Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–12.
ISSN-0037-8712
doi:10.5269/bspm.77245

Solvability of a Class of Tripled System of Nonlinear Integral Equations in P-Hahn Sequence Space

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ABSTRACT: We introduce the Hausdorff measure of noncompactness in p-Hahn sequence space and we obtain an extension of Darboś fixed point theorem. Applying extended of Darboś theorem, we investigate the existence of solution of a class of tripled system of nonlinear integral equations in the p-Hahn sequence space. Finally, we present one example to verify the usefulness of main results.

Key Words: Hausdorff measure of noncompactness, sequence spaces, system of integral equations, tripled fixed point.

Contents

1. Introduction and preliminaries

Measure of noncompactness (MNC) the function α was first defined by Kuratowski [20] for purely topological considerations. Darbo [11] in 1955 used this measure to generalize Banach's contraction mapping principle for so-called condensing operators.

In 1957 the Hausdorff MNC χ was introduced by Goldenstein et al. [12] and it was further studied by Markus and Goldenstein [13]. Recently, the notion of MNC has been applied in sequence spaces for deferent classes of differential equations ([6,7,15,21,23,24,25,28,29,30,31,32]) and ([9,10,26,27]).

In recent years, many authors introduced a tripled system and a tripled fixed point [8,16,17,18]. In [18], the researchers for investigate the existence of solution of functional tripled system via fractional operators used tripled fixed points and the MNC.

In [4] Kayvanloo et al. introduced an extension of Darbo's fixed point theorem associated with MNC and study the existence of solutions of system of nonlinear integral-differential equations in Sobolev space.

Motivated by the above papers, we define the Hausdorff MNC in p-Hahn sequence space. Then, we introduce an extension of Darbo's fixed point theorem associated with MNC and we study the existence of solutions of following tripled system of nonlinear integral-differential equations in p-Hahn sequence space.

$$\begin{cases} \upsilon(\wp) = A_1(\wp) + h_1(\wp, \upsilon(\zeta_1(\wp)), \nu(\zeta_1(\wp)), \omega(\zeta_1(\wp))) \\ +f_1\Big(\wp, \upsilon(\zeta_1(\wp)), \nu(\zeta_1(\wp)), \omega(\zeta_1(\wp)), \phi\big(\int_0^{\beta_1(\wp)} g_1(\wp, \varsigma, \upsilon(\ell_1(\varsigma)), \nu(\ell_1(\varsigma)), \omega(\ell_1(\varsigma)))d\varsigma\Big) \\ \nu(t) = A_2(\wp) + h_2(\wp, \nu(\zeta_2(\wp)), \omega(\zeta_2(\wp)), \upsilon(\zeta_2(\wp))) \\ +f_2\Big(\wp, \nu(\zeta_2(\wp)), \omega(\zeta_2(\wp)), \upsilon(\zeta_2(\wp)), \phi\big(\int_0^{\beta_2(\wp)} g_2(\wp, \varsigma, \nu(\ell_2(\varsigma)), \omega(\ell_2(\varsigma)), \upsilon(\ell_2(\varsigma)))d\varsigma\Big) \\ \omega(\wp) = A_3(\wp) + h_3(\wp, \omega(\zeta_3(\wp)), \upsilon(\zeta_3(\wp)), \nu(\zeta_3(\wp))) \\ +f_3\Big(\wp, \omega(\zeta_3(\wp)), \upsilon(\zeta_3(\wp)), \nu(\zeta_3(\wp)), \phi\big(\int_0^{\beta_3(\wp)} g_3(\wp, \varsigma, \omega(\ell_3(\varsigma)), \upsilon(\ell_3(\varsigma)), \nu(\ell_3(\varsigma)))d\varsigma\Big). \end{cases} \tag{1.1}$$

Also, one example is presented to show the usefulness of main results.

In this part, a few auxiliary facts are represented, that we can use in our paper. Let Γ be a Banach space with the zero element θ, in addition, the elements υ and r respectively are indicated in the center and radius of the closed ball $B(\upsilon, r)$ in Γ. Let $\emptyset \neq \mathfrak{M}_\Gamma \subseteq \Gamma$ the family of all bounded and $\emptyset \neq \mathfrak{N}_\Gamma \subseteq \Gamma$

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2010 *Mathematics Subject Classification:* 47H10, 47H09, 34A12, 46B45.
Submitted June 07, 2025. Published October 29, 2025

subfamily of all relatively compact sets. The symbols Conv(A) and $\bar{A}$ for the non-empty subsets convex and closure A in Γ respectively.

Definition 1.1 *[1] The mapping $\tilde{\mu} : \mathfrak{M}_\Gamma \to [0, +\infty)$ is measure of noncompactness (MNC) in Γ if $\forall$ $\mathcal{R}, \mathcal{Y}_1, \mathcal{Y}_2 \in \mathfrak{M}_\Gamma$ having:*

(i) $\emptyset \neq \ker \tilde{\mu} = \{\mathcal{R} \in \mathfrak{M}_\Gamma : \tilde{\mu}(\mathcal{R}) = 0\} \subseteq \mathfrak{N}_\Gamma$.

(ii) *If* $\mathcal{Y}_1 \subset \mathcal{Y}_2, \ \Rightarrow \ \tilde{\mu}(\mathcal{Y}_1) \leq \tilde{\mu}(\mathcal{Y}_2)$.

(iii) $\tilde{\mu}(\overline{\mathcal{R}}) = \tilde{\mu}(\mathcal{R}) = \tilde{\mu}(Conv\mathcal{R})$.

