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On (f,b)-Determinant Sequence of the Tridiagonal Matrices and their Properties

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ABSTRACT: In this paper, we introduce and study a new sequence with determinant functions, called (f, b)-determinant, where f is a real function and b a fixed real constant. Also, we explore a special type of tridiagonal matrices with f in the main diagonal, b in the subdiagonal, and we investigate the recursive pattern of their determinants. We focus on the solutions of the associated recurrence relations and the derivation of closed-form formulas. We examine different cases of tridiagonal matrices, including those with variable and constant coefficients, as well as the general solution for determinants in terms of the roots of the characteristic equations.

Key Words: Second-order recurrence, determinant, tridiagonal matrix.

Contents

1. Introduction and Background

A tridiagonal matrix is a special class of square matrix in which all elements are zero except those on the main diagonal, the diagonal above the main diagonal (superdiagonal), and the diagonal below the main diagonal (subdiagonal). For example, a 4×4 tridiagonal matrix may appear as follows:

$$\begin{bmatrix} a_1 & b_1 & 0 & 0 \\ c_1 & a_2 & b_2 & 0 \\ 0 & c_2 & a_3 & b_3 \\ 0 & 0 & c_3 & a_4 \end{bmatrix},$$

where a_1, a_2, a_3, a_4 are the elements of the main diagonal, b_1, b_2, b_3 are the elements of the superdiagonal, and c_1, c_2, c_3 are the elements of the subdiagonal.

Let us consider a particular kind of tridiagonal matrix $\mathcal{M}_n$ of order $n \geq 1$ defined by:

$$\mathcal{M}_n = \begin{bmatrix} a & b & 0 & 0 & \cdots & 0 & 0 & 0 \\ c & d & e & 0 & \cdots & 0 & 0 & 0 \\ 0 & c & d & e & \cdots & 0 & 0 & 0 \\ 0 & 0 & c & d & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & d & e & 0 \\ 0 & 0 & 0 & 0 & \cdots & c & d & e \\ 0 & 0 & 0 & 0 & \cdots & 0 & c & d \end{bmatrix}, \tag{1.1}$$

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where $a,\ b,\ c,\ d$ and e are non-zero real numbers. The importance of this type of study for a given matrix lies in the fact that it allows for an easy expression of its determinant. We now present the following result, which will be used throughout our paper concerning the determinant of an $n \times n$ tridiagonal matrix.

Lemma 1.1 *[1,5,11] The matrix $\mathcal{M}_n$ is tridiagonal, and for all $n \geq 2$ we have:*

$$|\mathcal{M}_{n+1}| = d|\mathcal{M}_n| - ce|\mathcal{M}_{n-1}| \ ,$$

where $\mathcal{M}_n$ is the tridiagonal matrix given by (1.1).

For the general case, this result was proved in Theorem 1 by [1].

Numerous researchers have extensively explored the use of tridiagonal matrix determinants to derive well-known integer sequences. Therefore, there are many known connections between the determinants of tridiagonal matrices and various number sequences, as discussed in [1,2,3,4,5,6,7,8,9,10,11], among others. In this article, we extend an application for these kind of matrices, constructing a sequence of determinants of tridiagonal matrices whose main diagonal was generated arbitrarily by any real function. So, we propose to present a new approach to matrix sequences as a possible extension of application.

Consider b a fixed constant (typically real) and let $f(x)$ be a real-valued function. We define the square (f, b)-tridiagonal matrix $A_{(f,b,n)}$ of order n by:

$$A_{(f,b,n)} = \begin{pmatrix} f(x) & 1 & 0 & \cdots & 0 \\ b & f(x) & 1 & \cdots & 0 \\ 0 & b & f(x) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & b & f(x) \end{pmatrix} . \tag{1.2}$$

The goal of this study is to find the determinant $D_n^{(f,b)}$ of the (f, b)-tridiagonal matrix $A_{(f,b,n)}$. For simplicity, we will just say the generalized (f, b)-determinant of $A_{(f,b,n)}$ and we write $D_n = D_n^{(f,b)}$.

The first six elements of the generalized (f, b)-determinant of $A_{(f,b,n)}$ are exhibited in Table 1. For convenience, we used $D_0 = 1$.

Table 1: Generalized (f, b)-determinant

D_0	D_1	D_2	D_3	D_4	D_5
1	$f(x)$	$f(x)^2 - b$	$f(x)^3 - 2bf(x)$	$f(x)^4 - 3bf(x)^2 + b^2$	$f(x)^5 - 4bf(x)^3 + 3b^2f(x)$

The main focus of this paper is to investigate some properties for the (f, b)-determinant D_n. In this introductory section, we reviewed the concepts related to tridiagonal matrices and a result that allows us to write the sequence of determinants $\{D_n\}_{n\geq 0}$ associated with (f, b)-tridiagonal matrix $A_{(f,b,n)}$. This work is made up of two more sections, as well as the final considerations. In Section 2 we study the homogeneous recurrence relation of order 2 associated with the sequence $\{D_n\}_{n\geq 0}$, thus obtaining a quadratic equation in which the roots depend on a discriminant Λ. In the three possible cases, namely: $\Lambda > 0$, $\Lambda = 0$, and $\Lambda < 0$ we find the Binet formula for the sequence $\{D_n\}_{n\geq 0}$ and present an example to illustrate the behavior of this sequence. While in Section 3 it will be presented some of the properties satisfied by the (f, b)-determinants sequence. In particular, we obtain some classical identities, including Tagiuri-Vajda's identity.

