Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44)** : 1–11.
ISSN-0037-8712
doi:10.5269/bspm.77223

Fixed Point Theorems for Generalized Contractions in b-Metric Spaces with Applications

Rahim Shah [1], Andleeb Akhtar [2], Isha Bibi [3] and Muhammad Irfan Qadir [4]

ABSTRACT: In this paper, we develop new fixed point results for generalized contractions in the framework of b-metric spaces. Several illustrative examples and a concrete application of integral equations are provided to demonstrate the utility and scope of the proposed theorems.

Key Words: Fixed point theory, generalized contractions, b-metric spaces, nonlinear analysis, existence and uniqueness, integral equations.

Contents

1. Introduction and Preliminaries

Fixed point theory plays a fundamental role in various branches of mathematics and has wide-ranging applications in science and engineering [1,2,3]. Since the introduction of metric spaces by M. Fréchet in 1906, the concept has evolved into a powerful analytical tool that is applied in fields such as image processing, protein modeling, and information retrieval systems. A cornerstone of this theory is the Banach contraction principle, which asserts the existence and uniqueness of fixed points for certain self-maps on complete metric spaces.

Over time, numerous researchers have extended the Banach principle either by modifying the contractive condition or by exploring broader categories of metric-like structures [4,5,6,7,8]. For example, Matthews [4] introduced partial metric spaces, allowing nonzero self-distances—a framework suitable for theoretical computer science. Branciari [5] presented a fixed point theorem involving integral-type inequalities. Mustafa and Sims [6] developed a generalized metric space framework with relaxed symmetry and triangular inequality conditions. Azam et al. [7] focused on fixed point theorems in complex-valued metric spaces, which are applicable in quantum systems. Mitrović and Radenović [8] examined Banach-type contractions in $bv(s)$-metric spaces that generalize b-metrics with additional variation measures.

Further enhancements to fixed point theory are found in a wide spectrum of generalizations [9,10,11,12,13,14,15,16,17,18,19,20]. Meir and Keeler [9] introduced contractive mappings governed by control functions. Nadler [10] extended Banach's result to multivalued mappings, which proved vital for fuzzy set theory. Rhoades' works [11,12] compared contractive mappings and explored weakly contractive maps. Samet et al. [13] proposed α-ψ-contractive mappings, combining several previous conditions into a unified structure. Wardowski [14] introduced the concept of F-contractions, which generated a new direction in the theory. Harandi [15] contributed to the understanding of partial and metric-like spaces with new fixed point results. Alghamdi et al. [16] explored coupled fixed points in b-metric-like spaces. Shukla [17] initiated a study on partial b-metric spaces, broadening the fixed point paradigm. Jleli and Samet in [18,19,20] developed several new generalized metric spaces with associated fixed point theorems.

Among these extensions, b-metric spaces have attracted considerable interest. These spaces modify the triangle inequality by incorporating a multiplicative constant $s \geq 1$. The concept was first introduced by Bakhtin [21] and later formalized and extended by Czerwik [22,23]. This generalization retains much

2020 *Mathematics Subject Classification:* 37C25, 47H10, 54H25.
Submitted June 06, 2025. Published December 29, 2025

of the utility of metric spaces while allowing for broader applicability. Several authors have developed fixed point results within this framework [24,25]. Shatanawi et al. [24] used comparison functions to define new contraction conditions on b-metric spaces. Younis et al. [25] integrated graphical structures with rectangular b-metric spaces, bridging ideas from topology and graph theory.

In addition to these works, several contributions by Zada, Shah, and collaborators offer important insights into generalized fixed point theory [27,28,29,30,31,32,33]. Zada et al. [27] introduced integral-type contractions for coupled coincidence points in G-metric spaces. Wang et al. [28] addressed self-mappings in dislocated metric spaces. Li et al. [29] studied fixed points in dislocated quasi-metric spaces. Shah et al. [30] proposed new coupled fixed point results for generalized contractive conditions. Zada et al. [31] analyzed ordered cone b-metric spaces. Shah and Zada [32] examined integral-type contractions for compatible maps. Shah [33] explored fixed point results in b-metric-like spaces with relaxed conditions.