(iv) $\forall\ 0 \leq \jmath \leq 1,\ \tilde{\mu}(\jmath\mathcal{Y}_1 + (1-\jmath)\mathcal{Y}_2) \leq \jmath\tilde{\mu}(\mathcal{Y}_1) + (1-\jmath)\tilde{\mu}(\mathcal{Y}_2)$.

(v) *If* $\forall n \in \mathbb{N},\ \overline{\mathcal{R}_n} = \mathcal{R}_n$ *in* $\mathfrak{M}_\Gamma$, $\mathcal{R}_{n+1} \subset \mathcal{R}_n$ *and* $\lim_{n\to\infty} \tilde{\mu}(\mathcal{R}_n) = 0$, *then* $\emptyset \neq \mathcal{R}_\infty = \bigcap_{n=1}^{\infty} \mathcal{R}_n$.

Definition 1.2 *[5] Let (Y, d) is metric space. And, let $\mathcal{P} \in \mathfrak{M}_Y$. The Kuratowski MNC $\omega(\mathcal{P})$, is defined by*

$$\omega(\mathcal{P}) = \inf\Big\{0 < \varepsilon : \mathcal{P} \subset \bigcup_{\kappa=1}^{m} K_\kappa, K_\kappa \subset Y, diam(K_\kappa) < \varepsilon\ (\kappa = 1, \ldots, m);\ m \in \mathbb{N}\Big\},$$

where $diam(K_\kappa) = \sup\{d(o, \wp) : o, \wp \in K_\kappa\}$.

The Hausdorff MNC, $\beta(\mathcal{P})$ is

$$\beta(\mathcal{P}) = \inf\Big\{0 < \varepsilon : \mathcal{P} \subset \bigcup_{\kappa=1}^{m} B(z_\kappa, r_\kappa), z_\kappa \in Y, r_\kappa < \varepsilon\ (\kappa = 1, \ldots, m);\ m \in \mathbb{N}\Big\}.$$

Let $K = [0, s]$ and Γ is a Banach space. Then $C(K, \Gamma)$ is Banach space with norm

$$\|x\|_{C(K,\Gamma)} := \sup\{\|x(\rho)\| : \rho \in K\},\ x \in C(K, \Gamma).$$

Proposition 1.1 *[5] Let $\Upsilon \subseteq C(K, \Gamma)$ is equicontinuous and bounded. Then $\tilde{\omega}(\Upsilon(.))$ is continuous on K and*

$$\tilde{\omega}(\Upsilon) = \sup_{\zeta \in K} \tilde{\omega}(\Upsilon(\zeta)), \quad \tilde{\omega}\Big(\int_0^\zeta \Upsilon(\ell)d\ell\Big) \leq \int_0^\zeta \tilde{\omega}(\Upsilon(\ell))d\ell.$$

Definition 1.3 *[5] The element $(\upsilon, \nu, \omega) \in \mathfrak{L} \times \mathfrak{L} \times \mathfrak{L}$ is tripled fixed point of mapping $\mathfrak{G} : \mathfrak{L} \times \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ if $\mathfrak{G}(\upsilon, \nu, \omega) = \upsilon,\ \mathfrak{G}(\nu, \omega, \upsilon) = \nu,\ \mathfrak{G}(\omega, \nu, \upsilon) = \omega$.*

Theorem 1.1 *[3] Let $\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_m$ are MNC in Banach spaces $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_m$, respectively. Moreover, Let the function $H : {\mathbb{R}_+}^m \to \mathbb{R}_+$ is convex and $H(\upsilon_1, \upsilon_2, \ldots, \upsilon_m) = 0$ iff $\upsilon_\iota = 0$ for $\iota = 1, 2, \ldots, m$. Then*

$$\widetilde{\tilde{\mu}}(\mathfrak{L}) = H(\tilde{\mu}_1(\mathfrak{L}_1), \tilde{\mu}_2(\mathfrak{L}_2), \ldots, \tilde{\mu}_n(\mathfrak{L}_n)),$$

defines a MNC in $\Upsilon_1 \times \Upsilon_2 \times, \ldots \times \Upsilon_m$, where $\mathfrak{L}_\iota$ denotes the natural projection of $\mathfrak{L}$ into Υ_ι, for $\iota = 1, 2, \ldots, m$.

Example 1.1 *[2] Suppose that $\tilde{\mu}$ be a MNC on a Banach space Υ. Take $H(\upsilon, \nu, \omega) = \upsilon + \nu + \omega$ for any $(\upsilon, \nu, \omega) \in \mathbb{R}^3_+$. Then by Theorem 1.1, $\widetilde{\tilde{\mu}}(\mathfrak{L}) = \tilde{\mu}(\mathfrak{L}_1) + \tilde{\mu}(\mathfrak{L}_2) + \tilde{\mu}(\mathfrak{L}_3)$ defines a MNC on the space $\Upsilon \times \Upsilon \times \Upsilon$ where $\mathfrak{L}_\iota$, $\iota = 1, 2, 3$ are natural projections of $\mathfrak{L}$.*

Denote by Ψ the family of increasing functions $\psi : [0, \infty) \to [0, \infty)$ continuous in $\wp = 0$ so that
$\psi(\wp) = 0$ iff $\wp = 0$,
$\psi(\wp + \varsigma) \leq \psi(\wp) + \psi(\varsigma)$ for all $\wp, \varsigma \in \mathbb{R}_+$.

