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Difference Compactness in Bitopological Spaces: Foundations from Difference Sets and Dual Views with Applications

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ABSTRACT: The notion of $D-$compactness in bitopological spaces, an inventive generalization that combines the structural variety of dual topologies with the adaptability of D-sets, is introduced and examined in this paper. Through a series of theorems and examples, we characterize pairwise $D-$compact spaces and establish basic properties of pairwise D-sets. A detailed analysis is conducted of the relationship between pairwise $D-$compactness and classical concepts like pairwise compactness and pairwise Hausdorff spaces. We investigate how pairwise $D-$compactness behaves under different topological operations. In addition to expanding the theoretical framework of bitopological spaces, this work lays the groundwork for applications in computational topology, Rough Set Theory, and multi-criteria decision making.

Key Words: Bitopological spaces, pairwise $D-$sets, $D-$covers, $D-$compactness, bitopological separation axioms, pairwise compact spaces, pairwise locally indiscrete spaces, pairwise continuous functions $D-$irresolute functions.

Contents

1. Introduction and Preliminaries

In the realm of general topology, the concept of compactness has long served as a cornerstone for analyzing the functional and structural properties of topological spaces. Its utility extends beyond foundational theory, offering critical insights into convergence, continuity, and other features. However, the transition to bitopological spaces provides more complex levels of spatial behaviors, that do not appear in single topological space.

Kelly's seminal work in 1963 [15] sparked ongoing interest in expanding this concept to bitopological spaces, yielding various generalizations including pairwise compactness [10,9,16,21], semi-compactness [12,14,2], pairwise local compactness [22,8,19], near pairwise compactness [20], local proper function of in bitopological spaces [7]. These constructs, making utilization of interactions between dual topologies. This gap is particularly apparent in the inadequate use of $D-$sets, introduced by Tong [23] as a difference of open sets. Initially intended to refine separation axioms (D_i, $s-D_i$, $pgpr-D_i$), $D-$sets have subsequently evolved into numerous instruments for exploring weaker topological hierarchies, from semi-open sets [17] to $b^\sharp-$open sets [24].

Recent studies, including Alrababah et al. [5] work on Difference Paracompactness, Atoom et al.'s studies on generalizations $[d,e]-$compactness spaces [6], and Mahmoud et al.'s investigations into soft bitopological $D-$sets [18], see also [3,4]. Yet, no systematic study of $D-$compactness has been carried out in bitopological spaces.

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This paper bridges that gap by introducing and analyzing pairwise $D-$compactness, a concept that combines dual topologies with the flexibility of $D-$covers. Our framework not only includes classical compactness to bitopological fields, but it also provides pathways for addressing unresolved issues in generalized topology and its applications.

The paper is laid out as follows: the following part of this section establishes foundational definitions and notations required for understanding the paper's contributions. Section 2 introduces the central concept of the paper about pairwise $D-$compactness and its properties. Section 3 explores the interplay between Pairwise D-Compactness and Separation axioms. whereas Section 4 examines Product spaces of Pairwise D-Compact Spaces. Finally, section 5 exploring how $D-$compactness in bitopological spaces can address challenges in various fields, in line with current trends in generalized topology.

The following definitions provides the essential mathematical foundation required for understanding the paper's contributions.

Definition 1.1 *[10] A cover $\hat{V}$ of $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ is termed pairwise open if $\hat{V} \subseteq \vartheta_1 \cup \vartheta_2$, and $\hat{V} \cap \vartheta_i \neq \emptyset$ for each $i = 1, 2$.*

If every pairwise open cover of $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ admits a finite subcover, then the space is termed pairwise compact.

Definition 1.2 *[13] A space $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ is termed pairwise T_1 if for any pair of distinct points $\mathfrak{z}$ and n, there exist a ϑ_1-open set D and a ϑ_2-open set F such that $\mathfrak{z} \in D$ but $n \notin D$, $n \in F$ but $\mathfrak{z} \notin F$.*

Definition 1.3 *[13] A space $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ is termed pairwise Hausdroff (or pairwise T_2) if for any two distinct points $\mathfrak{z}$ and n, there exist a ϑ_1-open set D and a ϑ_2-open set F such that $\mathfrak{z} \in D$ and $n \in F$, and $D \cap F = \emptyset$.*

Definition 1.4 *[10] A function $\Phi : (\mathfrak{Z}, \vartheta_1, \vartheta_2) \rightarrow (\mathfrak{N}, \beta_1, \beta_2)$ is termed pairwise continuous, if both $\Phi_1 \colon (\mathfrak{Z}, \vartheta_1) \rightarrow (\mathfrak{N}, \beta_1)$, and $\Phi_2 \colon (\mathfrak{Z}, \vartheta_2) \rightarrow (\mathfrak{N}, \beta_2)$ are continuous.*

Definition 1.5 *[10]: A function $\Phi \colon (\mathfrak{Z}, \vartheta_1, \vartheta_2) \rightarrow (\mathfrak{N}, \beta_1, \beta_2)$ is termed pairwise closed if both $\Phi_1 \colon (\mathfrak{Z}, \vartheta_1) \rightarrow (\mathfrak{N}, \beta_1)$, and $\Phi_2 \colon (\mathfrak{Z}, \vartheta_2) \rightarrow (\mathfrak{N}, \beta_2)$ are closed functions.*

As a result, if $\grave{P}_1$ is closed in ϑ_1, then $\Phi_1(\grave{P}_1)$ is closed in β_1; and if $\grave{P}_2$ is closed in ϑ_2, then $\Phi_2(\grave{P}_2)$ is closed in β_2.

Definition 1.6 *[13] A bitopological space $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ is termed pairwise locally compact if for every $\mathfrak{z} \in \mathfrak{Z}$, there either exists a ϑ_1-open neighborhood of $\mathfrak{z}$ whose ϑ_1-closure is pairwise compact, or a ϑ_2-open neighborhood of $\mathfrak{z}$ whose ϑ_2-closure is pairwise compact.*

Definition 1.7 *[11] A function $\Phi \colon (\mathfrak{Z}, \vartheta_1, \vartheta_2) \rightarrow (\mathfrak{N}, \beta_1, \beta_2)$ is termed a pairwise homomorphism if and only if both $\Phi_1 \colon (\mathfrak{Z}, \vartheta_1) \rightarrow (\mathfrak{N}, \beta_1)$, and $\Phi_2 \colon (\mathfrak{Z}, \vartheta_2) \rightarrow (\mathfrak{N}, \beta_2)$ are homomorphisms.*

Definition 1.8 *[4] A bitopological space $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ is termed a pairwise compact closed space if every ϑ_1-compact subset of $\mathfrak{Z}$ is ϑ_2-closed, every ϑ_2-compact subset of $\mathfrak{Z}$ is ϑ_1-closed.*

Definition 1.9 *[4] A function $\Phi \colon (\mathfrak{Z}, \vartheta_1, \vartheta_2) \rightarrow (\mathfrak{N}, \beta_1, \beta_2)$ is termed pairwise compact-preserving if for every pairwise compact subset $K \subseteq (\mathfrak{N}, \beta_1, \beta_2)$, the inverse image $\Phi^{-1}(K)$ is pairwise compact in $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$.*

Definition 1.10 *[4] A function $\Phi \colon (\mathfrak{Z}, \vartheta_1, \vartheta_2) \rightarrow (\mathfrak{N}, \beta_1, \beta_2)$ is termed a pairwise K-function if the inverse image of every pairwise compact subset of $(\mathfrak{N}, \beta_1, \beta_2)$ is pairwise compact in $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$; the image of every pairwise compact subset of $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ is pairwise compact in $(\mathfrak{N}, \beta_1, \beta_2)$.*

Definition 1.11 *[23] A subset $\grave{P}$ of a topological space $(\mathfrak{Z}, \vartheta)$ is termed a D-set if there exist open sets L and E in $(\mathfrak{Z}, \vartheta)$ such that $L \neq \mathfrak{Z}$, and $\grave{P} = L \setminus E$. In this case, $\grave{P}$ is described as a D-set generated by L and E. Notably, every open set $L \neq \mathfrak{Z}$ is trivially a D-set by taking $\grave{P} = L$ and $E = \emptyset$.*

2. A New Kind of Compactness: Characterizing Pairwise $D-$Compactness

This section introduces the central concept of the paper: pairwise D-sets and pairwise $D-$compactness in bitopological spaces. We define pairwise D-sets as subsets expressible as the difference $L \setminus E$, where L is a ϑ_1-open set ($L \neq \mathfrak{Z}$) and E is a ϑ_2-open set. Through examples and counterexamples, we explore the properties of these sets, including their behavior under set operations. Then, defines pairwise $D-$covers and pairwise $D-$compactness, establishing fundamental theorems about these concepts. We prove that all finite bitopological spaces are pairwise $D-$compact and examine the relationship between pairwise $D-$compactness and other compactness notions.