Recent work by Turab and Sintunavarat demonstrates the applicability of fixed point theory in modeling real-world behavioral phenomena. In [34], they addressed a learning avoidance model using Banach's theorem, while in [35], they modeled two-choice behavior in fish based on fixed point methods. Some recent advancements in fixed-point theory are recommended, particularly some papers presenting novel applications, such as [36,37,38]:

In this work, we focus on exploring a new class of generalized contractions within the framework of b-metric spaces. We develop several fixed point theorems under broader and more flexible contractive conditions, extending and refining existing results in the literature. To illustrate the theoretical significance of our findings, we present carefully constructed examples. Additionally, we demonstrate the practical relevance of one of our main theorems by applying it to establish the existence of a solution to a Fredholm-type integral equation. The novelty of our approach lies in introducing generalized cyclic and rational-type contraction conditions that unify and generalize various known results. By combining rigorous theoretical analysis with concrete examples and applications, our study contributes meaningfully to both abstract fixed point theory and its real-world implications. We begin by recalling the formal definition of a b-metric space:

Definition 1.1 [21,22,23] Let Z be a non-empty set. A function $D : Z \times Z \to \mathbb{R}^+$ is called a *b-metric* on Z if there exists a constant $s \geq 1$ such that for all $\mu, \omega, \nu \in Z$, the following conditions are satisfied:

1. $D(\mu,\omega) = 0 \Leftrightarrow \mu = \omega$,
2. $D(\mu,\omega) = D(\omega,\mu)$,
3. $D(\mu,\omega) \leq s\big[D(\mu,\nu) + D(\nu,\omega)\big]$.

The pair (Z, D) is then called a *b-metric space*. Note that when $s = 1$, the b-metric reduces to the usual metric.

Example 1.1 [26] Let (Z, d) be a metric space, and define a new function $D : Z \times Z \to [0,\infty)$ by

$$D(\mu,\omega) = [d(\mu,\omega)]^\eta, \quad \forall \mu,\omega \in Z,$$

where $\eta > 1$ is a fixed real number. Then (Z, D) is a b-metric space with coefficient $s = 2^{\eta-1}$.

In particular, if $Z = \mathbb{R}$ and $d(\mu,\omega) = |\mu - \omega|$ is the usual Euclidean metric, then

$$D(\mu,\omega) = (\mu - \omega)^2, \quad \forall \mu,\omega \in \mathbb{R},$$

defines a b-metric on $\mathbb{R}$ with $s = 2$. However, $(\mathbb{R}, D)$ is not a metric space because it does not satisfy the triangle inequality in the strict metric sense. For instance,

$$D(-2,2) = 16 > D(-2,0) + D(0,2) = 4 + 4 = 8,$$

thus violating condition (iii) when $s = 1$, confirming that this is not a standard metric.

2. Main Results

In this section, we present our main results.

Theorem 2.1 *Let (Z, D) be a complete b-metric space with coefficient $s \geq 1$, and let $A, B \subseteq Z$ be non-empty subsets. Suppose that the mapping $\tau : Z \to Z$ satisfies the generalized cyclic contraction condition:*

$$D(\tau\mu, \tau\omega) \leq kD(\mu, \omega) + \psi(D(\mu, \omega)), \tag{2.1}$$

for all $\mu \in A$, $\omega \in B$, where:

- $k \in [0, 1)$,
- $\psi : [0, \infty) \to [0, \infty)$ *is a continuous function such that* $\psi(t) \to 0$ *as* $t \to 0$.

Assume further that $\tau(A) \subseteq B$ and $\tau(B) \subseteq A$. Then, τ has a unique fixed point $\mu' \in A \cup B$. Moreover, for any $\mu_0 \in A$, the sequence $\mu_n = \tau^n(\mu_0)$ converges to μ'.

Proof: Let $\mu_0 \in A$, and define a sequence $\{\mu_n\}$ recursively by

$$\mu_{n+1} = \tau(\mu_n), \quad \text{for } n \geq 0.$$

By the cyclic property of τ, we have:

$$\mu_n \in A \Rightarrow \mu_{n+1} \in B, \quad \mu_n \in B \Rightarrow \mu_{n+1} \in A,$$

so the sequence alternates between A and B.

Using the contraction condition (2.1), for each $n \geq 0$, we get:

$$\begin{aligned} D(\mu_{n+1}, \mu_{n+2}) &= D(\tau\mu_n, \tau\mu_{n+1}) \\ &\leq kD(\mu_n, \mu_{n+1}) + \psi(D(\mu_n, \mu_{n+1})). \end{aligned} \tag{2.2}$$

Define $D_n := D(\mu_n, \mu_{n+1})$. Then from (2.2), we obtain:

$$D_{n+1} \leq kD_n + \psi(D_n).$$

Since $k \in [0, 1)$, ψ is continuous with $\psi(t) \to 0$ as $t \to 0$, and $D_n \geq 0$, it follows that $D_n \to 0$ as $n \to \infty$. Therefore,

$$D(\mu_n, \mu_{n+1}) \to 0 \quad \text{as } n \to \infty.$$

To show that $\{\mu_n\}$ is a Cauchy sequence, for any $m > n$, using the b-metric inequality repeatedly, we have:

$$D(\mu_n, \mu_m) \leq s \sum_{i=n}^{m-1} D(\mu_i, \mu_{i+1}).$$

Since the terms $D(\mu_i, \mu_{i+1})$ tend to zero, the right-hand side tends to zero as $n, m \to \infty$. Thus, $\{\mu_n\}$ is a Cauchy sequence.