2. Recursive relation and the Binet Formula

Consider the (f, b)-tridiagonal matrix $A_{(f,b,n)}$ given by Equation (1.2) and D_n its determinant. See that $D_0 = 1$, $D_1 = f(x)$ and by Lemma 1.1 we obtain the determinant D_n dependent on b and the function $f(x)$, as described in the next result:

Proposition 2.1 *Let n, b, $f(x)$ and $A_{(f,b,n)}$ be, respectively, a non-negative integer, a fixed real constant, a real-valued function and the (f,b)-tridiagonal matrix given by (1.2). For $n \geq 2$ the identity is true*

$$D_n = f(x) \cdot D_{n-1} - b \cdot D_{n-2}, \tag{2.1}$$

with initial conditions $D_0 = 1$ and $D_1 = f(x)$.

The difference equation associated with the (f,b)-determinant sequence $\{D_n\}_{n\geq 0}$ is given in (2.1), which has as its Horadam-type characteristic equation

$$r^2 - f(x)r + b = 0. \tag{2.2}$$

Our next result gives the ordinary generating function for the (f,b)-determinant sequence $\{D_n\}_{n\geq 0}$.

Proposition 2.2 *The ordinary generating function $G_{D_n}(z)$ for (f,b)-determinant sequence $\{D_n\}_{n\geq 0}$ is given by:*

$$G_{D_n}(z) = \frac{1}{1 - f(x)z + bz^2}.$$

Proof: The ordinary generating function for the (f,b)-determinant sequence $\{D_n\}_{n\geq 0}$ sequence is given by

$$G_{D_n}(z) = \sum_{n=0}^{\infty} D_n z^n = D_0 z^0 + D_1 z^1 + D_2 z^2 + D_3 z^3 + \cdots + D_n z^n + \cdots$$

By using the equality (2.1), we get

$$\begin{aligned}
\sum_{n=2}^{\infty} D_n z^n &= f \sum_{n=2}^{\infty} D_{n-1} z^n - b \sum_{n=2}^{\infty} D_{n-2} z^n \\
\sum_{n=2}^{\infty} D_n z^n &= f z \sum_{n=2}^{\infty} D_{n-1} z^{n-1} - b z^2 \sum_{n=2}^{\infty} D_{n-2} z^{n-2} \\
\sum_{n=0}^{\infty} D_n z^n - (D_0 + D_1 z) &= f z \left(\sum_{n=0}^{\infty} D_n z^n - D_0 \right) - b z^2 \sum_{n=0}^{\infty} D_n z^n
\end{aligned}$$

Therefore, $G_{D_n}(z) = \dfrac{D_0 + D_1 z - D_0 f z}{1 - fz + bz^2} = \dfrac{1}{1 - fz + bz^2}$, as required. □

We write Λ to denote the discriminant of Equation (2.2), that is, $\Lambda = f(x)^2 - 4b$. To find the Binet formula for the (f,b)-determinant sequence, let us consider separately three cases that may occur. So, from now on, we are going to consider three distinct cases: $\Lambda > 0$, $\Lambda = 0$, and $\Lambda < 0$.

2.1. First case

First, consider $\Lambda > 0$. In this case, the roots of Equation (2.2) are real and simple, and we present an auxiliary result that will be useful in our investigation.

Lemma 2.3 *If $\Lambda > 0$ and α_1 and α_2 are the roots of the characteristic Equation (2.2), then*

$$\alpha_1 = \frac{f(x) + \sqrt{\Lambda}}{2} \quad \text{and} \quad \alpha_2 = \frac{f(x) - \sqrt{\Lambda}}{2}, \tag{2.3}$$

where $\Lambda = f(x)^2 - 4b$.

For $\Lambda > 0$ and given α_1 and α_2 in Equation (2.3) we can establish the following result.

Proposition 2.4 (First Binet's Formula) *Let n be a non-negative integer, b a fixed real number, $f(x)$ a real-valued function. If $\Lambda > 0$, then for all $n \geq 0$, the following identity holds:*

$$D_n = A_1\alpha_1^n + A_2\alpha_2^n \,, \tag{2.4}$$

where α_1 and α_2 are given in Equation (2.3) and $A_1 = \frac{f(x)-\alpha_2}{\alpha_1-\alpha_2}$ and $A_2 = \frac{\alpha_1-f(x)}{\alpha_1-\alpha_2}$ are real numbers.

The following equalities hold true:

$$\begin{aligned} \alpha_1 + \alpha_2 &= f(x) \,, \\ \alpha_1 - \alpha_2 &= \sqrt{f(x)^2 - 4b} \,, \\ \alpha_1\alpha_2 &= b \,, \\ A_1A_2 &= \frac{-b}{f(x)^2-4b} = -\frac{\alpha_1\alpha_2}{(\alpha_1-\alpha_2)^2} \,. \end{aligned}$$

The proof of these identities is a simple routine and hence is omitted.