Definition 2.1 *In a bitopological space $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$, a subset $\grave{P}$ is termed a pairwise D-set if there exist a ϑ_1-open set L and a ϑ_2-open set E such that $L \neq \mathfrak{Z}$ and $\grave{P} = L \setminus E$. We say that $\grave{P}$ is generated by L and E. Note that every ϑ_1-open set $L \neq \mathfrak{Z}$ is a pairwise D-set by taking $E = \phi$.*

Example 2.2 *Consider the bitopological space $(\mathbb{R}, \vartheta_{std}, \vartheta_{ll})$, where ϑ_{std} is the standard topology on $\mathbb{R}$, and ϑ_{ll} is the lower limit topology.*

Let $L = (0, 2)$be a ϑ_{std}-open set, and $E = [1, 3)$ be a ϑ_{ll}-open set. Define the subset

$$\grave{P} = L \setminus E = (0, 2) \setminus [1, 3) = (0, 1).$$

Then $\grave{P} = (0, 1)$ is a pairwise D-set generated by L and E. Note that $L \neq \mathbb{R}$, satisfying the definition.

Remark 2.3 *A pairwise D-set need not be open in either ϑ_1 or ϑ_2, as it depends on the interplay between the two topologies. For example, $\grave{P} = (0, 1)$ is not ϑ_{ll}-open, since it cannot be written as a union of half-open intervals $[z, x)$.*

Proposition 2.4 *Consider $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ to be a bitopological space. Then: Every non-empty pairwise D-set $\grave{P} = L \setminus E$ satisfies $L \neq \emptyset$. If $\grave{P}_1 = L_1 \setminus E_1$ and $\grave{P}_2 = L_2 \setminus E_2$ are pairwise D-sets, then*

$$\grave{P}_1 \cap \grave{P}_2 = (L_1 \cap L_2) \setminus (E_1 \cup E_2)$$

is also a pairwise D-set, provided $L_1 \cap L_2 \neq \mathfrak{Z}$.

Example 2.5 *Consider in $(\mathbb{R}, \vartheta_{std}, \vartheta_{ll})$, $\grave{P}_1 = (0, 2) \setminus [1, 3) = (0, 1)$ and $\grave{P}_2 = (-1, 1.5) \setminus [0.5, 1) = (-1, 0.5) \cup [1, 1.5)$. Then*

$$\grave{P}_1 \cap \grave{P}_2 = (0, 0.5) = (0, 1.5) \setminus [0.5, 1.5),$$

which is a pairwise D-set generated by $L = (0, 1.5)$ and $E = [0.5, 1.5)$.

Any Finite union $\bigcup_{i=1}^{n} \grave{P}_i$ of pairwise D-sets $\grave{P}_i = L_i \setminus E_i$ are pairwise D-set if $\bigcup_i L_i \neq \mathfrak{Z}$.

Lemma 2.6 *In a pairwise Hausdorff bitopological space $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$, every $K \subseteq \mathfrak{Z}$ that is ϑ_1-compact, and ϑ_2-closed is a pairwise D-set.*

Example 2.7 *Consider the bitopological space $(\mathbb{R}, \vartheta_{std}, \vartheta_{ll})$. Let $K = [0, 1] \subset \mathbb{R}$. K is compact in ϑ_{std}, also K is ϑ_{ll}-closed. To illustrate: By the lemma, K is a pairwise D-set. To show that, let*

$$L = (-1, 2) \quad (\vartheta_{std}\text{-open}), \quad E = (-1, 0) \cup (1, 2) \quad (\vartheta_{ll}\text{-open}),$$

so that:

$$K = L \setminus E = [0, 1].$$

Conversely:

Remark 2.8 *If $\grave{P} \subset \mathfrak{Z}$ is a pairwise D-set, then there is a ϑ_1-open set L and a ϑ_2-open set E such that $\grave{P} = L \setminus E$, $L \neq \mathfrak{Z}$, and $\grave{P}$ satisfies additional topological properties (e.g., $\grave{P}$ is ϑ_1-compact and ϑ_2-closed).*

The following counterexample demonstrates this distinction.

Example 2.9 *Consider $(\mathbb{R}, \vartheta_{std}, \vartheta_{ll})$. Let $\grave{P} = (0,1) \subset \mathbb{R}$. In Example 2.2, we showed $\grave{P}$ is a pairwise D-set generated by $L = (0,2)$ (ϑ_{std}-open) and $E = [1,3)$ (ϑ_{ll}-open). However, $\grave{P}$ is not ϑ_{std}-compact. Also, it is not ϑ_{ll}-closed. Thus, $\grave{P}$ is a pairwise D-set that is neither ϑ_1-compact nor ϑ_2-closed, disproving the converse.*

Example 2.10 *Consider the bitopological space $(\mathfrak{Z} \times \mathfrak{N}, \vartheta_1 \times \vartheta_2, \vartheta_{cof} \times \vartheta_{cof})$, where $\mathfrak{Z} = \mathfrak{N} = \{z, x, a\}$ with $\vartheta_1 = \vartheta_2 = \{\emptyset, \mathfrak{Z}, \{z\}, \{z, x\}\}$.*

Case 1: *Consider $\mathfrak{Z} \times \mathfrak{N}$ with the product topology $\vartheta_1 \times \vartheta_2$. Define $L = \{z\} \times \{z, x\}$ ($\vartheta_1 \times \vartheta_2$-open), and $E = \{z\} \times \{z\}$ ($\vartheta_1 \times \vartheta_2$-open).*

Define the pairwise D-set:

$$\grave{P} = L \setminus E = \{(z, x)\}.$$

Here, $\grave{P}$ is a pairwise D-set but not $\vartheta_1 \times \vartheta_2$-open, as it cannot be written as a product of open sets from ϑ_1 and ϑ_2.

Case 2: *$\mathbb{R}^2$ with Cofinite Product Topology $\vartheta_{cof} \times \vartheta_{cof}$. Define $L = \mathbb{R}^2 \setminus \{(t, y)\}$ ($\vartheta_{cof} \times \vartheta_{cof}$-open), and $E = \mathbb{R}^2 \setminus \{(t, y), (z, x)\}$ ($\vartheta_{cof} \times \vartheta_{cof}$-open).*

Define the pairwise D-set:

$$\grave{P} = L \setminus E = \{(z, x)\}.$$

Here, $\grave{P}$ is a pairwise D-set but not open in $\vartheta_{cof} \times \vartheta_{cof}$, as singletons are closed in the cofinite topology.

Proposition 2.11 *The converse of the pairwise D-set definition does not hold universally. Specifically: A pairwise D-set need not be ϑ_1-compact or ϑ_2-closed. and a pairwise D-set need not be open in ϑ_1 or ϑ_2.*

Proof. Consider the pairwise D-set $\grave{P} = (0,1)$ in $(\mathbb{R}, \vartheta_{\text{std}}, \vartheta_{\text{ll}})$ suffices: $\grave{P}$ is ϑ_{std}-open but not ϑ_{std}-compact. $\grave{P}$ is not ϑ_{ll}-closed. And it is not ϑ_{ll}-open.

Thus, no additional topological properties beyond the definition are guaranteed for pairwise D-sets. ■

Definition 2.12 *In a bitopological space $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$, a cover $\widetilde{D} = \{D_\iota : \iota \in \Lambda\}$ is termed a pairwise D-cover if each D_ι is a pairwise D-set. Specifically,*

$$D_\iota = L_\iota \setminus E_\iota, \quad \text{where } L_\iota \in \vartheta_1,\ E_\iota \in \vartheta_2, \text{ and } L_\iota \neq \mathfrak{Z}.$$

for all $\iota \in \Lambda$.

While every ϑ_1-open cover or ϑ_2-open cover is a pairwise D-cover, the converse fails. For instance, in $(\mathbb{R}, \vartheta_{std}, \vartheta_{cof})$, the collection $\{\{\mathfrak{z}\} : \mathfrak{z} \in \mathbb{R}\}$ is a pairwise D-cover but not a ϑ_{std}- or ϑ_{cof}-open cover.

Remark 2.13 *A pairwise D-cover may involve sets from both ϑ_1 and ϑ_2, but it need not contain open sets from either topology.*

Remark 2.14 *In a pairwise compact space, a pairwise D-cover need not admit a finite subcover unless additional constraints are imposed.*

Proposition 2.15 *Let $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ be a bitopological space. If $\widetilde{D} = \{D_\iota\}_{\iota \in \Lambda}$ is a pairwise D-cover of $\mathfrak{Z}$, then*

$$\bigcup_{\iota \in \Lambda} D_\iota = \mathfrak{Z}.$$

However, the intersection $D_\iota \cap D_\beta$ of two pairwise D-sets need not cover any open subset.

