} \\ &= CN^p_{Q\cap G}(s) \vee CN^p_{Q\cap G}(u) \end{aligned}$$

Wich implies that $CN^p_{Q\cap G}(s.u^{-1}) \leq CN^p_{Q\cap G}(s) \vee CN^p_{Q\cap G}(u)$ Then $Q \cap G \in PFMG(S)$. □

Proposition 5 *Let $Q, G \in PFMG(S)$. Then:*

1. $CM^p_{Q\cup G}(s^{-1}) \geq CM^p_{Q\cup G}(s)$ *for all* $s \in S$;
2. $CN^p_{Q\cup G}(s^{-1}) \leq CN^p_{Q\cup G}(s)$ *for all* $s \in S$.

Proof: For any $s \in S$:

$$\begin{aligned} CM^p_{Q\cup G}(s^{-1}) &= \vee\{CM^p_Q(s^{-1}), CM^p_G(s^{-1})\} \\ &\geq \vee\{CM^p_Q(s), CM^p_G(s)\} \\ &= CM^p_{Q\cup G}(s) \end{aligned}$$

and

$$\begin{aligned} CN^p_{Q\cup G}(s^{-1}) &= \wedge\{CN^p_Q(s^{-1}), CN^p_G(s^{-1})\} \\ &\leq \wedge\{CN^p_Q(s), CN^p_G(s)\} \\ &= CN^p_{Q\cup G}(s). \end{aligned}$$

□

From this it is clear that, if $Q, G \in PFMG(S)$ then $Q \cup G \in PFMG(S)$ if and only if $CM^p_{Q\cup G}(s.u) \geq CM^p_{Q\cup G}(s) \wedge CM^p_{Q\cup G}(u)$ and $CN^p_{Q\cup G}(s.u) \leq CN^p_{Q\cup G}(s) \vee CN^p_{Q\cup G}(u)$.

5. Conclusion

This study has introduced the concept of Pythagorean Fuzzy Multigroup (PFMG), derived from the framework of Pythagorean Fuzzy Multisets (PFMS) and extending the established theory of Intuitionistic Fuzzy Multigroups. We have developed a coherent set of definitions, propositions, and theorems, supported by illustrative examples, to demonstrate the validity and structural properties of PFMGs. The formal results presented herein ensure consistency with the fundamental characteristics of algebraic group structures, while incorporating the flexibility offered by the Pythagorean fuzzy approach. This work thus provides a rigorous mathematical foundation for further exploration of algebraic operations within the Pythagorean fuzzy multiset framework.

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