Example 2.5 *We consider the real function f such that $f(x) = x^2 + 5$ and $b = 1$. Then the (f,b)-determinant sequence is given by*

$$D_n = (x^2+5)D_{n-1} - D_{n-2},$$

where $D_0(x) = 1$ and $D_1(x) = x^2 + 5$. The characteristic equation is:

$$r^2 - (x^2+5)r + 1 = 0 \,,$$

with discriminant

$$\Lambda = (x^2+5)^2 - 4 = x^4 + 10x^2 + 21 = (x^2+3)(x^2+7) > 0, \quad \textit{for all } x \in \mathbb{R}.$$

So $\alpha_1 = \frac{(x^2+5)+\sqrt{\Lambda}}{2}$ and $\alpha_2 = \frac{(x^2+5)-\sqrt{\Lambda}}{2}$. Thus, the general solution of the recurrence has the Binet form:

$$D_n = A_1\alpha_1^n + A_2\alpha_2^n,$$

where

$$A_1 = \frac{f(x)-\alpha_2(x)}{\alpha_1(x)-\alpha_2(x)} = \frac{(x^2+5+\sqrt{\Lambda})\sqrt{\Lambda}}{2\Lambda} \quad \textit{and} \quad A_2 = \frac{\alpha_1(x)-f(x)}{\alpha_1(x)-\alpha_2(x)} = \frac{(\sqrt{\Lambda}-x^2+5)\sqrt{\Lambda}}{2\Lambda}.$$

Figure 1 illustrates the asymptotic behavior of the sequence $\{D_n\}_{n\geq 0}$, given by

$$D_n = A_1\alpha_1^n + A_2\alpha_2^n,$$

for $f(x) = x^2 + 5$, $b = 1$, $n = 1, 2, \ldots, 20$, and $x = -1, -0.5, 0, 0.5, 1$.

2.2. Second case

Now consider $|f(x)| = 2\sqrt{b}$, in this case we have $\Lambda = 0$, our case two.

Lemma 2.6 *If $\Lambda = 0$ and β is the root of the characteristic Equation (2.2), then $\beta = \frac{f(x)}{2}$.*

As we have $\beta = \frac{f(x)}{2}$, so the (f,b)-determinant sequence $\{D_n\}_{n\geq 0}$ is given by:

$$D_n = (B_1 + B_2 n)\beta^n. \tag{2.5}$$

As $D_0 = 1$ and $D_1 = f(x)$, we obtain $B_1 = 1$ and $B_2 = 1$.

Under the previous discussion, the next result is given.

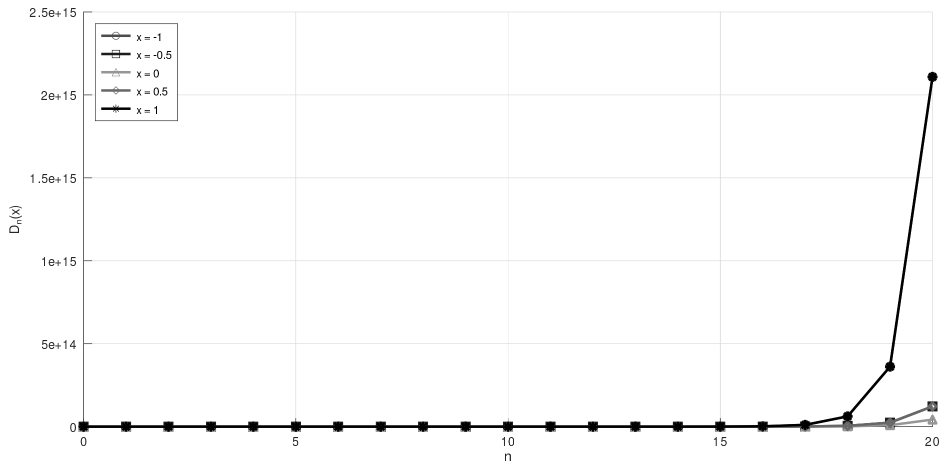

Figure 1: Solutions of $D_n = A_1\alpha_1^n + A_2\alpha_2^n$.

Proposition 2.7 (Second Binet's Formula) *Let n be a non-negative integer, b a fixed real number, $f(x)$ a real-valued function. If $\Lambda = 0$, then for $n \geq 0$, the following identity is valid:*

$$D_n = (1+n)\beta^n \,, \tag{2.6}$$

where β is a solution of (2.2).

In this case, the following equalities are obtained:

$$\begin{aligned} \beta + \beta &= f(x), \\ \beta\beta &= \left(\frac{f(x)}{2}\right)^2. \end{aligned}$$

Example 2.8 *Let $f(x) = 2$ be a real function, and set $b = 1$. The corresponding (f,b)-determinant sequence (D_n) is defined recursively by*

$$D_n = 2D_{n-1} - D_{n-2}, \quad \textit{for } n \geq 2,$$

with initial conditions $D_0 = 1$ and $D_1 = 2$. The associated characteristic equation is $r^2 - 2r + 1 = 0$, which has a discriminant $\Lambda = 2^2 - 4 = 0$, so the root $\beta = 1$. Since $\Lambda = 0$ for all $x \in \mathbb{R}$. Therefore, the sequence $\{D_n\}_{n\geq 0}$ can be expressed in closed form (Binet-type) solution $D_n = 1 + n$.

This example illustrates that, when the characteristic equation has a double real root, the (f,b)-determinant sequence becomes a linear sequence. In this case, all the determinants are strictly positive:

$$D_n = n + 1, \quad \text{for all } n \geq 0,$$

and the associated tridiagonal matrices are always invertible. In Figure 2, the strictly increasing behavior of D_n can be observed for $D_n = n + 1$ for all values of n and x.