As (Z, D) is complete, there exists $\mu' \in Z$ such that $\mu_n \to \mu'$ as $n \to \infty$.

We now verify that μ' is a fixed point of τ. Since τ is continuous under the b-metric topology induced by D, we have:

$$\tau\mu' = \lim_{n\to\infty} \tau(\mu_n) = \lim_{n\to\infty} \mu_{n+1} = \mu'.$$

To prove uniqueness, suppose there exists another fixed point $\omega' \in Z$, with $\omega' \neq \mu'$. Then:

$$\begin{aligned} D(\mu', \omega') &= D(\tau\mu', \tau\omega') \\ &\leq kD(\mu', \omega') + \psi(D(\mu', \omega')). \end{aligned}$$

Rearranging gives:

$$D(\mu', \omega')(1 - k) \leq \psi(D(\mu', \omega')).$$

Since $\psi(t) \to 0$ as $t \to 0$, the only possibility is $D(\mu', \omega') = 0$, which implies $\mu' = \omega'$.

Hence, τ has a unique fixed point $\mu' \in A \cup B$, and the sequence μ_n converges to μ'. □

Theorem 2.2 *Let (Z, D) be a complete b-metric space with constant $s \geq 1$. Suppose that the mapping $\tau : Z \to Z$ satisfies the generalized rational contraction condition:*

$$D(\tau\mu', \tau\omega') \leq \frac{kD(\mu, \omega)}{1 + \psi(D(\mu, \omega))}, \tag{2.3}$$

where:

1. *$k \in [0, 1)$,*
2. *$\psi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\psi(t) \to 0$ as $t \to 0$.*

Then τ has a unique fixed point $\mu' \in Z$, and for any initial point $\mu_0 \in Z$, the iterative sequence $\mu_{n+1} = \tau(\mu_n)$ converges to μ'.

Proof: Let $\mu_0 \in Z$ be arbitrary, and define a sequence $\{\mu_n\}$ by

$$\mu_{n+1} = \tau(\mu_n), \quad \text{for all } n \geq 0.$$

Assume that all successive terms are distinct; otherwise, if $\mu_{n+1} = \mu_n$ for some n, then μ_n is a fixed point.

Using the contractive condition (2.3), we get for all $n \geq 0$:

$$\begin{aligned} D(\mu_{n+1}, \mu_{n+2}) &= D(\tau\mu_n, \tau\mu_{n+1}) \\ &\leq \frac{kD(\mu_n, \mu_{n+1})}{1 + \psi(D(\mu_n, \mu_{n+1}))}. \end{aligned} \tag{2.4}$$

Define $D_n := D(\mu_n, \mu_{n+1})$. Then inequality (2.4) becomes:

$$D_{n+1} \leq \frac{kD_n}{1 + \psi(D_n)}. \tag{2.5}$$

Since $1 + \psi(D_n) > 1$ and $k \in [0, 1)$, it follows that

$$D_{n+1} < kD_n.$$

Hence, the sequence $\{D_n\}$ is positive and strictly decreasing. Moreover, since $D_n \geq 0$, it converges to some limit $L \geq 0$. Taking the limit on both sides of (2.5), we get:

$$L \leq \frac{kL}{1 + \psi(L)}.$$

If $L > 0$, then dividing both sides by L gives:

$$1 \leq \frac{k}{1 + \psi(L)},$$

which contradicts $k < 1$. Therefore, $L = 0$, and we conclude:

$$\lim_{n \to \infty} D(\mu_n, \mu_{n+1}) = 0.$$

Now, for $m > n$, using the b-metric triangle inequality:

$$D(\mu_n, \mu_m) \leq s \sum_{i=n}^{m-1} D(\mu_i, \mu_{i+1}).$$

Since the tail sum $\sum_{i=n}^{\infty} D(\mu_i, \mu_{i+1})$ converges to 0 as $n \to \infty$, the sequence $\{\mu_n\}$ is Cauchy. By completeness of (Z, D), there exists $\mu' \in Z$ such that

$$\mu_n \to \mu' \quad \text{as } n \to \infty.$$

We now verify that μ' is a fixed point of τ. Since τ is continuous under the b-metric D, we have:

$$\tau(\mu') = \lim_{n\to\infty} \tau(\mu_n) = \lim_{n\to\infty} \mu_{n+1} = \mu'.$$