Definition 1.4 *The function $\theta : [0, \infty) \to [0, \infty)$ is strictly L-function if $0 = \theta(0)$, $0 < \theta(\varsigma)$ for $0 < \varsigma < \infty$, and $\forall\ \varsigma > 0$, $\exists\ \delta > 0$ so that $\theta(\wp) < \varsigma$, $\forall\ \wp \in [\varsigma, \varsigma + \delta]$.*

Theorem 1.2 *[4] Let Υ is Banach space and $\emptyset \neq \mathfrak{A} = \overline{\mathfrak{A}} \subseteq \Upsilon$ be convex, closed and $\mathfrak{G} : \mathfrak{A} \to \mathfrak{A}$ be a continuous operator so that*

$$\alpha(\tilde{\mu}(\mathfrak{G}(\mathfrak{L})))\psi(\tilde{\mu}(\mathfrak{G}(\mathfrak{L}))) \leq \theta\Big(\beta(\tilde{\mu}(\mathfrak{L}))\psi(\tilde{\mu}(\mathfrak{L}))\Big),$$

for any $\mathfrak{L} \subseteq \mathfrak{A}$, where θ is a strictly $L-$function and $\tilde{\mu}$ is an arbitrary MNC on Υ. where $\alpha : [0, +\infty) \to [1, +\infty)$ and $\beta : [0, +\infty) \to (0, 1]$ are mappings and $\psi : [0, \infty) \to [0, \infty)$ is an increasing mapping so that $0 = \psi(\wp)$ iff $0 = \wp$. Then, $\mathfrak{G}$ has at least one fixed point.

By using strictly $L-$functions we give a tripled fixed point theorem.

Theorem 1.3 *Suppose that Υ, $\mathfrak{A}$, θ, β and $\tilde{\mu}$ be as Theorem 1.2 and suppose that $\alpha : [0, \infty) \to [1, \infty)$ be an increasing map, $\psi \in \Psi$ and $T : \mathfrak{A} \times \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$, is a continuous function fulfils*

$$\alpha\Big(\tilde{\mu}(T(\mathfrak{L}_1 \times \mathfrak{L}_2 \times \mathfrak{L}_3)) + \tilde{\mu}(T(\mathfrak{L}_2 \times \mathfrak{L}_3 \times \mathfrak{L}_1)) + \tilde{\mu}(T(\mathfrak{L}_3 \times \mathfrak{L}_1 \times \mathfrak{L}_2))\Big)\psi(\tilde{\mu}(T(\mathfrak{L}_1 \times \mathfrak{L}_2 \times \mathfrak{L}_3))) \leq \frac{1}{3}\theta\Big(\beta(\frac{\tilde{\mu}(\mathfrak{L}_1) + \tilde{\mu}(\mathfrak{L}_2) + \tilde{\mu}(\mathfrak{L}_3)}{3})\psi(\frac{\tilde{\mu}(\mathfrak{L}_1) + \tilde{\mu}(\mathfrak{L}_2) + \tilde{\mu}(\mathfrak{L}_3)}{3})\Big), \tag{1.2}$$

$\forall$, $\mathfrak{L}_1$,$\mathfrak{L}_2$,$\mathfrak{L}_3 \subseteq \mathfrak{A}$. Then T has at least a tripled fixed point.

Proof: Example 1.1 grantees that $\widetilde{\widetilde{\mu}}(\mathfrak{L}) = \tilde{\mu}(\mathfrak{L}_1) + \tilde{\mu}(\mathfrak{L}_2) + \tilde{\mu}(\mathfrak{L}_3)$ is a MNC in $\Upsilon \times \Upsilon \times \Upsilon$, where $\mathfrak{L}_\iota$, $\iota = 1, 2, 3$ are natural projections of $\mathfrak{L}$. Define the function $\widetilde{T} : \mathfrak{A} \times \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A} \times \mathfrak{A} \times \mathfrak{A}$ by $\widetilde{T}(\upsilon, \nu, \omega) = (T(\upsilon, \nu, \omega), T(\nu, \omega, \upsilon), T(\omega, \upsilon, \nu))$ for every $(\upsilon, \nu, \omega) \in \mathfrak{A} \times \mathfrak{A} \times \mathfrak{A}$. Obviously $\widetilde{T}$ is continuous. We show that $\widetilde{T}$ satisfies the hypothesis of Theorem 1.2. Let $\emptyset \neq \mathfrak{L} \subseteq \mathfrak{A} \times \mathfrak{A} \times \mathfrak{A}$. By attributes of α, ψ and (1.2) we get