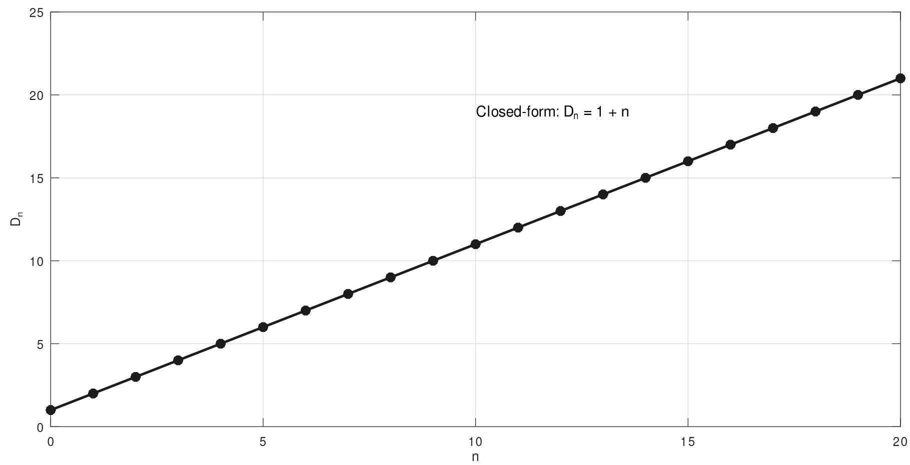

Figure 2: Solutions of $D_n = n + 1$

2.3. Third case

Finally, when $\Lambda < 0$, that is, in this case, $|f(x)| < 2\sqrt{b}$ and the roots γ_1 and γ_2 of the characteristic Equation (2.2) are complex conjugates.

Lemma 2.9 *Let γ_1 and γ_2 be the roots of the characteristic Equation (2.2). If $\Lambda < 0$, then*

$$\gamma_1 = \frac{f(x)}{2} + i\frac{\sqrt{-\Lambda}}{2} \quad \text{and} \quad \gamma_2 = \frac{f(x)}{2} - i\frac{\sqrt{-\Lambda}}{2}, \tag{2.7}$$

where $\Lambda = f(x)^2 - 4b$.

Let $\theta(x) \in (0, \pi)$ be defined by:

$$\theta(x) = \arccos\left(\frac{f(x)}{2\sqrt{b}}\right),$$

and note that the modulus of each root is $\sqrt{b}$. Using the initial conditions $D_0 = 1$ and $D_1 = f(x)$, as well as the fact that $\cos(\theta(x)) = \dfrac{f(x)}{2\sqrt{b}}$ and $\sin(\theta(x)) = \dfrac{\sqrt{4b - f(x)^2}}{2\sqrt{b}}$, the general real solution of the recurrence is given by the following result.

Proposition 2.10 *[Third Binet's Formula] Let n be a non-negative integer, b a fixed real number, $f(x)$ a real-valued function. If $\Lambda < 0$, then for $n \geq 0$, the following identity is valid:*

$$D_n = \left(\sqrt{b}\right)^n \left[\cos(n\theta(x)) + \frac{f(x)}{\sqrt{4b - f(x)^2}}\sin(n\theta(x))\right], \tag{2.8}$$

where $\cos(\theta(x)) = \dfrac{f(x)}{2\sqrt{b}}$ and $\sin(\theta(x)) = \dfrac{\sqrt{4b - f(x)^2}}{2\sqrt{b}}$.

Here is an example to illustrate the point.

Example 2.11 *Take the real function $f(x) = 2\cos(x)$, and let $b = 1$. The associated (f, b)-determinant sequence $\{D_n(x)\}$ satisfies the linear recurrence relation*

$$D_n = 2\cos(x)D_{n-1} - D_{n-2}, \quad \text{for } n \geq 2,$$

with initial terms $D_0(x) = 1$ *and* $D_1(x) = 2\cos(x)$. *The recurrence has the characteristic equation* $r^2 - [2\cos(x)]r + 1 = 0$, *with discriminant* $\Lambda = [2\cos(x)]^2 - 4 < 0$. *Therefore, the discriminant is negative for all* $x \in (0, \pi) \setminus \{\frac{\pi}{2}\}$, *implying complex conjugate roots. The roots can be expressed in polar form as*

$$\alpha_1 = e^{ix}, \qquad \gamma_2 = e^{-ix}\,.$$

Thus, the general real solution takes the form

$$D_n(x) = \cos(nx) + \cot(x)\sin(nx),$$

where the coefficient $\cot(x)$ *arises from the initial conditions* $D_0 = 1$ *and* $D_1 = 2\cos(x)$. *This solution is valid for* $x \in (0, \pi) \setminus \{k\pi\}$, *where* $\cot(x)$ *is defined.*

In Figure 3, the oscillatory behavior of D_n can be observed for $D_n = \cos(nx) + \cot(x)\sin(nx)$ for different values of n and x.