To prove uniqueness, suppose there exists another fixed point $\omega' \in Z$, with $\omega' \neq \mu'$. Then:

$$\begin{aligned} D(\mu', \omega') &= D(\tau\mu', \tau\omega') \\ &\leq \frac{kD(\mu', \omega')}{1 + \psi(D(\mu', \omega'))}. \end{aligned}$$

Dividing both sides by $D(\mu', \omega') > 0$, we get:

$$1 \leq \frac{k}{1 + \psi(D(\mu', \omega'))},$$

which contradicts $k < 1$. Therefore, $D(\mu', \omega') = 0$, implying $\mu' = \omega'$. Hence, the fixed point is unique. □

Example 2.1 Consider the set $Z = \mathbb{R}$ equipped with the function $D : Z \times Z \to [0, \infty)$ defined by

$$D(\mu, \omega) = |\mu - \omega|^p,$$

where $p > 0$ is fixed. This function D defines a b-metric on Z with constant $s = 2^p$.

Define the mapping $\tau : Z \to Z$ by

$$\tau(\mu) = \frac{\mu}{2}.$$

This mapping represents a discrete-time dynamical system exhibiting exponential decay toward zero. We analyze the asymptotic behavior of the iterative sequence

$$\mu_{n+1} = \tau(\mu_n)$$

using Theorem 2.2 (the generalized rational contraction).

Let the auxiliary function $\psi : [0, \infty) \to [0, \infty)$ be given by

$$\psi(t) = t^q,$$

for some $q > 0$.

To verify the contraction condition of Theorem 2.2, consider any $\mu, \omega \in Z$. We compute:

$$D(\tau\mu, \tau\omega) = \left|\frac{\mu}{2} - \frac{\omega}{2}\right|^p = \frac{|\mu - \omega|^p}{2^p},$$

and

$$\frac{kD(\mu, \omega)}{1 + \psi(D(\mu, \omega))} = \frac{k|\mu - \omega|^p}{1 + |\mu - \omega|^{pq}},$$

for some constant $k \in [0, 1)$.

By choosing an appropriate k, the inequality

$$D(\tau\mu, \tau\omega) \leq \frac{kD(\mu, \omega)}{1 + \psi(D(\mu, \omega))}$$

holds for all $\mu, \omega \in \mathbb{R}$. Since (Z, D) is complete and $\psi(t) \to 0$ as $t \to 0$, all conditions of Theorem 2.2 are fulfilled.

Interpretation: This example models a recursive process that halves the input value at each step, such as a damped system or a root-finding algorithm with step halving. The fixed point $\mu' = 0$ corresponds to the equilibrium state. The theorem guarantees that, for any initial $\mu_0 \in \mathbb{R}$, the sequence $\{\tau^n(\mu_0)\}$ converges to this equilibrium. This illustrates how the generalized contraction principle ensures convergence and uniqueness in iterative computational methods.

Theorem 2.3 *Let (Z, D) be a complete b-metric space with constant $s \geq 1$, and let $\tau : Z \to Z$ be a mapping satisfying the generalized contraction condition:*

$$D(\tau\mu, \tau\omega) \leq kD(\mu, \omega) + \psi_1(D(\mu, \omega))\psi_2(D(\mu, \omega)), \tag{2.6}$$

where:

1. $k \in [0, 1)$,
2. $\psi_1, \psi_2 : [0, \infty) \to [0, \infty)$ *are continuous functions with* $\psi_1(0) = \psi_2(0) = 0$.

Then τ has a unique fixed point $\mu' \in Z$, and for any initial point $\mu_0 \in Z$, the sequence defined recursively by $\mu_{n+1} = \tau(\mu_n)$ converges to μ'.

Proof: Let $\mu_0 \in Z$ be arbitrary, and define a sequence $\{\mu_n\}$ by:

$$\mu_{n+1} = \tau(\mu_n), \quad \text{for } n = 0, 1, 2, \ldots$$

From the contraction condition (2.6), it follows for all $n \geq 0$ that:

$$\begin{aligned} D(\mu_{n+1}, \mu_{n+2}) &= D(\tau\mu_n, \tau\mu_{n+1}) \\ &\leq kD(\mu_n, \mu_{n+1}) + \psi_1(D(\mu_n, \mu_{n+1}))\psi_2(D(\mu_n, \mu_{n+1})). \end{aligned} \tag{2.7}$$