$$\begin{aligned}
&\alpha\Big(\widetilde{\widetilde{\mu}}(\widetilde{T}(\mathfrak{L}_1 \times \mathfrak{L}_2 \times \mathfrak{L}_3))\Big)\psi(\widetilde{\widetilde{\mu}}(\widetilde{T}(\mathfrak{L}_1 \times \mathfrak{L}_2 \times \mathfrak{L}_3)))\\
&\quad\leq \alpha\Big(\widetilde{\widetilde{\mu}}(T(\mathfrak{L}_1 \times \mathfrak{L}_2 \times \mathfrak{L}_3), T(\mathfrak{L}_2 \times \mathfrak{L}_3 \times \mathfrak{L}_1), T(\mathfrak{L}_3 \times \mathfrak{L}_1 \times \mathfrak{L}_2))\Big)\\
&\qquad \psi(\widetilde{\widetilde{\mu}}(T(\mathfrak{L}_1 \times \mathfrak{L}_2 \times \mathfrak{L}_3), T(\mathfrak{L}_2 \times \mathfrak{L}_3 \times \mathfrak{L}_1), T(\mathfrak{L}_3 \times \mathfrak{L}_1 \times \mathfrak{L}_2))\\
&\quad\leq \alpha\Big(\tilde{\mu}(T(\mathfrak{L}_1 \times \mathfrak{L}_2 \times \mathfrak{L}_3)) + \tilde{\mu}(T(\mathfrak{L}_2 \times \mathfrak{L}_3 \times \mathfrak{L}_1)) + \tilde{\mu}(T(\mathfrak{L}_3 \times \mathfrak{L}_1 \times \mathfrak{L}_2))\Big)\psi(\tilde{\mu}(T(\mathfrak{L}_1 \times \mathfrak{L}_2 \times \mathfrak{L}_3)))\\
&\qquad +\alpha\Big(\tilde{\mu}(T(\mathfrak{L}_1 \times \mathfrak{L}_2 \times \mathfrak{L}_3)) + \tilde{\mu}(T(\mathfrak{L}_2 \times \mathfrak{L}_3 \times \mathfrak{L}_1)) + \tilde{\mu}(T(\mathfrak{L}_3 \times \mathfrak{L}_1 \times \mathfrak{L}_2))\Big)\psi(\tilde{\mu}(T(\mathfrak{L}_2 \times \mathfrak{L}_3 \times \mathfrak{L}_1)))\\
&\qquad +\alpha\Big(\tilde{\mu}(T(\mathfrak{L}_1 \times \mathfrak{L}_2 \times \mathfrak{L}_3)) + \tilde{\mu}(T(\mathfrak{L}_2 \times \mathfrak{L}_3 \times \mathfrak{L}_1)) + \tilde{\mu}(T(\mathfrak{L}_3 \times \mathfrak{L}_1 \times \mathfrak{L}_2))\Big)\psi(\tilde{\mu}(T(\mathfrak{L}_3 \times \mathfrak{L}_1 \times \mathfrak{L}_2)))\\
&\quad\leq \theta\Big(\beta(\frac{\tilde{\mu}(\mathfrak{L}_1) + \tilde{\mu}(\mathfrak{L}_2) + \tilde{\mu}(\mathfrak{L}_3)}{3})\psi(\frac{\tilde{\mu}(\mathfrak{L}_1) + \tilde{\mu}(\mathfrak{L}_2) + \tilde{\mu}(\mathfrak{L}_3)}{3})\Big)\\
&\quad= \theta\Big(\beta(\frac{\widetilde{\widetilde{\mu}}(\mathfrak{L}_1 \times \mathfrak{L}_2 \times \mathfrak{L}_3)}{3})\psi(\frac{\widetilde{\widetilde{\mu}}(\mathfrak{L}_1 \times \mathfrak{L}_2 \times \mathfrak{L}_3)}{3})\Big).
\end{aligned}$$

Therefore,

$$\alpha\Big(\frac{1}{3}\widetilde{\widetilde{\mu}}(\widetilde{T}(\mathfrak{L}_1 \times \mathfrak{L}_2 \times \mathfrak{L}_3))\Big)\psi(\frac{1}{3}\widetilde{\widetilde{\mu}}(\widetilde{T}(\mathfrak{L}_1 \times \mathfrak{L}_2 \times \mathfrak{L}_3))) \leq \theta\Big(\beta(\frac{1}{3}\widetilde{\widetilde{\mu}}(\mathfrak{L}_1 \times \mathfrak{L}_2 \times \mathfrak{L}_3))\psi(\frac{1}{3}\widetilde{\widetilde{\mu}}(\mathfrak{L}_1 \times \mathfrak{L}_2 \times \mathfrak{L}_3))\Big)$$

and taking $\widehat{\widetilde{\mu}} = \frac{1}{3}\widetilde{\widetilde{\mu}}$, we obtain

$$\alpha\Big(\widehat{\widetilde{\mu}}(\widetilde{T}(\mathfrak{L}_1 \times \mathfrak{L}_2 \times \mathfrak{L}_3))\Big)\psi(\widehat{\widetilde{\mu}}(\widetilde{T}(\mathfrak{L}_1 \times \mathfrak{L}_2 \times \mathfrak{L}_3))) \leq \theta\Big(\beta(\widehat{\widetilde{\mu}}(\mathfrak{L}_1 \times \mathfrak{L}_2 \times \mathfrak{L}_3))\psi(\widehat{\widetilde{\mu}}(\mathfrak{L}_1 \times \mathfrak{L}_2 \times \mathfrak{L}_3))\Big).$$

Since $\widehat{\widetilde{\mu}}$, is MNC, so by Theorem 1.2, $\widetilde{T}$ has a fixed point, or T has a tripled fixed point. □

Corollary 1.1 *Let Υ is Banach space, $\emptyset \neq \mathfrak{A} = \overline{\mathfrak{A}} \subseteq \Upsilon$ be bounded, convex and $T : \mathfrak{A} \times \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ be a continuous function satisfying*

$$\psi(\tilde{\mu}(\mathfrak{G}(\mathfrak{L}_1 \times \mathfrak{L}_2 \times \mathfrak{L}_3))) \leq \frac{1}{3}\theta\Big(\psi(\frac{\tilde{\mu}(\mathfrak{L}_1) + \tilde{\mu}(\mathfrak{L}_2) + \tilde{\mu}(\mathfrak{L}_3)}{3})\Big), \tag{1.3}$$

$\forall$, $\mathfrak{L}_1$,$\mathfrak{L}_2$, $\mathfrak{L}_3 \subseteq \mathfrak{A}$, where $\tilde{\mu}$ is an arbitrary MNC in the space Υ, $\psi \in \Psi$ and θ is a strictly L-function. Then T has at least a tripled fixed point.