Proposition 2.16 *The refinement of a pairwise D-cover by other pairwise D-sets remains a pairwise D-cover.*

Lemma 2.17 *In a pairwise T_1-space $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$, every pairwise D-cover consisting of singleton sets $\{\mathfrak{z}\}$, where $\mathfrak{z} \in \mathfrak{Z}$ is also a pairwise D-cover in the cofinite topology ϑ_{cof} paired with any other topology.*

Example 2.18 *Consider the bitopological space $(\mathbb{R}^2, \vartheta_{std} \times \vartheta_{std}, \vartheta_{cof} \times \vartheta_{cof})$. Define $L_\iota = \mathbb{R}^2 \setminus \{(\mathfrak{z}_\iota, y_\iota)\}$ ($\vartheta_{cof} \times \vartheta_{cof}$-open), and $E_\iota = \emptyset$ ($\vartheta_{std} \times \vartheta_{std}$-open).*
Then $D_\iota = L_\iota \setminus E_\iota = \mathbb{R}^2 \setminus \{(\mathfrak{z}_\iota, y_\iota)\}$. The collection $\{D_\iota\}$ covers $\mathbb{R}^2$ but is not a $\vartheta_{std} \times \vartheta_{std}$-open cover.

Definition 2.19 *A bitopological space $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ is pairwise D-compact if every pairwise D-cover of $\mathfrak{Z}$ has a finite subcover.*

Remark 2.20 *Pairwise D-compactness is strictly weaker than pairwise compactness.*

Remark 2.21 *A space can be pairwise D-compact even if neither $(\mathfrak{Z}, \vartheta_1)$ nor $(\mathfrak{Z}, \vartheta_2)$ is D-compact individually.*

Proposition 2.22 *Let $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ be pairwise D-compact and pairwise Hausdorff space. Then every ϑ_1-closed and ϑ_2-compact subset is pairwise D-compact.*

Proposition 2.23 *Finite topological sums of pairwise D-compact spaces are pairwise D-compact.*

Lemma 2.24 *In a pairwise D-compact space $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$, every pairwise D-set is ϑ_1-compact and ϑ_2-closed.*

Example 2.25 *Let $\mathfrak{Z} = \mathbb{R}$, $\vartheta_1 = \{\emptyset, \mathbb{R}, \{1\}, \mathbb{R} \setminus \{1\}\}$, and $\vartheta_2 = \vartheta_{cof}$. Then, all pairwise D-sets in ϑ_1 are finite or cofinite. And any pairwise D-cover must include $\mathbb{R} \setminus \{1\}$ and finitely many other sets. Hence, $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ is pairwise D-compact.*

Example 2.26 *Let $\mathfrak{Z} = \mathbb{R}$, $\vartheta_1 = \vartheta_{std}$, and $\vartheta_2 = \vartheta_{discrete}$. For $n \in \mathbb{N}$, define:*

$$D_n = (-n, n) \setminus \{0\},$$

where $(-n, n)$ is ϑ_1-open and $\{0\}$ is ϑ_2-closed. The collection $\widetilde{D} = \{D_n : n \in \mathbb{N}\}$ is a pairwise D-cover with no finite subcover, as

$$(\bigcup_{i=1}^{k} D_{n_i} = (-N, N) \setminus \{0\} \neq \mathbb{R}$$

for any $N \in \mathbb{N}$.

Theorem 2.27 *If $\mathfrak{Z}$ is a finite set, then $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ is pairwise D-compact for any bitopology $(\vartheta_1, \vartheta_2)$ on $\mathfrak{Z}$.*

Proof. Consider $\mathfrak{Z} = \{\mathfrak{z}_1, \mathfrak{z}_2, \ldots, \mathfrak{z}_n\}$ to be finite. And let $\widetilde{D} = \{D_\iota : \iota \in \Lambda\}$ be a pairwise D-cover, where every $D_\iota = L_\iota \setminus E_\iota$ with $L_\iota \in \vartheta_1$, $E_\iota \in \vartheta_2$, and $L_\iota \neq \mathfrak{Z}$. For every $\mathfrak{z}_i \in \mathfrak{Z}$, select $D_i \in \widetilde{D}$ such that $\mathfrak{z}_i \in D_i$. The collection $\{D_1, D_2, \ldots, D_n\}$ is a finite subcover of $\widetilde{D}$. Thus, $\mathfrak{Z}$ is pairwise D-compact. ∎

Remark 2.28 *The intersection of any two pairwise D-sets is a pairwise D-set.*

Corollary 2.29 *In any bitopological space $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$, the finite intersection of pairwise D-sets is a pairwise D-set.*

Proof. By induction; for two pairwise D-sets $D_1 = L_1 \setminus E_1$ and $D_2 = L_2 \setminus E_2$, their intersection is:

$$D_1 \cap D_2 = (L_1 \cap L_2) \setminus (E_1 \cup E_2),$$

where $L_1 \cap L_2 \in \vartheta_1$, $E_1 \cup E_2 \in \vartheta_2$, and $L_1 \cap L_2 \neq \mathfrak{Z}$.

Now, assume the intersection of n pairwise D-sets is a pairwise D-set. For $n+1$, the intersection with an additional pairwise D-set follows similarly. Thus, finite intersections preserve pairwise D-sets. ∎

Remark 2.30 *The union of pairwise D-sets may not be a pairwise D-set.*

Example 2.31 *Consider* $\mathfrak{Z} = \{z, x, a\}$ *with* $\vartheta_1 = \{\emptyset, \mathfrak{Z}, \{z\}, \{a\}, \{z, a\}\}$, *and* $\vartheta_2 = \{\emptyset, \mathfrak{Z}, \{x\}, \{a\}, \{x, a\}\}$. *Define:*

$$D_1 = \{z\} = \{z, a\} \setminus \{x, a\} \quad (\vartheta_1\text{-open} \setminus \vartheta_2\text{-open}),$$

$$D_2 = \{x\} = \{x, a\} \setminus \{a\} \quad (\vartheta_2\text{-open} \setminus \vartheta_1\text{-open}).$$

Both D_1 *and* D_2 *are pairwise D-sets. However:*

$$D_1 \cup D_2 = \{z, x\},$$

which is not a pairwise D-set, as no $L \in \vartheta_1$ *or* $E \in \vartheta_2$ *satisfies* $L \setminus E = \{z, x\}$ *with* $L \neq \mathfrak{Z}$.

Definition 2.32 *A bitopological space* $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ *is called pairwise locally indiscrete if every* ϑ_1*-open set is* ϑ_1*-clopen and every* ϑ_2*-open set is* ϑ_2*-clopen.*

Example 2.33 *Let* $\mathfrak{Z} = \{1, 2, 3\}$ *with*

$$\vartheta_1 = \{\mathfrak{Z}, \emptyset, \{1\}, \{2, 3\}\}, \quad \vartheta_2 = \{\mathfrak{Z}, \emptyset, \{2\}, \{1, 3\}\}.$$

Here, $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ *is pairwise locally indiscrete because* ϑ_1*-open sets* $\mathfrak{Z}, \emptyset, \{1\}, \{2, 3\}$ *are* ϑ_1*-closed, and* ϑ_2*-open sets* $\mathfrak{Z}, \emptyset, \{2\}, \{1, 3\}$ *are* ϑ_2*-closed.*
However, the space is not pairwise discrete, as $\{3\}$ *is not open in* ϑ_1 *or* ϑ_2.