Let us denote $\delta_n := D(\mu_n, \mu_{n+1})$. Then inequality (2.7) becomes:

$$\delta_{n+1} \leq k\delta_n + \psi_1(\delta_n) \cdot \psi_2(\delta_n).$$

Since $k < 1$, we have $k\delta_n < \delta_n$ for all sufficiently large n, and as $\psi_1(0) = \psi_2(0) = 0$ and both functions are continuous, it follows that $\psi_1(\delta_n)\psi_2(\delta_n) \to 0$ as $\delta_n \to 0$. Hence, $\{\delta_n\}$ is a non-negative sequence that converges to zero:

$$\delta_n \to 0 \quad \text{as } n \to \infty.$$

That is,

$$D(\mu_n, \mu_{n+1}) \to 0 \quad \text{as } n \to \infty.$$

To prove that $\{\mu_n\}$ is a Cauchy sequence, let $m > n$. Using the b-metric inequality, we obtain:

$$D(\mu_n, \mu_m) \leq s \sum_{i=n}^{m-1} D(\mu_i, \mu_{i+1}).$$

Since the right-hand side is a finite sum of terms tending to zero, for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$D(\mu_n, \mu_m) < \epsilon.$$

Therefore, $\{\mu_n\}$ is a Cauchy sequence. By completeness of (Z, D), there exists $\mu' \in Z$ such that:

$$\mu_n \to \mu' \quad \text{as } n \to \infty.$$

We now verify that μ' is a fixed point of τ. By continuity of τ, we have:

$$\tau(\mu') = \tau\left(\lim_{n\to\infty} \mu_n\right) = \lim_{n\to\infty} \tau(\mu_n) = \lim_{n\to\infty} \mu_{n+1} = \mu'.$$

Hence, μ' is indeed a fixed point of τ.

To establish uniqueness, suppose μ' and ω' are both fixed points of τ. Applying (2.6) gives:

$$\begin{aligned} D(\mu', \omega') &= D(\tau\mu', \tau\omega') \\ &\leq kD(\mu', \omega') + \psi_1(D(\mu', \omega'))\psi_2(D(\mu', \omega')). \end{aligned} \tag{2.8}$$

Set $\Delta := D(\mu', \omega')$. Then from (2.8), we obtain:

$$\Delta \leq k\Delta + \psi_1(\Delta) \cdot \psi_2(\Delta).$$

Rewriting this gives:

$$(1-k)\Delta \leq \psi_1(\Delta) \cdot \psi_2(\Delta).$$

If $\Delta > 0$, the right-hand side is strictly positive, while the left is strictly positive as well. But since $\psi_1(0) = \psi_2(0) = 0$ and both functions are continuous, this inequality can only hold if $\Delta = 0$. Hence:

$$D(\mu', \omega') = 0 \Rightarrow \mu' = \omega'.$$

Therefore, the fixed point μ' is unique, and the sequence $\{\mu_n\}$ converges to this unique fixed point. The proof is complete. □

Example 2.2 Let $Z = [0, \infty)$, and define a b-metric $D : Z \times Z \to [0, \infty)$ by

$$D(\mu, \omega) = |\mu - \omega| + |\mu - \omega|^2,$$

which satisfies the b-metric condition with constant $s = 2$.

Consider the self-mapping $\tau : Z \to Z$ given by

$$\tau(\mu) = \frac{\mu}{3}.$$

This mapping represents a simple contraction system where each iteration reduces the input value by a factor of 3. We aim to demonstrate that τ satisfies the assumptions of Theorem 2.3 (generalized product-type contraction), and that it converges to a unique fixed point.

Define two continuous auxiliary functions $\psi_1, \psi_2 : [0, \infty) \to [0, \infty)$ as

$$\psi_1(t) = \frac{t}{2}, \quad \psi_2(t) = \frac{t}{3},$$

and let $k = \frac{1}{3}$.

For any $\mu, \omega \in Z$, we calculate:

$$\begin{aligned} D(\tau\mu, \tau\omega) &= \left|\frac{\mu}{3} - \frac{\omega}{3}\right| + \left|\frac{\mu}{3} - \frac{\omega}{3}\right|^2 \\ &= \frac{1}{3}|\mu - \omega| + \frac{1}{9}|\mu - \omega|^2. \end{aligned}$$

Now, the right-hand side of the contraction condition is:

$$\begin{aligned} kD(\mu, \omega) + \psi_1(D(\mu, \omega)) \cdot \psi_2(D(\mu, \omega)) &= \frac{1}{3}(|\mu - \omega| + |\mu - \omega|^2) \\ &\quad + \left(\frac{|\mu - \omega| + |\mu - \omega|^2}{2}\right)\left(\frac{|\mu - \omega| + |\mu - \omega|^2}{3}\right). \end{aligned}$$

It is evident that

$$D(\tau\mu, \tau\omega) \leq kD(\mu, \omega) + \psi_1(D(\mu, \omega)) \cdot \psi_2(D(\mu, \omega))$$

holds for all $\mu, \omega \in Z$. Therefore, the contraction condition of Theorem 2.3 is satisfied.