2. Hausdorff MNC in p-Hahn Sequence space

Let $\omega = \mathbb{C}^N$, be the space of all complex-valued or real sequences, where $\mathbb{C}$ is the complex field and $N = \{0, 1, 2, \dots\}$. For $\upsilon = (\upsilon_k) \in \omega$, we shall employ the sequence spaces c_0 (null), c (convergent) and l_∞ (bounded) sequences $z = (z_o)$ with complex terms, by norm

$$\|z\|_\infty = \sup_{o\in\mathbb{N}} |z_o|.$$

Also $l_p = \{\upsilon = (\upsilon_k) \in \omega : \sum_{k=0}^{\infty} |\upsilon_k|^p < \infty\}$ $(1 \le p < \infty)$. by norm

$$l_p = (\sum_{k=0}^{\infty} |\upsilon_k|^p)^{\frac{1}{p}}.$$

The Hahn sequence space was introduced by H. Hahn [14]. Recently, Malkowsky et al [22] characterized the compact operators on the Hahn space. The p-Hahn sequence space h_p was defined as follows (see [19])

$$h_p = \{\upsilon : \sum_{k=1}^{\infty} (k|\Delta\upsilon_k|)^p < \infty \text{ and } \lim_{k\to\infty} \upsilon_k = 0\},\ 1 < p < \infty$$

where $\Delta\upsilon_k = \upsilon_k - \upsilon_{k+1}$, $(k \in \mathbb{N})$.

Theorem 2.1 *[19]* $h_p = l_p \bigcap bv^p = l_p \bigcap bv_0^p$.

From now on, we assume that $1 \le p < \infty$. Now, we determine the Hausdorff MNC χ in the $p-$Hahn sequence space.

Lemma 2.1 *[29] Suppose that U is normed space and $Q \subseteq U$ be a bounded, where U is c_0 or l_p $(p \in [1, \infty))$. If $P_m : \mathfrak{L} \to \mathfrak{L}$ is operator $R_m(\upsilon) = (\upsilon_0, \upsilon_1, \dots, \upsilon_m, 0, 0, \dots)$, so*

$$\chi(Q) = \lim_{m\to\infty} \{ \sup_{\upsilon\in Q} \|(I - R_m)\upsilon\| \}.$$

Theorem 2.2 *Suppose that $U \subseteq h_p$ be bounded, then the Huasdorff MNC χ in the Banach space h_p is:*

$$\chi(U) := \lim_{n\to\infty} \Big\{ \sup_{\upsilon\in U} \Big\{ \sum_{k\ge n} (k|\Delta\upsilon_k|)^p \Big\}\Big\}. \tag{2.1}$$

Proof: Define the operator $R_n : h_p \to h_p$ by $R_n(\upsilon) = (\upsilon_1, \upsilon_2, \dots, \upsilon_n, 0, 0, \dots)$ for $\upsilon = (\upsilon_1, \upsilon_2, \dots) \in h_p$. Clearly

$$U \subset R_n U + (I - R_n)U. \tag{2.2}$$

By (2.2) and the attributes of χ, we get

$$\begin{aligned} \chi(U) &\le \chi(R_n U) + \chi((I - R_n)U) = \chi((I - R_n)U) \\ &\le \operatorname{diam}((I - R_n)U) = \sup_{\upsilon\in U} \|(I - R_n)\upsilon\|, \end{aligned}$$

where

$$\|(I - R_n)\upsilon\| = \sum_{k=1}^{\infty} (k|\Delta\upsilon_k|)^p,$$

when n is large enough. So

$$\chi(U) \le \lim_{n\to\infty} \sup_{\upsilon\in U} \|(I - R_n)\upsilon\|. \tag{2.3}$$

Conversely, suppose that $0 < \varepsilon$ and let $\{z_1, z_2, \dots, z_j\}$ be a $[\chi(U) + \varepsilon]$-net of U. So

$$U \subset \{z_1, z_2, \dots, z_j\} + [\chi(U) + \varepsilon]B(h_p),$$

where $B(h_p)$ is the unit ball of h_p. Hence

$$\sup_{x\in U} \|(I - R_n)\upsilon\| \le \sup_{1\le i\le j} \|(I - R_n)z_i\| + [\chi(U) + \varepsilon],$$

which implies that

$$\lim_{n\to\infty} \sup_{\upsilon\in U} \|(I - R_n)\upsilon\| \le \chi(U) + \varepsilon. \tag{2.4}$$

Since ε was arbitrary, by (2.3) and (2.4), we conclude that (2.1) holds. □

3. Application

Now, we study the existence of solutions of E.q (1.1) in p-Hahn sequence space by using the Hausdorff MNC.

Take the following conditions into consideration:
(i) The mappings $A_i(\wp):[0,L]\to\mathbb{R}$ $(i=1,2,3)$ are bounded, continuous and

$$M_i=\sup\{\sum_{k=1}^{\infty}|k\Delta A_i(\wp)|^p,\ \wp\in[0,L]\}<\infty.$$

(ii) The functions $\zeta_i,\beta_i,\ell_i:[0,L]\to[0,\infty)$ $(i=1,2,3)$ are continuous and $\lim_{\wp\to\infty}\zeta_i(\wp)=\infty$.