Proposition 2.34 *Every pairwise discrete space (where all sets are* ϑ_1*-open and* ϑ_2*-open) is pairwise locally indiscrete, but the converse fails.*

Proof. In a pairwise discrete space, all sets are clopen in both ϑ_1 and ϑ_2, satisfying pairwise local indiscreteness. The converse fails as shown in the example above, where $\{3\}$ is not open. ∎

Remark 2.35 *In a pairwise locally indiscrete space, every pairwise D-set* $D = L \setminus E$ *is* ϑ_1*-clopen.*

Corollary 2.36 *In a pairwise locally indiscrete space, the union of finitely many pairwise D-sets is a pairwise D-set.*

Proof. let $\widetilde{D} = \{D_\iota = L_\iota \setminus E_\iota : \iota \in \Lambda\}$, where $L_\iota \in \vartheta_1$ and $E_\iota \in \vartheta_2$. Define $L = \bigcup_\iota L_\iota$ and $E = \bigcap_\iota E_\iota$. Then

$$\bigcup_\iota D_\iota = \bigcup_\iota (L_\iota \setminus E_\iota) = L \setminus E,$$

which is a pairwise D-set since $L \in \vartheta_1$ and $E \in \vartheta_2$. ∎

Theorem 2.37 *Every pairwise D-compact bitopological space* $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ *is pairwise compact.*

Proof. Let $\widetilde{L} = \{L_\iota : \iota \in \Lambda\}$ be a pairwise open cover of $\mathfrak{Z}$, where every L_ι belongs to ϑ_1 or ϑ_2. Since every pairwise open set is a pairwise D-set (by taking $E = \emptyset$), $\widetilde{L}$ is also a pairwise D-cover. By pairwise D-compactness, $\widetilde{L}$ has a finite subcover. Hence, $\mathfrak{Z}$ is pairwise compact. ∎

Example 2.38 *Let* $\mathfrak{Z} = \mathbb{R}$, $\vartheta_1 = \vartheta_{cof}$, *and* $\vartheta_2 = \vartheta_{cof}$. *Then, any pairwise open cover has a finite subcover.*
Define pairwise D-sets $D_{\mathfrak{z}} = (\mathbb{R} \setminus \{y\}) \setminus (\mathbb{R} \setminus \{\mathfrak{z}, y\}) = \{\mathfrak{z}\}$, *where* $L = \mathbb{R} \setminus \{y\} \in \vartheta_1$, $E = \mathbb{R} \setminus \{\mathfrak{z}, y\} \in \vartheta_2$, *and* $L \neq \mathfrak{Z}$.
The collection $\widetilde{D} = \{\{\mathfrak{z}\} : \mathfrak{z} \in \mathbb{R}\}$ *is a pairwise D-cover with no finite subcover, contradicting pairwise D-compactness.*

Example 2.39 *Let* $\mathfrak{Z} = \mathbb{R}$, $\vartheta_1 = \vartheta_{std}$, *and* $\vartheta_2 = \vartheta_{dis}$. *The* ϑ_2*-open cover* $\{\{\mathfrak{z}\} : \mathfrak{z} \in \mathbb{R}\}$ *has no finite subcover.* $\mathfrak{Z}$ *is not pairwise compact and therefore not pairwise D-compact.*

Lemma 2.40 *In a pairwise D-compact space, every pairwise D-cover has a finite subcover. In a pairwise compact space, every pairwise open cover has a finite subcover.*

Proposition 2.41 *Pairwise D-compactness and pairwise compactness are distinct: Pairwise D-compactness $\Rightarrow$ pairwise compactness. And pairwise compactness $\not\Rightarrow$ pairwise D-compactness.*

Proof. The forward direction follows from the theorem 2.37. The converse failure is validated by Example 2.38, where the space is pairwise compact but not pairwise D-compact. ■

Theorem 2.42 *Every pairwise locally indiscrete and pairwise compact bitopological space $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ is pairwise D-compact.*

Proof. Consider $\widetilde{D} = \{D_\iota : \iota \in \Lambda\}$ to be a pairwise D-cover of $\mathfrak{Z}$, where every $D_\iota = L_\iota \setminus E_\iota$ with $L_\iota \in \vartheta_1$, $E_\iota \in \vartheta_2$, and $L_\iota \neq \mathfrak{Z}$.

In a pairwise locally indiscrete space: L_ι is ϑ_1-clopen, E_ι is ϑ_2-clopen.
Thus, $D_\iota = L_\iota \setminus E_\iota = L_\iota \cap E_\iota^c$ is ϑ_1-clopen. The collection $\{D_\iota\}$ forms a ϑ_1-clopen cover. By pairwise compactness, it has a finite subcover $\{D_{\iota_1}, \ldots, D_{\iota_n}\}$. Hence, $\mathfrak{Z}$ is pairwise D-compact. ■

Example 2.43 *Let $\mathfrak{Z} = \mathbb{R}$, with $\vartheta_1 = \vartheta_2 = \{\emptyset, \mathbb{R}\}$. The space is Pairwise locally indiscrete, Pairwise compact, Hence, pairwise D-compact.*

Example 2.44 *Let $\mathfrak{Z} = \mathbb{R}$, $\vartheta_1 = \{\emptyset, \mathfrak{Z}, \mathfrak{Z}\setminus\{2\}, \{2\}\}$, and $\vartheta_2 = \{\emptyset, \mathfrak{Z}, \mathfrak{Z}\setminus\{3\}, \{3\}\}$. Then, ϑ_1-open sets are ϑ_1-clopen, and ϑ_2-open sets are ϑ_2-clopen, any pairwise open cover must include $\mathfrak{Z}$ or finitely many clopen sets, Thus, $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ is pairwise compact and pairwise locally indiscrete, hence pairwise D-compact.*

Lemma 2.45 *In a pairwise locally indiscrete space, every pairwise D-cover is a ϑ_1-clopen cover and a ϑ_2-clopen cover.*

Proof. For $D_\iota = L_\iota \setminus E_\iota$: L_ι is ϑ_1-clopen and E_ι is ϑ_2-clopen, D_ι is ϑ_1-clopen (as L_ι is closed in ϑ_1 and E_ι is closed in ϑ_1, hence $L_\iota \setminus E_\iota$ is ϑ_1-clopen).

Similarly, D_ι is ϑ_2-clopen. ■

Proposition 2.46 *If $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ is pairwise locally indiscrete and pairwise compact, then every pairwise D-set is ϑ_1-clopen and ϑ_2-clopen. Also, the space is pairwise zero-dimensional.*

Theorem 2.47 *Let $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ be a bitopological space and $\grave{P} \subseteq \mathfrak{Z}$. If D_ι is a pairwise D-set in $\mathfrak{Z}$, then $D_\iota \cap \grave{P}$ is a pairwise D-set in the subspace $(\grave{P}, \vartheta_{1_{\grave{P}}}, \vartheta_{2_{\grave{P}}})$, where $\vartheta_{1_{\grave{P}}}$ and $\vartheta_{2_{\grave{P}}}$ are the induced bitopologies on $\grave{P}$.*

Proof. Consider $D_\iota = L \setminus E$ to be a pairwise D-set in $\mathfrak{Z}$, where $L \in \vartheta_1$, $E \in \vartheta_2$, and $L \neq \mathfrak{Z}$. Then:

$$D_\iota \cap \grave{P} = (L \setminus E) \cap \grave{P} = (L \cap \grave{P}) \setminus (E \cap \grave{P}).$$

Since $L \cap \grave{P}$ is $\vartheta_{1_{\grave{P}}}$-open and $E \cap \grave{P}$ is $\vartheta_{2_{\grave{P}}}$-open, $D_\iota \cap \grave{P}$ is a pairwise D-set in $(\grave{P}, \vartheta_{1_{\grave{P}}}, \vartheta_{2_{\grave{P}}})$. ■

Remark 2.48 *Pairwise D-sets are preserved under intersection with subspaces only when the ϑ_1-open set L satisfies $L \cap \grave{P} \neq \grave{P}$.*

Example 2.49 *Consider $\mathfrak{Z} = \mathbb{R}$, with $\vartheta_1 = \vartheta_{std}$, $\vartheta_2 = \vartheta_{cof}$, and $\grave{P} = [0, 1]$. Define*

$$D = (0, 2) \setminus \{1\} \quad (\vartheta_1\textit{-open} \setminus \vartheta_2\textit{-closed}).$$

Then,

$$D \cap \grave{P} = (0, 1) \setminus \{1\} = (0, 1),$$

which is a pairwise D-set in $(\grave{P}, \vartheta_{1_{\grave{P}}}, \vartheta_{2_{\grave{P}}})$. Here, $L \cap \grave{P} = (0, 1) \neq \grave{P}$, satisfying the critical condition.

Lemma 2.50 *Let $\grave{P} \subseteq \mathfrak{Z}$. If $L \in \vartheta_1$ and $L \cap \grave{P} \neq \grave{P}$, then $D \cap \grave{P}$ is a valid pairwise D-set in $(\grave{P}, \vartheta_{1_{\grave{P}}}, \vartheta_{2_{\grave{P}}})$.*

Proposition 2.51 *Pairwise D-sets are stable under subspace inclusion if the ϑ_1-open set L satisfies $L \cap \grave{P} \neq \grave{P}$.*

Theorem 2.52 *Let $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ be a pairwise D-compact bitopological space. Then every ϑ_1-closed subspace $\grave{P} \subseteq \mathfrak{Z}$ is pairwise D-compact.*

Proof. Let $\grave{P} \subseteq \mathfrak{Z}$ be ϑ_1-closed. Consider $\widetilde{D} = \{D_\iota : \iota \in \Lambda\}$ be a pairwise D-cover of $\grave{P}$, where $D_\iota = L_\iota \setminus E_\iota$ with $L_\iota \in \vartheta_1$, $E_\iota \in \vartheta_2$, and $L_\iota \neq \mathfrak{Z}$.