Application Insight: The mapping τ can be viewed as a basic model of a damped iterative process, such as one used in control systems, economic decay models, or convergence algorithms. The b-metric structure, which includes both linear and quadratic components, reflects sensitivity to deviations of varying magnitude. The unique fixed point $\mu' = 0$ acts as the system's equilibrium. Consequently, for any initial state $\mu_0 \in Z$, the iterative sequence $\tau^n(\mu_0)$ converges to μ'.

This example demonstrates how Theorem 2.3 can guarantee convergence in nonlinear systems, showcasing a real application rather than just hypothesis verification.

3. Application

In this section, we apply Theorem 2.2 to prove the existence and uniqueness of a solution to a Fredholm integral equation. Let $Z = C[a, b]$ denote the space of all real-valued continuous functions defined on the closed interval $[a, b]$. Define a b-metric $D : Z \times Z \to [0, \infty)$ by

$$D(\mu, \omega) = \sup_{t \in [a,b]} |\mu(t) - \omega(t)|^\eta,$$

where $0 < \eta \leq 1$. It is well known that $(C[a, b], D)$ forms a complete b-metric space under this definition.

We consider the Fredholm integral equation:

$$\mu(t) = \int_a^b N(t, \tau, \mu(\tau))\, d\tau + f(t), \quad t \in [a, b], \tag{3.1}$$

where

- $N : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function (the kernel),
- $f : [a, b] \to \mathbb{R}$ is a given continuous function,
- $\mu \in C[a, b]$ is the unknown function.

Define the operator $\tau : Z \to Z$ by

$$\tau(\mu)(t) = \int_a^b N(t, \tau, \mu(\tau))\, d\tau + f(t).$$

We now verify that τ satisfies the generalized rational contraction condition described in Theorem 2.2, which will imply the existence of a unique fixed point $\mu' \in Z$ such that $\tau(\mu') = \mu'$, thus solving the integral equation (3.1).

Assume that the kernel N satisfies a Lipschitz condition with respect to its third variable: there exists a constant $M > 0$ such that for all $t, \tau \in [a, b]$ and $x, y \in \mathbb{R}$,

$$|N(t, \tau, x) - N(t, \tau, y)| \leq M|x - y|.$$

Then for any $\mu, \omega \in Z$, we estimate:

$$\begin{aligned}
D(\tau\mu, \tau\omega) &= \sup_{t \in [a,b]} \left| \int_a^b \big(N(t, \tau, \mu(\tau)) - N(t, \tau, \omega(\tau))\big)\, d\tau \right|^\eta \\
&\leq \sup_{t \in [a,b]} \left(\int_a^b M|\mu(\tau) - \omega(\tau)|\, d\tau \right)^\eta \\
&= M^\eta (b - a)^\eta \left(\sup_{\tau \in [a,b]} |\mu(\tau) - \omega(\tau)| \right)^\eta \\
&= kD(\mu, \omega),
\end{aligned}$$

where $k = M^\eta (b - a)^\eta$. If $M(b - a) < 1$, then $k < 1$, ensuring a contraction.

Define $\psi(t) = t^\eta$, which is continuous and satisfies $\psi(t) \to 0$ as $t \to 0$. Then,

$$D(\tau\mu, \tau\omega) \leq \frac{kD(\mu, \omega)}{1 + \psi(D(\mu, \omega))},$$

which confirms that τ satisfies the generalized rational contraction condition in Theorem 2.2. Therefore, τ has a unique fixed point $\mu' \in Z$, and for any initial function $\mu_0 \in Z$, the iterates $\tau^n(\mu_0)$ converge to μ'. Hence, the integral equation (3.1) admits a unique continuous solution.

Example 3.1 Let $a = 0$, $b = 1$, and consider the Fredholm integral equation

$$\mu(t) = \int_0^1 \frac{1}{2}\mu(\tau)\sin(t\tau)\,d\tau + \cos(t), \quad t \in [0,1].$$

Here, the kernel and inhomogeneous term are given by

$$N(t,\tau,\mu(\tau)) = \frac{1}{2}\mu(\tau)\sin(t\tau), \quad f(t) = \cos(t).$$

We verify that the kernel satisfies a Lipschitz condition:

$$|N(t,\tau,x) - N(t,\tau,y)| = \frac{1}{2}|\sin(t\tau)| \cdot |x-y| \leq \frac{1}{2}|x-y|.$$

Thus, the Lipschitz constant is $M = \frac{1}{2}$. Taking $\eta \in (0,1]$, we compute the contraction coefficient:

$$k = M^\eta (b-a)^\eta = \left(\frac{1}{2}\right)^\eta < 1.$$

Using the b-metric

$$D(\mu,\omega) = \sup_{t\in[0,1]} |\mu(t) - \omega(t)|^\eta,$$

we conclude that all conditions of Theorem 2.2 are satisfied. Therefore, the operator

$$\tau(\mu)(t) = \int_0^1 \frac{1}{2}\mu(\tau)\sin(t\tau)\,d\tau + \cos(t)$$

has a unique fixed point $\mu' \in C[0,1]$, which is the unique solution to the integral equation.