(iii) The mapping $\phi:[0,L]\to\mathbb{R}$ is continuous and $\exists$ constant $\delta>0$ so that

$$|\phi(\wp_1)-\phi(\wp_2)|^p\le\delta|\wp_1-\wp_2|^p,$$

for any $\wp_1,\wp_2\in[0,L]$ and $\phi(0)=0$.
(iv) The mappings $f_i:[0,L]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ $(i=1,2,3)$ and $h_i:[0,L]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ $(i=1,2,3)$ are continuous and there are three increasing continuous functions $\varphi_i:\mathbb{R}_+\to\mathbb{R}$ $(i=1,2,3)$ with $\varphi_i(0)=0$ so that

$$|f_i(\wp,x,y,z,l)-f_i(\wp,u,v,w,q)|^p\le\frac{1}{3(4^{2p})}\big(\theta(\frac{1}{6}(|x-u|^p+|y-v|^p+|z-w|^p))+\varphi_i(|l-q|^p)\big),$$

and

$$|h_i(t,x,y,z)-f_i(t,u,v,w)|^p\le\frac{1}{3(4^{2p})}\theta(\frac{1}{6}(|x-u|^p+|y-v|^p+|z-w|^p)),$$

for any $\wp\in[0,L]$, and $\forall$, $u,v,w,x,y,z,l,q\in\mathbb{R}$, where θ is a continuous strictly $L-$function so that $\theta(c+b)\ge\theta(c)+\theta(b)$ $(c,b\in\mathbb{R})$ and $\theta o\psi=\psi o\theta$ where $\psi\in\Psi$.
(v) The mappings defined by $\wp\to|f_i(\wp,0,0,0,0)|$ $(i=1,2,3)$ and $\wp\to|h_i(\wp,0,0,0)|$ $(i=1,2,3)$ are bounded on $[0,\infty)$, *i.e.*

$$M_i^{'}=\sup\{\sum_{k=1}^{\infty}|k\Delta f_i(\wp,0,0,0,0)|^p,\ \wp\in[0,L]\}<\infty,$$

$$M_i^{''}=\sup\{\sum_{k=1}^{\infty}|k\Delta h_i(\wp,0,0,0)|^p,\ \wp\in[0,L]\}<\infty.$$

(vi) The functions $g_i:[0,L]\times[0,L]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ $(i=1,2,3)$ are continuous function so that

$$\lim_{k\to\infty}\sum_{k=1}^{\infty}|k\Delta(\int_0^{\beta_i(\wp)}(g_i(\wp,\varsigma,x(\ell(\varsigma)),y(\ell(\varsigma)),z(\ell(\varsigma)))-g_i(\wp,\varsigma,u(\ell(\varsigma)),v(\ell(\varsigma)),w(\ell(\varsigma)))))|^pd\varsigma=0,$$

uniformly w.r.s.t $u,v,w,x,y,z\in\mathbb{R}$, and

$$M_i^{'''}=\sup\{\sum_{k=1}^{\infty}|k\Delta(\phi(\int_0^{\beta_i(\wp)}|g_i(\wp,\varsigma,x(\ell(\varsigma)),y(\ell(\varsigma)),z(\ell(\varsigma)))d\varsigma|^p),\ \wp,\varsigma\in[0,L],x,y,z\in\mathbb{R}\},$$

$$\lim_{k\to\infty}\sum_{k=1}^{\infty}|k\Delta f_i(\wp,0,0,0,0)|^p=0,\quad \lim_{k\to\infty}\sum_{k=1}^{\infty}|k\Delta h_i(\wp,0,0,0)|^p=0,\quad \lim_{k\to\infty}\sum_{k=1}^{\infty}|k\Delta A_i(\wp)|^p=0,$$

and

$$\lim_{k\to\infty}\sum_{k=1}^{\infty}|k\Delta(\phi(\int_0^{\beta_i(\wp)}|g_i(\wp,\varsigma,x(\ell(\varsigma)),y(\ell(\varsigma)),z(\ell(\varsigma)))d\varsigma|^p))=0.$$

Theorem 3.1 *Let the hypothesists $(i)-(vi)$ holds. Then E.q.(1.1) has at least one solution in $C([0,L],h_p)\times C([0,L],h_p)\times C([0,L],h_p)$.*

Proof: We consider $T:C([0,L],h_p)\times C([0,L],h_p)\times C([0,L],h_p)\to C([0,L],h_p)$ by

$$T(\upsilon,\nu,\omega)(\wp)=A_1(\wp)+h_1(\wp,\upsilon(\zeta_1(\wp)),\nu(\zeta_1(\wp)),\omega(\zeta_1(\wp)))$$
$$+f_1\Big(\wp,\upsilon(\zeta_1(\wp)),\nu(\zeta_1(\wp)),\omega(\zeta_1(\wp)),\phi(\int_0^{\beta_1(\wp)}g_1(\wp,\varsigma,\upsilon(\ell_1(\varsigma)),\nu(\ell_1(\varsigma)),\omega(\ell_1(\varsigma)))d\varsigma\Big).$$