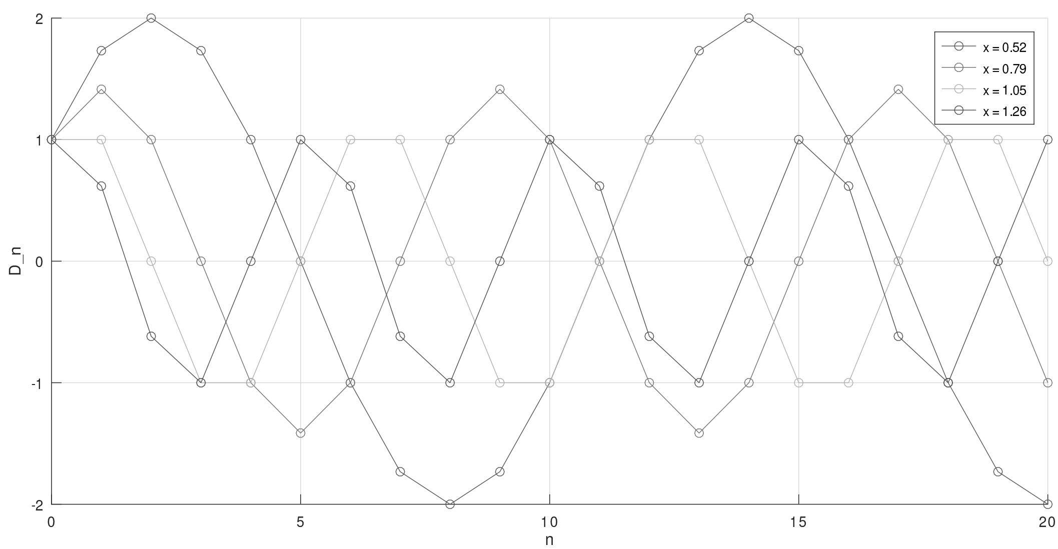

Figure 3: Solutions of $D_n = \cos(nx) + \cot(x)\sin(nx)$.

3. Some identities

In this section, we will provide some identities for the (f, b)-determinant sequence by considering the Binet formulas for the $\{D_n\}_{n\geq 0}$.

3.1. The case of $\Lambda > 0$

First, consider the Binet formula with $\Lambda > 0$, namely, Proposition 2.4. Applying the first Binet formula (Equation (2.4)) and by a direct calculation, we get:

Proposition 3.1 *Let* m *and* n *be natural numbers with* $m \geq n$, *and let* $\Lambda > 0$. *Then,*

(a) $D_m + D_n = A_1\alpha_1^n(\alpha_1^{m-n} + 1) + A_2\alpha_2^n(\alpha_2^{m-n} + 1)$,

(b) $D_m - D_n = A_1\alpha_1^n(\alpha_1^{m-n} - 1) + A_2\alpha_2^n(\alpha_2^{m-n} - 1)$,

where $\{D_n\}_{n\geq 0}$ *is the* (f, b)*-determinant sequence,* α_1 *and* α_2 *are given in Equation* (2.3) *and* $A_1 = \frac{f(x)-\alpha_2}{\alpha_1-\alpha_2}$ *and* $A_2 = \frac{\alpha_1 - f(x)}{\alpha_1-\alpha_2}$ *are real numbers.*

As a direct consequence of Propositions 3.1 we have the following results.

Corollary 3.2 *Let m, k and n be natural numbers with $m \geq n$, and let $\Lambda > 0$. Then the following identities hold:*

(a) $D_{m+k} + D_{n+k} = A_1\alpha_1^{n+k}(\alpha_1^{m-n} + 1) + A_2\alpha_2^{n+k}(\alpha_2^{m-n} + 1)$,

(b) $D_{m+k} - D_{n+k} = A_1\alpha_1^{n+k}(\alpha_1^{m-n} - 1) + A_2\alpha_2^{n+k}(\alpha_2^{m-n} - 1)$,

(c) $D_{2n} + D_n = A_1\alpha_1^{n}(\alpha_1^{n} + 1) + A_2\alpha_2^{n}(\alpha_2^{n} + 1)$,

(d) $D_{2n} - D_n = A_1\alpha_1^{n}(\alpha_1^{n} - 1) + A_2\alpha_2^{n}(\alpha_2^{n} - 1)$,

(e) $D_{n+1} + D_n = A_1\alpha_1^{n}(\alpha_1 + 1) + A_2\alpha_2^{n}(\alpha_2 + 1)$,

(f) $D_{n+1} - D_n = A_1\alpha_1^{n}(\alpha_1 - 1) + A_2\alpha_2^{n}(\alpha_2 - 1)$,

where $\{D_n\}_{n\geq 0}$ denotes the (f,b)-determinant sequence, and α_1 and α_2 are the characteristic roots defined in Equation (2.3). The coefficients A_1 and A_2 are real-valued functions given by $A_1 = \frac{f(x)-\alpha_2}{\alpha_1-\alpha_2}$ and $A_2 = \frac{\alpha_1-f(x)}{\alpha_1-\alpha_2}$.

Next, we establish the Tagiuri-Vajda identity for the $\{D_n\}_{n\geq 0}$ sequence.

Theorem 3.3 (Firt Tagiuri-Vajda's identity) *Let m, h and k be natural numbers, and let $\Lambda > 0$. Then,*

$$D_{m+h}D_{m+k} - D_mD_{m+h+k} = \frac{(\alpha_1\alpha_2)^{m+1}}{(\alpha_1-\alpha_2)^2}(\alpha_1^k - \alpha_2^k)(\alpha_1^h - \alpha_2^h)\,, \tag{3.1}$$

where $\{D_n\}_{n\geq 0}$ is the (f,b)-determinant sequence.