Since $\grave{P}$ is ϑ_1-closed, $\mathfrak{Z} \setminus \grave{P}$ is ϑ_1-open. Define

$$D_0 = \mathfrak{Z} \setminus \grave{P} = L_0 \setminus \emptyset \quad \text{where } L_0 = \mathfrak{Z} \setminus \grave{P} \in \vartheta_1.$$

Then $\widetilde{D} \cup \{D_0\}$ is a pairwise D-cover of $\mathfrak{Z}$. By pairwise D-compactness, there exists a finite subcover $\{D_1, \ldots, D_n, D_0\}$. Removing D_0, the remaining sets $\{D_1, \ldots, D_n\}$ covers $\grave{P}$. Hence, $\grave{P}$ is pairwise D-compact. ■

Lemma 2.53 *In a pairwise D-compact space, the complement of any closed set is a pairwise D-set.*

Proof. For $\grave{P} \subseteq \mathfrak{Z}$ ϑ_1-closed, $\mathfrak{Z} \setminus \grave{P}$ is ϑ_1-open. Then:

$$\mathfrak{Z} \setminus \grave{P} = L \setminus \emptyset \quad \text{where } L = \mathfrak{Z} \setminus \grave{P} \in \vartheta_1 \text{ and } L \neq \mathfrak{Z}.$$

Hence, $\mathfrak{Z} \setminus \grave{P}$ is a pairwise D-set. ■

Proposition 2.54 *Pairwise D-compactness is hereditary with respect to closed subspaces.*

Theorem 2.55 *Let $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ be a pairwise D-compact bitopological space. Then every ϑ_1-closed subspace $\grave{P} \subseteq \mathfrak{Z}$ is pairwise compact.*

Proof. Let $\grave{P} \subseteq \mathfrak{Z}$ be ϑ_1-closed. Consider a pairwise open cover

$$\widetilde{L} = \{L_\iota : \iota \in \Lambda\}$$

of $\grave{P}$, where each L_ι is open in either $\vartheta_{1_{\grave{P}}}$ or $\vartheta_{2_{\grave{P}}}$. Extend $\widetilde{L}$ to a pairwise open cover of $\mathfrak{Z}$ by adding $\mathfrak{Z} \setminus \grave{P}$, which is ϑ_1-open.

Since $\mathfrak{Z}$ is pairwise D-compact, there exists a finite subcover $\widetilde{L}^* = \{L_1, \ldots, L_n, \mathfrak{Z} \setminus \grave{P}\}$. Removing $\mathfrak{Z} \setminus \grave{P}$, the collection $\{L_1, \ldots, L_n\}$ covers $\grave{P}$. Hence, $\grave{P}$ is pairwise compact. ■

Example 2.56 *Let $\mathfrak{Z} = [0, 5]$ with $\vartheta_1 = \vartheta_{std}$, $\vartheta_2 = \vartheta_{cof}$. Then $\mathfrak{Z}$ is pairwise D-compact. The subspace $\grave{P} = [0, 1]$ is pairwise compact: Any pairwise open cover of $\grave{P}$ has a finite subcover.*

Lemma 2.57 *In a pairwise D-compact space $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$, every closed subspaces inherit pairwise compactness.*

Proposition 2.58 *Pairwise compactness is hereditary with respect to closed subspaces in pairwise D-compact spaces.*

3. The Geometry of Pairwise $D-$Compactness: Exploring Separation Properties

This section examines the interaction between pairwise $D-$compactness and separation axioms, showing that this property is compatible with weaker separation axioms even when stronger ones fail.

Definition 3.1 *Let $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ be a bitopological space.*

1. *$\mathfrak{Z}$ is pairwise D_0 if for any two distinct points $\mathfrak{z}, \mathfrak{n} \in \mathfrak{Z}$, there exists a pairwise D-set $D_{x\mathfrak{n}}$ such that either $\mathfrak{z} \in D_{\mathfrak{z}\mathfrak{n}}$ and $\mathfrak{n} \notin D_{\mathfrak{z}\mathfrak{n}}$, or $\mathfrak{n} \in D_{\mathfrak{z}\mathfrak{n}}$ and $\mathfrak{z} \notin D_{\mathfrak{z}\mathfrak{n}}$.*

2. $\mathfrak{Z}$ *is pairwise* D_1 *if for any two distinct points* $\mathfrak{z}, \mathfrak{n} \in \mathfrak{Z}$, *there exist pairwise* D*-sets* G *and* H *such that*
$$\mathfrak{z} \in G \setminus H \quad \text{and} \quad \mathfrak{n} \in H \setminus G.$$

3. $\mathfrak{Z}$ *is pairwise* D_2 *if for any two distinct points* $\mathfrak{z}, \mathfrak{n} \in \mathfrak{z}$, *there exist disjoint pairwise* D*-sets* G *and* H *such that*
$$\mathfrak{z} \in G \quad \text{and} \quad \mathfrak{n} \in H.$$

Remark 3.2 *Pairwise* D_0 *generalizes the classical* T_0 *axiom using pairwise* D*-sets instead of open sets.*

Remark 3.3 *Pairwise* D_1 *spaces satisfy pairwise* D_0, *but the converse fails (Example 3.5).*

Remark 3.4 *Pairwise* D_2 *implies pairwise* D_1, *which implies pairwise* D_0.

Example 3.5 *Consider* $\mathfrak{Z} = \{z, x, a\}$, *with* $\vartheta_1 = \{\emptyset, \mathfrak{Z}, \{z\}, \{x\}\}$, *and* $\vartheta_2 = \{\emptyset, \mathfrak{Z}, \{a\}\}$. *Pairwise* D*-sets;* $\{z\} = \{z\} \setminus \emptyset$, $\{x\} = \{x\} \setminus \emptyset$, $\{a\} = \mathfrak{Z} \setminus \{z, x\}$.
$\mathfrak{Z}$ *is pairwise* D_0; *for* z *and* x, *use* $D = \{z\}$; *for* z *and* a, *use* $D = \{z\}$; *for* x *and* a, *use* $D = \{x\}$.
But $\mathfrak{Z}$ *is not pairwise* D_1; *No* D*-sets* G, H *exist to separate* a *from* z *or* x.

Example 3.6 *Consider* $\mathfrak{Z} = \{z, x\}$, *with* $\vartheta_1 = \{\emptyset, \mathfrak{Z}, \{z\}\}$, $\vartheta_2 = \{\emptyset, \mathfrak{Z}, \{x\}\}$.
Pairwise D*-sets;* $\{z\} = \{z\} \setminus \emptyset$, $\{x\} = \{x\} \setminus \emptyset$.
$\mathfrak{Z}$ *is pairwise* D_1: *For* z *and* x, $G = \{z\}$, $H = \{x\}$ *satisfy* $z \in G \setminus H$ *and* $x \in H \setminus G$. *But* $\mathfrak{Z}$ *is not pairwise* D_2; *The* D*-sets* $\{z\}$ *and* $\{x\}$ *intersect since* $\mathfrak{Z}$ *has only two points.*

Example 3.7 *Consider* $\mathfrak{Z} = \mathbb{R}$, *with* $\vartheta_1 = \vartheta_{std}$, $\vartheta_2 = \vartheta_{ll}$. *For distinct* $\mathfrak{z}, \mathfrak{n} \in \mathbb{R}$, *let*
$$G = (\mathfrak{z} - \epsilon, \mathfrak{z} + \epsilon) \setminus \{\mathfrak{n}\} \quad (\vartheta_1\text{-open} \setminus \vartheta_2\text{-closed}),$$
$$H = [\mathfrak{n}, \mathfrak{n} + \epsilon) \setminus \{\mathfrak{z}\} \quad (\vartheta_2\text{-open} \setminus \vartheta_1\text{-closed}).$$
Then $G \cap H = \emptyset$, $\mathfrak{z} \in G$, $\mathfrak{n} \in H$.

Proposition 3.8 *A pairwise* D_1 *space is pairwise* D_0.

Proposition 3.9 *A pairwise* D_2 *space is pairwise* D_1.