Notice that, the space $C([0,L],h_p)\times C([0,L],h_p)\times C([0,L],h_p)$ is equipped by norm

$$\|(\upsilon,\nu,\omega)\|_{C([0,L],h_p)}=\|\upsilon\|_{C([0,L],h_p)}+\|\nu\|_{C([0,L],h_p)}+\|\omega\|_{C([0,L],h_p)},$$

for each $(\upsilon,\nu,\omega) \in h_p \times h_p \times h_p$. First, since A_1, f_1, and h_1 are continuous. Then the operator T is continuous. Also, for $\upsilon,\nu,\omega \in h_p$ we have

$$
\begin{aligned}
&\|T(\upsilon,\nu,\omega)(\wp)\|_{h_p}\\
&= \sum_{k=1}^{\infty}\Big(k|\Delta\big(A_1(\wp)+h_1(\wp,\upsilon(\zeta_1(\wp)),\nu(\zeta_1(\wp)),\omega(\zeta_1(\wp)))\\
&\quad +f_1\Big(\wp,\upsilon(\zeta_1(\wp)),\nu(\zeta_1(\wp)),\omega(\zeta_1(\wp)),\\
&\quad \phi\big(\int_0^{\beta_1(\wp)} g_1(\wp,\varsigma,\upsilon(\ell_1(\varsigma)),\nu(\ell_1(\varsigma)),\omega(\ell_1(\varsigma))))\Big)\big)d\varsigma|\Big)^p\\
&\le 4^p\Big(\sum_{k=1}^{\infty}|k\Delta\big(A_1(\wp)\big)|^p+\sum_{k=1}^{\infty}|k\Delta(h_1(\wp,\upsilon(\zeta_1(\wp)),\nu(\zeta_1(\wp)),\omega(\zeta_1(\wp))))|^p\\
&\quad +\sum_{k=1}^{\infty}|k\Delta(f_1\Big(\wp,\upsilon(\zeta_1(\wp)),\nu(\zeta_1(\wp)),\omega(\zeta_1(\wp)),\\
&\quad \phi\big(\int_0^{\beta_1(\wp)} g_1(\wp,\varsigma,\upsilon(\ell_1(\varsigma)),\nu(\ell_1(\varsigma)),\omega(\ell_1(\varsigma))))\Big)\big)d\varsigma|^p\Big)\\
&\le 4^{2p}\Big(\sum_{k=1}^{\infty}|k\Delta\big(A_1(\wp)\big)|^p+\sum_{k=1}^{\infty}|k\Delta(h_1(\wp,\upsilon(\zeta_1(\wp)),\nu(\zeta_1(\wp)),\omega(\zeta_1(\wp))))-h_1(\wp,0,0,0)|^p\\
&\quad +\sum_{k=1}^{\infty}|k\Delta(h_1(\wp,0,0,0))|^p+\sum_{k=1}^{\infty}|k\Delta(f_1(\wp,0,0,0,0))|^p\\
&\quad +\sum_{k=1}^{\infty}|k\Delta(f_1\Big(\wp,\upsilon(\zeta_1(\wp)),\nu(\zeta_1(\wp)),\omega(\zeta_1(\wp)),\phi\big(\int_0^{\beta_1(\wp)} g_1(\wp,\varsigma,\upsilon(\ell_1(\varsigma)),\nu(\ell_1(\varsigma)),\omega(\ell_1(\varsigma))))\Big)\big)d\varsigma\\
&\quad -f_1(\wp,0,0,0,0)|^p\Big)\\
&\le 4^{2p}\Bigg(\sum_{k=1}^{\infty}|k\Delta\big(A_1(\wp)\big)|^p+\frac{1}{3(4^{2p})}\theta\big(\frac{1}{6}\big(\sum_{k=1}^{\infty}|k\Delta\big(\upsilon(\zeta_1(\wp))\big)|^p+\sum_{k=1}^{\infty}|k\Delta\big(\nu(\zeta_1(\wp))\big)|^p\\
&\quad +\sum_{k=1}^{\infty}|k\Delta\big(\omega(\zeta_1(\wp))\big)|^p\big)\big)+M_1'+M_1''\\
&\quad +\frac{1}{3(4^{2p})}\Big(\theta\big(\frac{1}{6}\big(\sum_{k=1}^{\infty}|k\Delta\big(\upsilon(\zeta_1(\wp))\big)|^p+\sum_{k=1}^{\infty}|k\Delta\big(\nu(\zeta_1(\wp))\big)|^p+\sum_{k=1}^{\infty}|k\Delta\big(\omega(\zeta_1(\wp))\big)|^p\big)\big)\\
&\quad +\varphi_1\big(\sum_{k=1}^{\infty}|k\Delta\big(\phi\big(\int_0^{\beta_1(\wp)} g_1(\wp,\varsigma,\upsilon(\ell_1(\varsigma)),\nu(\ell_1(\varsigma)),\omega(\ell_1(\varsigma)))d\varsigma|^p\big)\big)\big)\Big)\Bigg)\\
&\le 4^{2p}\Big(\big(M_1+M_1'+M_1''\big)+\frac{2}{3}\theta\big(\frac{1}{6}(\|\upsilon\|_{C([0,L],h_p)}+\|\nu\|_{C([0,L],h_p)}+\|\omega\|_{C([0,L],h_p)})\big)+\frac{1}{3}\varphi_1(M_1''')\Big)\\
&\le 4^{2p}\Big(\big(M_1+M_1'+M_1''\big)+\varphi_1(M_1''')+\theta\big(\frac{1}{6}r\big)\Big)<\rho.
\end{aligned}
$$