Interpretation: This equation models a process where the current state $\mu(t)$ depends on a weighted average of past states modulated by a sinusoidal kernel. The application of Theorem 2.2 ensures that such a system has a unique stable state and that numerical approximation methods based on successive iterations will converge to this state.

4. Conclusion

In this paper, we have developed a collection of fixed point theorems for a new class of generalized contractions in the setting of b-metric spaces. These results offer meaningful extensions and refinements of existing theorems by introducing broader contraction conditions, including rational-type and product-type mappings.

To highlight the practical relevance of our theoretical contributions, we have included illustrative examples that model computational and dynamical systems. Furthermore, we demonstrated an application of one of the main theorems by establishing the existence and uniqueness of a solution to a Fredholm integral equation, thereby underscoring the utility of our framework in solving real analytical problems.
Future Work: The flexibility of the contraction frameworks proposed in this study opens up several promising avenues for future research. These include:

- Extending the results to multivalued (set-valued) mappings in b-metric or other generalized metric spaces;
- Studying fixed point results under weaker assumptions, such as discontinuous or non-measurable mappings;
- Applying the developed theorems to a broader class of functional equations, including Hammerstein and Volterra-type integral equations;
- Investigating algorithmic implementations and convergence behavior of iterative schemes based on the proposed theoretical models.

Such developments would not only deepen the theoretical understanding of generalized contractions but also broaden their applicability in nonlinear analysis and computational mathematics.

Competing interests

The authors declare that they have no competing interests.

Availability of data and Metarials

No data were used to support this study.

References

1. Karapinar, Erdogan, et al. *Applying new fixed point theorems on fractional and ordinary differential equations*, Advances in Difference Equations, 2019, 1–25, (2019) https://doi.org/10.1186/s13662-019-2354-3.

2. Kilbas, Anatoly A., et al. *Theory and Application of Fractional Differential Equations*, Elsevier, (2006). https://doi.org/10.1016/S0304-0208(06)80001-0.

3. Khamsi, Mohamed A., and William A. Kirk. *An Introduction to Metric Spaces and Fixed Point Theory*, John Wiley and Sons, (2001). https://doi.org/10.1002/9781118033074.

4. Matthews, S. G. *Partial metric topology. Annals of the New York Academy of Sciences*, 728, 1994, 183–197. https://doi.org/10.1111/j.1749-6632.1994.tb44144.x.

5. Branciari, A. *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces*, Publicationes Mathematicae, 57, 31–37, (2000).

6. Mustafa, Z., and B. Sims. *A new approach to generalized metric spaces*, Journal of Nonlinear and Convex Analysis, 7, 289–297, (2006).

7. Azam, A., et al. *Common fixed point theorems in complex valued metric spaces*, Numerical Functional Analysis and Optimization, 32, 243–253, (2011). https://doi.org/10.1080/01630563.2011.533046.

8. Mitrović, Z. D., and Radenović. S. *The Banach and Reich contractions in bv(s)-metric spaces*, Journal of Fixed Point Theory and Applications, 19, 3087–3095, (2017). https://doi.org/10.1007/s11784-017-0469-2.

9. Meir, A., and Keeler. E. *A theorem on contractive mappings*, Journal of Mathematical Analysis and Applications 28, 326–329, (1969).

10. Nadler, Sam B. *Multivalued contraction mappings.* Pacific Journal of Mathematics, 30, 475–488, (1969).

11. Rhoades, B. E. *A comparison of various definitions of contractive mappings*, Transactions of the American Mathematical Society, 226, 257–290, 1977.

12. Rhoades, B. E. *Some theorems on weakly contractive maps*, Nonlinear Analysis: Theory, Methods and Applications, 47, 4, 2683–2693, (2001).

13. Samet, B., et al. *Fixed point theorems for α-ψ-contractive type mappings*, Nonlinear Analysis: Theory, Methods and Applications, 75, 2154–2165, (2012).

14. Wardowski, D. *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory and Applications, 2012, 94, (2012).

15. Haranadi, A. A. *Metric-like spaces, partial metric spaces and fixed points*, Fixed Point Theory and Applications, 2012, 204, (2012).

16. Alghamdi, M. A., et al. *Fixed point and coupled fixed point theorems on b-metric-like spaces*, Journal of Inequalities and Applications, 2013, 402.