Proof: Using Equation (2.4), we have:

$$\begin{aligned} D_{m+h}D_{m+k} &= \left(A_1\alpha_1^{m+h} + A_2\alpha_2^{m+h}\right)\left(A_1\alpha_1^{m+k} + A_2\alpha_2^{m+k}\right) \\ &= A_1^2\alpha_1^{2m+h+k} + A_2^2\alpha_2^{2m+h+k} + A_1A_2(\alpha_1\alpha_2)^m\left(\alpha_1^h\alpha_2^k + \alpha_1^k\alpha_2^h\right), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} D_mD_{m+h+k} &= \left(A_1\alpha_1^{m} + A_2\alpha_2^{m}\right)\left(A_1\alpha_1^{m+h+k} + A_2\alpha_2^{m+h+k}\right) \\ &= A_1^2\alpha_1^{2m+h+k} + A_2^2\alpha_2^{2m+h+k} + A_1A_2(\alpha_1\alpha_2)^m\left(\alpha_1^{h+k} + \alpha_2^{h+k}\right). \end{aligned} \tag{3.3}$$

Subtracting (3.3) of the (3.2), we obtain:

$$\begin{aligned} D_{m+h}D_{m+k} - D_mD_{m+h+k} &= A_1A_2(\alpha_1\alpha_2)^m\left(\alpha_1^h\alpha_2^k + \alpha_1^k\alpha_2^h - \alpha_1^{h+k} - \alpha_2^{h+k}\right) \\ &= A_1A_2(\alpha_1\alpha_2)^m\left(\alpha_1^h(\alpha_2^k - \alpha_1^k) + \alpha_2^h(\alpha_1^k - \alpha_2^k)\right) \\ &= A_1A_2(\alpha_1\alpha_2)^m\left((\alpha_1^k - \alpha_2^k)(\alpha_2^h - \alpha_1^h)\right) \\ &= -\frac{(\alpha_1\alpha_2)^{m+1}}{(\alpha_1-\alpha_2)^2}\left((\alpha_1^k - \alpha_2^k)(\alpha_2^h - \alpha_1^h)\right), \end{aligned}$$

which completes the proof, since $A_1A_2 = -\dfrac{\alpha_1\alpha_2}{(\alpha_1-\alpha_2)^2}$. □

The following results, the d'Ocagne, Catalan, and Cassini identities, for the (f,b)-determinant sequence were established as a consequence of the Tagiuri-Vajda identity.

Proposition 3.4 (First D'Ocagne's identity) *Let m and $n \geq m$ be natural numbers, and let $\Lambda > 0$. Then the following identity holds:*

$$D_{m+1}D_n - D_mD_{n+1} = \frac{(\alpha_1\alpha_2)^{m+1}(\alpha_1^{n-m} - \alpha_2^{n-m})}{\alpha_1-\alpha_2},$$

where $\{D_n\}_{n\geq 0}$ denotes the (f,b)-determinant sequence.

Proof: Firstly, consider $k = n - m$ and $h = 1$ in Equation (3.1), and obtain the result. □

Proposition 3.5 (First Catalan's identity) *Let m and $n \leq m$ be natural numbers, and let $\Lambda > 0$. Then the following identity holds:*

$$(D_m)^2 - D_{m-n}D_{m+n} = \frac{(\alpha_1\alpha_2)^{m+1-n}(\alpha_1^n - \alpha_2^n)^2}{(\alpha_1 - \alpha_2)^2}, \tag{3.4}$$

where $\{D_n\}_{n\geq 0}$ denotes the (f,b)-determinant sequence.

Proof: Making $h = -n$ and $k = n$ in Equation (3.1). □

Now, by making $n = 1$ in Equation (3.4), and using $\alpha_1\alpha_2 = b$, we get the Cassini identity.

Proposition 3.6 (First Cassini's identity) *Let m be natural numbers and let $\Lambda > 0$. Then the following identity holds:*

$$(D_m)^2 - D_{m-1}D_{m+1} = b^m,$$

where $\{D_n\}_{n\geq 0}$ denotes the (f,b)-determinant sequence.

3.2. The case of $\Lambda = 0$

Now consider the Binet type formula with $\Lambda = 0$, as given in Proposition 2.7. By applying the second Binet formula (Equation (2.6)) and performing a straightforward computation, we obtain:

Proposition 3.7 *Let m and n be natural numbers with $m \geq n$, and let $\Lambda = 0$. Then,*

(a) $D_m + D_n = [(1+m)\beta^{m-n} + (1+n)]\beta^n$,

(b) $D_m - D_n = [(1+m)\beta^{m-n} - (1+n)]\beta^n$,

where $\{D_n\}_{n\geq 0}$ is the (f,b)-determinant sequence, and $\beta = \frac{f(x)}{2}$ is a real function.

As a direct consequence of Propositions 3.7, the following results are derived.

Corollary 3.8 *Let m, k and n be natural numbers such that $m \geq n$, and let $\Lambda = 0$. Then the following identities hold:*

(a) $D_{m+k} + D_{n+k} = [(1+m+k)\beta^{m-n} + (1+n+k)]\,\beta^{n+k}$,

(b) $D_{m+k} - D_{n+k} = [(1+m+k)\beta^{m-n} - (1+n+k)]\,\beta^{n+k}$,

(c) $D_{2n} + D_n = [(1+2n)\beta^n + (1+n)]\,\beta^n$,

(d) $D_{2n} - D_n = [(1+2n)\beta^n - (1+n)]\,\beta^n$,

(e) $D_{n+1} + D_n = [(2+n)\beta + (1+n)]\,\beta^n$,

(f) $D_{n+1} - D_n = [(2+n)\beta - (1+n)]\,\beta^n$,

where $\{D_n\}_{n\geq 0}$ denotes the (f,b)-determinant sequence, and $\beta = \frac{f(x)}{2}$ is a real function.