Lemma 3.10 *In a pairwise* D_1 *space, every singleton is a pairwise* D*-set.*

Proof. Let $\mathfrak{z} \in \mathfrak{Z}$. For any $\mathfrak{n} \neq \mathfrak{z}$, there exist D-sets G, H such that $\mathfrak{z} \in G \setminus H$ and $\mathfrak{n} \in H \setminus G$. Thus, $\{\mathfrak{z}\} = \bigcap_{\mathfrak{n} \neq \mathfrak{z}} G_{\mathfrak{n}}$, which is a pairwise D-set. ∎

Theorem 3.11 *Let* K *be a pairwise* D*-compact subset of a pairwise locally indiscrete pairwise* D_2*-space* $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$. *Then for every* $\mathfrak{z} \notin K$, *there exist pairwise* D*-sets* $D_{\mathfrak{z}}$ *and* D_K *such that:*
$$\mathfrak{z} \in D_{\mathfrak{z}}, \quad K \subseteq D_K, \quad \text{and} \quad D_{\mathfrak{z}} \cap D_K = \emptyset.$$

Proof. Consider $\mathfrak{z} \in \mathfrak{Z} \setminus K$. Since $\mathfrak{Z}$ is pairwise D_2, for every $\mathfrak{n} \in K$, there exist disjoint pairwise D-sets $D_{1\mathfrak{n}} \in \vartheta_1$ and $D_{2\mathfrak{n}} \in \vartheta_2$ such that:
$$\mathfrak{z} \in D_{1\mathfrak{n}}, \quad \mathfrak{n} \in D_{2\mathfrak{n}}, \quad D_{1\mathfrak{n}} \cap D_{2\mathfrak{n}} = \emptyset.$$
The collection $\widetilde{D} = \{D_{2\mathfrak{n}} : \mathfrak{n} \in K\}$ forms a pairwise D-cover of K. By pairwise D-compactness, there exists a finite subcover $\{D_{2y_1}, \dots, D_{2y_n}\}$. Define
$$D_{\mathfrak{n}} = \bigcup_{i=1}^{n} D_{2\mathfrak{n}_i} \quad (\vartheta_2\text{-open} \setminus \vartheta_1\text{-closed}), \quad D_{\mathfrak{z}} = \bigcap_{i=1}^{n} D_{1\mathfrak{n}_i} \quad (\vartheta_1\text{-open} \setminus \vartheta_2\text{-closed}).$$
Since every $D_{1\mathfrak{n}_i}$ is ϑ_1-clopen, and disjoint from $D_{2\mathfrak{n}_i}$, it follows that $D_{\mathfrak{z}} \cap D_{\mathfrak{n}} = \emptyset$. ∎

Lemma 3.12 *In a pairwise locally indiscrete space, finite unions of ϑ_2-clopen sets are ϑ_2-clopen.*

Proposition 3.13 *Pairwise D-compact subsets in pairwise locally indiscrete D_2-spaces are pairwise normal.*

Theorem 3.14 *Let K be a pairwise D-compact subset of a pairwise Hausdorff bitopological space $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$. For every $\mathfrak{z} \notin K$, there exist a ϑ_1-open set $L_{\mathfrak{z}}$ and a ϑ_2-open set E_K such that*

$$\mathfrak{z} \in L_{\mathfrak{z}}, \quad K \subseteq E_K, \quad \textit{and} \quad L_{\mathfrak{z}} \cap E_K = \emptyset.$$

Proof. By pairwise Hausdorffness, for every $\mathfrak{n} \in K$, there exist disjoint sets $L_{\mathfrak{n}} \in \vartheta_1$ and $E_{\mathfrak{n}} \in \vartheta_2$ with $\mathfrak{z} \in L_{\mathfrak{n}}$ and $\mathfrak{n} \in E_{\mathfrak{n}}$. The collection $\{E_{\mathfrak{n}} : \mathfrak{n} \in K\}$ is a ϑ_2-open cover of K. By pairwise D-compactness, there exists a finite subcover $\{E_{\mathfrak{n}_1}, \dots, E_{\mathfrak{n}_n}\}$. Define

$$E_K = \bigcup_{i=1}^{n} E_{\mathfrak{n}_i} \quad (\vartheta_2\text{-open}), \quad L_{\mathfrak{z}} = \bigcap_{i=1}^{n} L_{\mathfrak{n}_i} \quad (\vartheta_1\text{-open}).$$

If $L_{\mathfrak{z}} \cap E_K \neq \emptyset$, then $L_{\mathfrak{n}_i} \cap E_{\mathfrak{n}_i} \neq \emptyset$ for some i, contradicting pairwise Hausdorffness. Thus, $L_{\mathfrak{z}} \cap E_K = \emptyset$. ■

Theorem 3.15 *Any pairwise D-compact subset of a pairwise Hausdorff space $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ is ϑ_1-closed.*

Proof. For $\mathfrak{z} \in \mathfrak{Z} \setminus K$, Theorem 3.14 gives a ϑ_1-open set $L_{\mathfrak{z}}$ such that $L_{\mathfrak{z}} \subseteq \mathfrak{Z} \setminus K$. Hence, $\mathfrak{Z} \setminus K$ is ϑ_1-open, implying K is ϑ_1-closed. ■

Theorem 3.16 *Any pairwise D-compact subset of a pairwise locally indiscrete pairwise D_2-space $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ is ϑ_1-closed.*

Proof. For $\mathfrak{Z} \in \mathfrak{Z} \setminus K$, Theorem 3.14 provides pairwise D-sets $D_{\mathfrak{z}} \in \vartheta_1$ and $D_K \in \vartheta_2$ with $\mathfrak{z} \in D_{\mathfrak{z}}$, $K \subseteq D_K$, and $D_{\mathfrak{z}} \cap D_K = \emptyset$. Since $D_{\mathfrak{z}}$ is ϑ_1-clopen (by pairwise local indiscreteness), $\mathfrak{Z} \setminus K$ is ϑ_1-open. ■

Lemma 3.17 *In a pairwise Hausdorff space, disjoint ϑ_1-open and ϑ_2-open sets separate points and closed sets.*

Proposition 3.18 *Pairwise D-compactness is preserved under ϑ_1-closed subspaces in pairwise Hausdorff spaces.*

4. Extending Pairwise $D-$Compactness: Behavior Under Product Topologies

Throughout this section, all bitopological spaces are pairwise Hausdorff unless stated otherwise.

Theorem 4.1 *Let $\Phi\colon (\mathfrak{Z}, \vartheta_1, \vartheta_2) \to (\mathfrak{N}, \beta_1, \beta_2)$ be a pairwise continuous surjection. If $\mathfrak{Z}$ is pairwise D-compact, then $\mathfrak{N}$ is pairwise D-compact.*

Proof. Let $\widetilde{D}_{\mathfrak{N}} = \{D_\iota\}_{\iota \in \Lambda}$ be a pairwise D-cover of $\mathfrak{N}$, where $D_\iota = L_\iota \setminus E_\iota$ with $L_\iota \in \beta_1$, $E_\iota \in \beta_2$. Since Φ is pairwise continuous, $f^{-1}(D_\iota) = f^{-1}(L_\iota) \setminus f^{-1}(E_\iota)$ is pairwise D-set in $\mathfrak{Z}$. The collection $\{\Phi^{-1}(D_\iota)\}$ forms a pairwise D-cover of $\mathfrak{Z}$. So by pairwise D-compactness, there exists a finite subcover $\{\Phi^{-1}(D_{\iota_1}), \dots, \Phi^{-1}(D_{\iota_n})\}$. Thus, $\{D_{\iota_1}, \dots, D_{\iota_n}\}$ covers $\mathfrak{N}$. ■

Definition 4.2 *A function $\Phi : (\mathfrak{Z}, \vartheta_1, \vartheta_2) \to (\mathfrak{N}, \beta_1, \beta_2)$ is pairwise irresolute if*

$$\Phi^{-1}(C) \textit{ is } \vartheta_i\textit{-closed for every } \beta_i\textit{-closed set } C \subseteq \mathfrak{N} forall i = 1, 2.$$

Definition 4.3 *A function $\Phi\colon (\mathfrak{Z}, \vartheta_1, \vartheta_2) \to (\mathfrak{N}, \beta_1, \beta_2)$ is pairwise D-irresolute if for every pairwise D-set $D \subseteq \mathfrak{N}$, $\Phi^{-1}(D)$ is a pairwise D-set in $\mathfrak{Z}$.*

Theorem 4.4 *Let a pairwise D-irresolute surjection $\Phi : (\mathfrak{Z}, \vartheta_1, \vartheta_2) \to (\mathfrak{N}, \beta_1, \beta_2)$. If $\mathfrak{Z}$ is pairwise compact, then $\mathfrak{N}$ is pairwise D-compact.*

Proof. Consider $\widetilde{D}_{\mathfrak{N}} = \{D_\iota\}$ to be a pairwise D-cover of $\mathfrak{N}$. By pairwise D-irresoluteness, $\{\Phi^{-1}(D_\iota)\}$ is a pairwise open cover of $\mathfrak{Z}$. Since $\mathfrak{Z}$ is pairwise compact, there exists a finite subcover $\{\Phi^{-1}(D_{\iota_1}), \dots, \Phi^{-1}(D_{\iota_n})\}$. Thus, $\{D_{\iota_1}, \dots, D_{\iota_n}\}$ covers $\mathfrak{N}$. ■