17. Shukla, S. *Partial b-metric spaces and fixed point theorems*, Mediterranean Journal of Mathematics, 11, 703–711, (2014).

18. Jleli, M., et al. *Further generalizations of the Banach contraction principle*, Journal of Inequalities and Applications, 1, 439, (2014).

19. Jleli, M., and B. Samet. *A generalized metric space and related fixed point theorems*, Fixed Point Theory and Applications, 2015, 61.

20. Jleli, M., and B. Samet. *On a new generalization of metric spaces*, Journal of Fixed Point Theory and Applications, 20, 128, (2018).

21. Bakhtin, I. A. *The contraction mapping principle in almost metric spaces*, Functional Analysis, 30, 26–37, (1989).

22. Czerwik, S. *Contraction mappings in b-metric spaces.Acta Mathematica et Informatica Universitatis Ostraviensis*, 30, 5–11, (1993).

23. Czerwik, S. *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti del Seminario Matematico e Fisico dell'Università di Modena, 46, 263–276, 1998.

24. Shatanawi, W., et al. *Contraction conditions using comparison functions on b-metric spaces*, Fixed Point Theory and Applications, 135, (2014).

25. Younis, M., et al. *Revisiting graphical rectangular b-metric spaces*, Asian-European Journal of Mathematics, 15, 2250072, (2022).

26. Khamsi, M., and N. Hussain. *KKM mappings in metric type spaces*, Nonlinear Analysis: Theory, Methods and Applications, 73, 9, 3123–3129, (2010).

27. Zada, A., et al. *Integral type contraction and coupled coincidence fixed point theorems for two pairs in G-metric spaces*, Hacettepe Journal of Mathematics and Statistics, 45, 5, 1475–1484, (2016).

28. Wang, P., et al. *Some common fixed point results for two pairs of self-maps in dislocated metric spaces*, Journal of Computational Analysis and Applications, 25, 8, 1410–1424, (2018).

29. Li, S., et al. *Fixed point theorems in dislocated quasi-metric spaces*, Journal of Nonlinear Sciences and Applications, 10, 4695–4703, (2017).

30. Shah, R., et al. *New common coupled fixed point results of integral type contraction in generalized metric spaces*, Journal of Analysis and Number Theory, 4, 2, 145–152, (2016).

31. Zada, A., et al. *Fixed point theorems in ordered cone b-metric spaces*, Scientific Studies and Research, Series Mathematics and Informatics, 26, 1, 109–120, (2016).

32. Shah, R., and A. Zada. *Some common fixed point theorems of compatible maps with integral type contraction in G-metric spaces*, Proceedings of IAM, 5, 1, 64–74, (2016).

33. Shah, R. *Some new fixed point results in b-metric-like spaces*, Palestine Journal of Mathematics, 11, 1, 378–384, (2022).

34. Turab, A., Sintunavarat, W. *On the solution of the traumatic avoidance learning model approached by the Banach fixed point theorem*, J. Fixed Point Theory Appl, 50, (2020). https://doi.org/10.1007/s11784-020-00788-3

35. Turab, A., Sintunavarat, W. *On analytic model for two–choice behavior of the paradise fish based on the fixed point method*, J. Fixed Point Theory Appl, 21, 56, (2019). https://doi.org/10.1007/s11784-019-0694-y

36. Younis, M., Öztürk, M. *Some novel proximal point results and applications*, Universal Journal of Mathematics and Applications, 8, 1, 8-20, (2025). https://doi.org/10.32323/ujma.1597874

37. Guo, L., et al. *Convergence results for controlled contractions and applications to a system of Volterra integral equations*, Applied Mathematics in Science and Engineering, 32(1), (2024). https://doi.org/10.1080/27690911.2024.2424909

38. Ahmad, H.; Din, F.U.; Younis, M.; Guran, L. *Analyzing the chaotic dynamics of a fractional order Dadras–Momeni system using relaxed contractions*, Fractal Fract. 2024, 8, 699. https://doi.org/10.3390/fractalfract8120699

[1]*Department of Mathematics, Kohsar University Murree, Murree, Pakistan.*

[2]*Department of Mathematics, Kohsar University Murree, Murree, Pakistan.*

[3]*Department of Mathematics, Kohsar University Murree, Murree, Pakistan.*

[4]*Department of Mass Communication and Media Studies, Kohsar University Murree, Murree, Pakistan.*

E-mail address: [1]`rahimshah@kum.edu.pk, shahraheem1987@gmail.com`

[2]`andleebakhtar20@gmail.com`

[3]`abbasiishaasiq@gmail.com`

[4]`irfan.qadir@kum.edu.pk`