We now proceed to establish the second Tagiuri–Vajda identity for the (f,b)-determinant sequence.

Theorem 3.9 *Let m, h and k be natural numbers, and let $\Lambda = 0$. Then,*

$$D_{m+h}D_{m+k} - D_mD_{m+h+k} = hk\beta^{2m+h+k}\,,$$

where $\{D_n\}_{n\geq 0}$ is the (f,b)-determinant sequence.

Proof: Using Equation (2.6), we have

$$\begin{aligned} D_{m+h}D_{m+k} &= \big(1+1(m+h)\big)\beta^{m+h}\big(1+1(m+k)\big)\beta^{m+k} \\ &= \big(1^2+2m+1(h+k)+1^2(m+h)(m+k)\big)\beta^{2m+h+k} \\ &= \big(1+2m+(h+k)+(m+h)(m+k)\big)\beta^{2m+h+k} \\ &= \big((m+1)^2+k(m+1)+h(m+1)+hk\big)\beta^{2m+h+k}, \end{aligned} \tag{3.5}$$

and also,

$$\begin{aligned} D_mD_{m+h+k} &= \big(1+1(m)\big)\beta^{m}\big(1+1(m+h+k)\big)\beta^{m+h+k} \\ &= \big(1^2+1(m+h+k)+1m+1^2m(m+h+k)\big)\beta^{2m+h+k} \\ &= \big(1+(m+h+k)+m+m(m+h+k)\big)\beta^{2m+h+k} \\ &= \big((m+1)^2+h(m+1)+k(m+1)\big)\beta^{2m+h+k}. \end{aligned} \tag{3.6}$$

Subtracting Equations (3.6) of the (3.5),

$$D_{m+h}D_{m+k} - D_mD_{m+h+k} = hk\beta^{2m+h+k},$$

and we have the result. □

In a similar way that we have done with the Propositions 3.4, 3.5 and 3.6, we obtain the following results.

Proposition 3.10 (Second D'Ocagne's identity) *Let m and $n \geq m$ be natural numbers, and let $\Lambda = 0$. Then the following identity holds:*

$$D_{m+1}D_{n+1} - D_{m+1}D_n = (n-m)\beta^{m+n+1},$$

where $\{D_n\}_{n\geq 0}$ denotes the (f,b)-determinant sequence.

In the next two results, we also use the fact that $\beta = \frac{f(x)}{2}$.

Proposition 3.11 (Second Catalan's identity) *Let m and $n \leq m$ be natural numbers, and let $\Lambda = 0$. Then the following identity holds:*

$$(D_m)^2 - D_{m-n}D_{m+n} = n^2\left(\frac{f(x)}{2}\right)^{2m},$$

where $\{D_n\}_{n\geq 0}$ denotes the (f,b)-determinant sequence.

Proposition 3.12 (Second Cassini's identity) *Let m be natural numbers and let $\Lambda = 0$. Then the following identity holds:*

$$(D_m)^2 - D_{m-1}D_{m+1} = \left(\frac{f(x)}{2}\right)^{2m},$$

where $\{D_n\}_{n\geq 0}$ denotes the (f,b)-determinant sequence.

3.3. The case of $\Lambda < 0$

To prove the Theorem 3.14, we will need the following classics and well-known identities:

$$\cos\phi\cos\omega = \frac{1}{2}\big(\cos(\phi+\omega)+\cos(\phi-\omega)\big), \tag{3.7}$$

$$\sin\phi\sin\omega = \frac{1}{2}\big(\cos(\phi-\omega)-\cos(\phi+\omega)\big), \tag{3.8}$$

$$\sin\phi\cos\omega = \frac{1}{2}\big(\sin(\phi+\omega)+\sin(\phi-\omega)\big), \tag{3.9}$$

$$\cos\phi\sin\omega = \frac{1}{2}\big(\sin(\phi+\omega)+\sin(\omega-\phi)\big). \tag{3.10}$$

Now, consider the Binet type formula with $\Lambda < 0$, as presented in Proposition 2.10. Using the third Binet formula given in Equation (2.8), and after a direct computation, we arrive at the following result:

Proposition 3.13 *Let m and n be natural numbers with $m \geq n$, and let $\Lambda < 0$. Then,*

(a) $D_m + D_n = (\sqrt{b})^n \left((\sqrt{b})^{m-n} + 1\right)\left[\cos(n\theta(x)) + \frac{f(x)}{\sqrt{4b-f(x)^2}}\sin(n\theta(x))\right]$,

(b) $D_m - D_n = (\sqrt{b})^n \left((\sqrt{b})^{m-n} - 1\right)\left[\cos(n\theta(x)) + \frac{f(x)}{\sqrt{4b-f(x)^2}}\sin(n\theta(x))\right]$,

where $\{D_n\}_{n\geq 0}$ is the (f,b)-determinant sequence, and $\cos(\theta(x)) = \dfrac{f(x)}{2\sqrt{b}}$ *and* $\sin(\theta(x)) = \dfrac{\sqrt{4b-f(x)^2}}{2\sqrt{b}}$.