Theorem 4.5 *Let a pairwise perfect function $\Phi : (\mathfrak{Z}, \vartheta_1, \vartheta_2) \to (\mathfrak{N}, \beta_1, \beta_2)$ with $\mathfrak{Z}$ pairwise locally indiscrete. Then*

$$\mathfrak{Z} \text{ is pairwise } D\text{-compact} \iff \mathfrak{N} \text{ is pairwise } D\text{-compact.}$$

Proof. ($\Rightarrow$): Follows from Theorem 4.1
($\Leftarrow$): Consider $\widetilde{D}$ to be a pairwise D-cover of $\mathfrak{Z}$. For $\mathfrak{n} \in \mathfrak{N}$, $\Phi^{-1}(\mathfrak{n})$ is pairwise compact, so there exists a finite subcover $\bigcup_{\iota \in \Lambda_{\mathfrak{n}}} D_\iota$. Define

$$O_{\mathfrak{n}} = \mathfrak{N} \setminus \Phi\left(\mathfrak{Z} \setminus \bigcup_{\iota \in \Lambda_{\mathfrak{n}}} D_\iota\right) \quad (\beta_1\text{-open}).$$

The collection $\{O_{\mathfrak{n}} : \mathfrak{n} \in \mathfrak{N}\}$ covers $\mathfrak{N}$. By pairwise D-compactness of $\mathfrak{N}$, there exists a finite subcover $\{O_{\mathfrak{n}_1}, \dots, O_{\mathfrak{n}_n}\}$. Then

$$\mathfrak{Z} = \bigcup_{i=1}^{n} \Phi^{-1}(O_{\mathfrak{n}_i}) \subseteq \bigcup_{i=1}^{n} \bigcup_{\iota \in \Lambda_{\mathfrak{n}_i}} D_\iota,$$

yields a finite subcover for $\mathfrak{Z}$. ■

Remark 4.6 *In pairwise locally indiscrete spaces, pairwise D-sets are clopen in their respective topologies, ensuring $\Phi^{-1}(O_{\mathfrak{n}})$ inherits necessary properties.*

Theorem 4.7 *Let $\Phi\colon (\mathfrak{Z}, \vartheta_1, \vartheta_2) \to (\mathfrak{N}, \beta_1, \beta_2)$ be a pairwise perfect function. If $\mathfrak{N}$ is pairwise D-compact, then $\mathfrak{Z}$ is pairwise compact.*

Proof. Consider $\widetilde{L} = \{L_\iota\}_{\iota \in \Lambda}$ to be a pairwise open cover of $\mathfrak{Z}$. For $\mathfrak{n} \in \mathfrak{N}$, $\Phi^{-1}(\mathfrak{n})$ is pairwise compact, so there exists $\{L_{\iota_1^{\mathfrak{n}}}, \dots, L_{\iota_{n_{\mathfrak{n}}}^{\mathfrak{n}}}\}$ covering $\Phi^{-1}(\mathfrak{n})$. Define

$$W_{\mathfrak{n}} = \bigcup_{i=1}^{n_{\mathfrak{n}}} L_{\iota_i^{\mathfrak{n}}}.$$

Since Φ is pairwise closed, $\Phi(\mathfrak{Z} \setminus W_{\mathfrak{n}})$ is β_i-closed for $i = 1, 2$. Thus

$$E_{\mathfrak{n}} = \mathfrak{N} \setminus \Phi(\mathfrak{Z} \setminus W_{\mathfrak{n}})$$

is β_i-open and contains $\mathfrak{n}$. The collection $\{E_{\mathfrak{n}} : \mathfrak{n} \in \mathfrak{N}\}$ is a pairwise open cover of $\mathfrak{N}$. Since $\mathfrak{N}$ is pairwise D-compact , hence pairwise compact, there exists a finite subcover $\{E_{\mathfrak{n}_1}, \dots, E_{\mathfrak{n}_k}\}$. Then

$$\mathfrak{Z} = \bigcup_{j=1}^{k} f^{-1}(E_{\mathfrak{n}_j}) \subseteq \bigcup_{j=1}^{k} W_{\mathfrak{n}_j} = \bigcup_{j=1}^{k} \bigcup_{i=1}^{n_{\mathfrak{n}_j}} L_{\iota_i^{\mathfrak{n}_j}},$$

which is a finite subcover of $\widetilde{L}$. Hence, $\mathfrak{Z}$ is pairwise compact. ■

Definition 4.8 *Let $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ and $(\mathfrak{N}, \beta_1, \beta_2)$ be bitopological spaces. The pairwise product space is $(\mathfrak{Z} \times \mathfrak{N}, \vartheta_1 \times \beta_1, \vartheta_2 \times \beta_2)$, where $\vartheta_i \times \beta_i$ denotes the product topology formed by open sets in ϑ_i and β_i.*

Theorem 4.9 *Let $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ be pairwise compact and $(\mathfrak{N}, \beta_1, \beta_2)$ be pairwise Hausdorff. Then the projection $P_{\mathfrak{n}}\colon \mathfrak{Z} \times \mathfrak{N} \to \mathfrak{N}$ is pairwise perfect.*

Proof. $\pi_{\mathfrak{N}}$ is continuous in both topologies as preimages satisfy $\pi_{\mathfrak{N}}^{-1}(E) = \mathfrak{Z} \times E$ the pairwise continuity. $\pi_{\mathfrak{N}}^{-1}(\mathfrak{n}) = \mathfrak{Z} \times \{\mathfrak{n}\} \cong \mathfrak{Z}$ is pairwise compact.
Let $C \subseteq \mathfrak{Z} \times \mathfrak{N}$ be $\vartheta_i \times \beta_i$-closed. let $\mathfrak{n} \notin \pi_{\mathfrak{N}}(C)$. For every $\mathfrak{z} \in \mathfrak{Z}$, $(\mathfrak{z}, \mathfrak{n}) \notin C$. By pairwise Hausdorffness, there exist $L_{\mathfrak{z}} \in \vartheta_j$ containing $\mathfrak{z}$, $E_{\mathfrak{z}} \in \beta_i$ containing y, $L_{\mathfrak{z}} \times E_{\mathfrak{z}} \cap C = \emptyset$

Compactness of $\mathfrak{Z}$ gives finite subcover $\{L_{\mathfrak{z}_1}, \ldots, L_{\mathfrak{z}_n}\}$. Then $E = \bigcap_{k=1}^{n} E_{\mathfrak{z}_k}$ is a β_i-neighborhood of $\mathfrak{n}$ disjoint from $\pi_{\mathfrak{N}}(C)$. Thus, $\pi_{\mathfrak{N}}(C)$ is closed. ■

Theorem 4.10 *Let $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ and $(\mathfrak{N}, \beta_1, \beta_2)$ be pairwise Hausdorff, pairwise locally indiscrete spaces where $\mathfrak{Z}$ is pairwise compact and $\mathfrak{N}$ is pairwise D-compact. Then $\mathfrak{Z} \times \mathfrak{N}$ is pairwise D-compact.*

Proof. By Theorem 4.9, $\pi_{\mathfrak{N}} : \mathfrak{Z} \times \mathfrak{N} \to \mathfrak{N}$ is pairwise perfect. Since $\mathfrak{Z} \times \mathfrak{N}$ is pairwise locally indiscrete, and $\mathfrak{N}$ is pairwise D-compact, Theorem 4.5 implies $\mathfrak{Z} \times \mathfrak{N}$ is pairwise D-compact. ■

Corollary 4.11 *The product of a pairwise compact Hausdorff space and a pairwise locally indiscrete D-compact space is pairwise compact.*

Theorem 4.12 *Let $(\mathfrak{Z}, \vartheta_1, \vartheta_2)$ be pairwise D-compact and $(\mathfrak{N}, \beta_1, \beta_2)$ be pairwise Hausdorff. The projection $P_{\mathfrak{n}} : \mathfrak{Z} \times \mathfrak{N} \to \mathfrak{N}$ is pairwise perfect.*

Proof. By product topology properties; the pairwise continuity satesfied. $\pi_{\mathfrak{N}}^{-1}(\mathfrak{n}) = \mathfrak{Z} \times \{\mathfrak{n}\} \cong \mathfrak{Z}$ is pairwise D-compact, hence pairwise compact by Theorem 2.37. By Theorem 4.9 given closed $C \subseteq \mathfrak{Z} \times \mathfrak{N}$, $P_{\mathfrak{n}}(C)$ is closed in $\mathfrak{N}$ as $\mathfrak{N}$ is pairwise Hausdorff. ■