The next result is the third Tagiuri–Vajda identity for the (f,b)-determinant sequence.

Theorem 3.14 *Let m, h and k be natural numbers, and let $\Lambda < 0$. Then,*

$$D_{m+h}D_{m+k} - D_mD_{m+h+k} = \frac{1}{2}\big(\cos(h-k)\theta(x) - \cos(h+k)\theta(x)\big)\left(E^2+1\right)\left(\sqrt{b}\right)^{2m+h+k},$$

where $\{D_n\}_{n\geq 0}$ denotes the (f,b)-determinant sequence and $E = \dfrac{f(x)}{\sqrt{4b-f(x)^2}}$.

Proof: By use $E = \dfrac{f(x)}{\sqrt{4b-f(x)^2}}$ in Equation (2.8), we have

$$\begin{aligned} D_{m+h}D_{m+k} &= \left(\sqrt{b}\right)^{2m+h+k}\big(\cos(m+h)\theta(x) + E\sin(m+h)\theta(x)\big)\big(\cos(m+k)\theta(x) + E\sin(m+k)\theta(x)\big) \\ &= \left(\sqrt{b}\right)^{2m+h+k}\big(\cos(m+h)\theta(x)\cdot\cos(m+k)\theta(x) + E\cos(m+h)\theta(x)\cdot\sin(m+k)\theta(x) \\ &\quad + E\sin(m+h)\theta(x)\cdot\cos(m+k)\theta(x) + E^2\sin(m+h)\theta(x)\cdot\sin(m+k)\theta(x)\big). \end{aligned} \tag{3.11}$$

And also,

$$\begin{aligned} D_mD_{m+h+k} &= \left(\sqrt{b}\right)^{2m+h+k}\big(\cos(m)\theta(x) + E\sin(m)\theta(x)\big)\big(\cos(m+h+k)\theta(x) + E\sin(m+h+k)\theta(x)\big) \\ &= \left(\sqrt{b}\right)^{2m+h+k}\big(\cos(m)\theta(x)\cdot\cos(m+h+k)\theta(x) + E\cos(m)\theta(x)\cdot\sin(m+h+k)\theta(x) \\ &\quad + E\sin(m)\theta(x)\cdot\cos(m+h+k)\theta(x) + E^2\ \sin(m)\theta(x)\cdot\sin(m+h+k)\theta(x)\big). \end{aligned} \tag{3.12}$$

Using Equations (3.7)-(3.10) and subsequently subtracting Equations (3.11) and (3.12), we obtain

$$D_{m+h}D_{m+k} - D_mD_{m+h+k} = \frac{1}{2}\big(\cos(h-k)\theta(x) - \cos(h+k)\theta(x)\big)\left(E^2+1\right)\left(\sqrt{b}\right)^{2m+h+k},$$

which verifies the result. □

One more time, in a manner analogous to the approach taken in Propositions 3.4, 3.5, and 3.6, we proceed similarly below and derive the following results.

Proposition 3.15 (Third D'Ocagne's identity) *Let m and $n \geq m$ be natural numbers, and let $\Lambda < 0$. Then the following identity holds:*

$$D_{m+1}D_n - D_mD_{n+1} = \frac{1}{2}\big(\cos(m-n+1)\theta(x) - \cos(n-m+1)\theta(x)\big)\left(E^2+1\right)\left(\sqrt{b}\right)^{m+n+1},$$

where $\{D_n\}_{n\geq 0}$ denotes the (f,b)-determinant sequence.

Proposition 3.16 (Third Catalan's identity) *Let m and $n \leq m$ be natural numbers, and let $\Lambda < 0$. Then the following identity holds:*

$$(D_m)^2 - D_{m-n}D_{m+n} = \frac{1}{2}\Big(1 - \cos(2n\theta(x))\Big)(E^2+1)\left(\sqrt{b}\right)^{2m},$$

where $\{D_n\}_{n\geq 0}$ denotes the (f,b)-determinant sequence.

Proposition 3.17 (Third Cassini's identity) *Let m be natural numbers and let $\Lambda < 0$. Then the following identity holds:*

$$(D_m)^2 - D_{m-1}D_{m+1} = \frac{1}{2}\Big(1 - \cos(2\theta(x))\Big)(E^2+1)\left(\sqrt{b}\right)^{2m}$$

where $\{D_n\}_{n\geq 0}$ denotes the (f,b)-determinant sequence.

4. Final considerations

In this work, we analyzed the sequence of determinants $\{D_n\}_{n\geq 0}$, referred to as the (f,b)-determinant, which represents the determinant of a tridiagonal matrix of order n, whose main diagonal is defined by a function of the real variable x. Binet-type formulas were expressed and presented for the different cases of the discriminant of the associated characteristic equation of the recurrence, along with numerical examples that illustrate the application of the obtained results. Furthermore, we explored several classical identities related to the recurrence D_n, including the Tagiuri-Vajda, D'Ocagne, and Cassini formulas.

The motivation for this study lies in the applicability of tridiagonal matrices in numerical, physical, and engineering contexts, especially in the discretizations of differential equations. It is expected that the contributions presented here will serve as a foundation for future investigations involving tridiagonal matrices with similar structures or higher-order recurrences.

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