Theorem 4.13 *Let $\{(\mathfrak{Z}_i, \vartheta_{i1}, \vartheta_{i2})\}_{i=1}^{n}$ be pairwise locally indiscrete bitopological spaces. Then*

$$\prod_{i=1}^{n} \mathfrak{Z}_i \text{ is pairwise } D\text{-compact} \iff \text{each } \mathfrak{Z}_i \text{ is pairwise } D\text{-compact.}$$

Proof. ($\Rightarrow$): Pairwise continuous projections preserve pairwise D-compactness.
($\Leftarrow$): Induction using pairwise perfect projections and Theorem 4.12. ■

Theorem 4.14 *Every pairwise continuous function $\Phi\colon (\mathfrak{Z}, \vartheta_1, \vartheta_2) \to (\mathfrak{N}, \beta_1, \beta_2)$ from a pairwise D-compact space to a pairwise Hausdorff space is closed.*

Proof. Let $C \subseteq \mathfrak{Z}$ be ϑ_1-closed. By Theorem 2.52, C is pairwise D-compact. Since Φ is pairwise continuous, $\Phi(C)$ is pairwise D-compact in $\mathfrak{N}$. By Theorem 3.15, $\Phi(C)$ is β_1-closed. ■

Theorem 4.15 *Let $\Phi\colon (\mathfrak{Z}, \vartheta_1, \vartheta_2) \to (\mathfrak{N}, \beta_1, \beta_2)$ be pairwise continuous, where $\mathfrak{Z}$ is pairwise D-compact and $\mathfrak{N}$ is pairwise Hausdorff. Then*

$$\Phi(CL(\grave{P})) = CL(\Phi(\grave{P})) \quad \forall \grave{P} \subseteq \mathfrak{Z}.$$

Proof. ($\subseteq$): Pairwise continuity implies $\Phi(\mathrm{CL}(\grave{P})) \subseteq \mathrm{CL}(\Phi(\grave{P}))$.
($\supseteq$): $\mathrm{CL}(\grave{P})$ is pairwise D-compact by theorem 2.52, so $\Phi(\mathrm{CL}(\grave{P}))$ is closed by theorem 4.14, containing $\mathrm{CL}(\Phi(\grave{P}))$. ■

5. The Impact of Pairwise $D-$Compactness: Applications Across Diverse Fields

Beyond its appeal in theory, the concept of D-compactness in bitopological spaces provides a useful framework for addressing practical issues involving several, typically conflicting viewpoints or criteria. Consider it a set of mathematical techniques for dealing with situations where there are two points of view. This section examines the ways in which this theory can offer novel perspectives and useful benefits in a variety of domains.

There are many choices that involve compromises in life. Selecting a project may require balancing future earnings against environmental impact, just as choosing a job may require balancing contribute against commute time. Such scenarios can be remarkably actually modeled in bitopological spaces. One topology ϑ_1 can be used to symbolize the things we want to achieve or maximize such as a short commute or a high profit, while the second topology ϑ_2 can be used to symbolize the things we want to avoid or limit such as environmental damage or high risk.

A pairwise D-set $L \setminus E$, where $L \in \vartheta_1$ and $E \in \vartheta_2$ well captures a desirable region in this framework: options that satisfy the positive criteria L while avoiding the negative ones E. Pairwise D-compactness provides an important insight: if the decision space has this property, then any method of covering all possible options with these important regions can be reduced to a finite number of types of representative regions. The creation of effective algorithms depends on this finiteness.

For example, consider a city planner selecting sites for new public parks. While ϑ_2 denotes areas zoned for industrial use, ϑ_1 could represent areas with a high population density. Strongly populated $L \in \vartheta_1$ but not industrial zones $E \in \vartheta_2$ would be identified by a pairwise D-set. The planner knows they only need to examine a limited number of basic location types if the city map, represented as a bitopological space, is pairwise D-compact. This makes the difficult optimization problem computationally manageable.

This method offers an objective approach to managing multiple objectives and has great potential in financial modeling by balancing risk and return, resource allocation by maximizing usage under constraints, and strategic planning.

Often, work with imprecise or incomplete information. Tools for approximating sets based on available data are provided by rough set theory. This is an appropriate location for bitopological spaces, where ϑ_1 may be the topology produced by lower approximations and ϑ_2 by the complement of upper approximations.

Partially known concepts could then be represented by a pairwise D-set, which excludes a set of definite non-members related to E while including a core set of definite members L. A complex knowledge base based on such approximations could be represented or reasoned about using a limited number of basic rules or patterns if pairwise D-compactness were present.

This has implications for building systems that effectively manage confusion and incomplete data, identifying patterns in noisy or imprecise datasets, and making well-informed decisions even in the face of uncertainty are all impacted by this.

The computational viability and representational power of reasoning with such approximations are guaranteed by D-compactness.

Modern datasets are often massive and high-dimensional. The primary objective of computational topology is to comprehend the fundamental structure and layout of this data. To analyze data using two distinct notions of closeness or dimension at the same time, bitopological spaces could be used. For instance, ϑ_1 based on a local distance metric and ϑ_2 based on a global clustering structure.

Data points that relate to an a specific local cluster L but are different from a specific large-scale feature E may be found using a pairwise D-set. Based on pairwise D-compactness, a limited number of representative features or regions can capture the data collection's fundamental topological structure when viewed through this dual lens.

This could enhance topological Data Analysis by creating algorithms that better capture topological features at multiple scales, constructing lower-dimensional representations while maintaining global structure and local neighborhoods, and creating algorithms that use several, possibly incompatible similarity criteria to group data.

Theoretical assures provided by D-compactness may result in more reliable and computationally practical techniques for examining the hidden shapes in complex data.

These possible uses provide an extensive number of opportunities for further investigation into the newly emerging field of D-compactness in bitopological spaces. Investigating the computational complexity of verifying pairwise D-compactness, creating visible algorithms based on these concepts for particular issues in each domain, and examining connections to other generalized compactness concepts within these applied contexts are important methods for future research. There is still much to learn about the relationship between the two topologies chosen and the resulting D-compactness properties in real-world models.

6. Conclusion

This article deliberately generalizes classical compactness to bitopological spaces using an innovative concept of pairwise D-compactness, which is defined using D-sets. By axiomatizing this framework, we establish its fundamental properties, such as an evident classification of pairwise D-compactness independent of classical compactness; thorough relationships between D-compactness, and hereditary separation axioms; and applications demonstrating its utility in multi-layered networks, quantum state order and and hybrid cyber-physical systems. The theory brings together disparate approaches to dual-topology analysis while preserving key intuitive properties of compactness, providing a scalable tool for modern systems with layered interactions that defy single-topology models.

This article introduces pairwise D-compactness as a generalized stability framework for bitopological spaces, unifying classical compactness and dual-topology interactions. First, it uses D-sets to redefine compactness. Second, a hierarchy of separation axioms is established for pairwise D_0, D_1, and D_2-spaces. This hierarchy is adapted for multi-layered separability and mirrors the classical T_i-axioms. Third, the results of structural stability preservation extend Tychonoff-like theorems to dual topologies by demonstrating that closed subspaces, continuous images, and finite products of D-compact spaces maintain compactness under mild conditions. Lastly, functional implications demonstrate that D-irresolute and perfect maps maintain compactness, which makes them valuable for designing hybrid systems and algorithmic stability.

Pairwise D-compactness has predictive power that goes well beyond pure topology. through the formalization of compactness in computational topology, rough set theory, and multi-criteria decision making. Stability, robustness, and efficiency can be modeled using the mathematical language provided by this framework.

In conclusion, the notion of pairwise D-compactness extends the limits of topology while respecting its applied traditions, offering a flexible perspective for analyzing complexity in a multi-layered, increasingly interconnected world. Its practical adaptability and mathematical simplicity make it a solid framework for future studies where interdisciplinary problems call for instruments that balance abstract theory with the structural complexity of the real world. Also, it may lead to future research in defining and characterizing related concepts like pairwise D-Lindelöf or pairwise D-countably compact spaces within the bitopological spaces, investigating variations like soft pairwise D-compactness or fuzzy pairwise D-compactness to handle uncertainty in various mathematical settings, further exploring the properties of functions that preserve or reflect pairwise D-compactness, such as variations of pairwise D-irresolute or D-continuous functions, and applying the theoretical results on pairwise D-compactness to develop concrete algorithms and models in fields like Multi-Criteria Decision Making, Digital Topology, Computational Topology for data analysis, and possibly Rough Set Theory are some examples of possible research.

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