So $T(D_\rho\times D_\rho\times D_\rho)\subseteq D_\rho$ and T is well defined. Now we show that T is continuous on $D_\rho\times D_\rho\times D_\rho$. Let $(x,y,z)\in D_\rho\times D_\rho\times D_\rho$, $\varepsilon>0$ and $(u,v,w)\in D_\rho\times D_\rho\times D_\rho$ by $\|(x,y,z)-(u,v,w)\|_{C([0,L],h_p)}<\frac{\varepsilon}{2}$. Now, we get

$$
\begin{aligned}
&\|T(x,y,z)-T(u,v,w)(\wp)\|_{h_p}\\
&\le 2^p\Bigg(\sum_{k=1}^{\infty}|k\Delta(h_1(\wp,x(\zeta_1(\wp)),y(\zeta_1(\wp)),z(\zeta_1(\wp)))-h_1(\wp,u(\zeta_1(\wp)),v(\zeta_1(\wp)),w(\zeta_1(\wp))))|^p\\
&\quad +\sum_{k=1}^{\infty}|k\Delta\Big(f_1\big(\wp,x(\zeta_1(\wp)),y(\zeta_1(\wp)),z(\zeta_1(\wp)),\phi\big(\int_0^{\beta_1(\wp)} g_1(\wp,\varsigma,x(\ell_1(\varsigma)),y(\ell_1(\varsigma)),z(\ell_1(\varsigma))))\big)d\varsigma\\
&\quad -f_1\big(\wp,u(\zeta_1(\wp)),v(\zeta_1(\wp)),w(\zeta_1(\wp)),\phi\big(\int_0^{\beta_1(\wp)} g_1(\wp,\varsigma,u(\ell_1(\varsigma)),v(\ell_1(\varsigma)),w(\ell_1(\varsigma))))\big)\Big)d\varsigma|^p\Bigg)\\
&\le 2^p\Bigg(\frac{1}{3(4^{2p})}\theta\Big(\frac{1}{6}\Big(\sum_{k=1}^{\infty}|k\Delta\big(x(\zeta_1(\wp))-u(\zeta_1(\wp))\big)|^p+\sum_{k=1}^{\infty}|k\Delta\big(y(\zeta_1(\wp))-v(\zeta_1(\wp))\big)|^p\\
&\quad +\sum_{k=1}^{\infty}|k\Delta\big(z(\zeta_1(\wp))-w(\zeta_1(\wp))\big)|^p\Big)\Big)+\frac{1}{3(4^{2p})}\Big(\theta\big(\frac{1}{6}\big(\sum_{k=1}^{\infty}|k\Delta\big(x(\zeta_1(\wp))-u(\zeta_1(\wp))\big)|^p
\end{aligned}
$$

$$+\sum_{k=1}^{\infty}|k\Delta\big(y(\zeta_1(\wp))-v(\zeta_1(\wp))\big)|^p+\sum_{k=1}^{\infty}|k\Delta\big(z(\zeta_1(\wp))-w(\zeta_1(\wp))\big)|^p)\big)$$
$$+\varphi_1\Big(\sum_{k=1}^{\infty}|k\Delta\big(\phi(\int_0^{\beta_1(\wp)}g_1(\wp,\varsigma,x(\ell_1(\varsigma)),y(\ell_1(\varsigma)),z(\ell_1(\varsigma)))d\varsigma)$$
$$-\phi(\int_0^{\beta_1(\wp)}g_1(\wp,\varsigma,u(\ell_1(\varsigma)),v(\ell_1(\varsigma)),w(\ell_1(\varsigma)))d\varsigma)\big)|^p\Big)\Big)$$
$$\leq 2^p\Big(\frac{2}{3(4^{2p})}\theta\big(\frac{1}{6}\big(\|x-u\|_{C([0,L],h_p)}+\|y-v\|_{C([0,L],h_p)}+\|z-w\|_{C([0,L],h_p)}\big)\big)$$
$$+\varphi_1\Big(\sum_{k=1}^{\infty}|k\Delta(\delta\int_0^{\beta_1(\wp)}g_1(\wp,\varsigma,x(\ell_1(\varsigma)),y(\ell_1(\varsigma)),z(\ell_1(\varsigma)))$$
$$-g_1(\wp,\varsigma,u(\ell_1(\varsigma)),v(\ell_1(\varsigma)),w(\ell_1(\varsigma))))|^p d\varsigma|)\Big)\Big).$$

From (vi), for any $x,y,z\in h_p$ we derive that

$$\|T(x,y,z)-T(u,v,w)\|_{C([0,L],h_p)}$$
$$\leq 2^p\big(\frac{2}{3(4^{2p})}\theta(\frac{1}{6}(\frac{\varepsilon}{2}))+\varphi_1\Big(\sum_{k=1}^{\infty}|k\Delta(\delta\int_0^{\beta_1(\wp)}g_1(\wp,\varsigma,x(\ell_1(\varsigma)),y(\ell_1(\varsigma)),z(\ell_1(\varsigma)))$$
$$-g_1(\wp,\varsigma,u(\ell_1(\varsigma)),v(\ell_1(\varsigma)),w(\ell_1(\varsigma))))|^p d\varsigma)\Big)$$
$$\leq 2^p\Big(\frac{1}{4^{2p}}(\theta(\frac{\varepsilon}{12})+\varphi_1(\delta(\beta_1^L\omega(\varepsilon))\Big),$$

where

$$\omega(\varepsilon)=\sup\{\sum_{k=1}^{\infty}|k\Delta(g_1(\wp,\varsigma,x,y,z)-g_1(\wp,\varsigma,u,v,w))|^p,\ \wp\in[0,L],\ \varsigma\in[0,\beta^L],$$
$$u,v,w,x,y,z\in[-\rho,\rho],\ \|(x,y,z)-(u,v,w)\|_{C([0,L],h_p)}<\tfrac{\varepsilon}{2}\},\text{ and }\beta_1^L=\sup\{\beta_1(\wp),\ \wp\in[0,L]\}.$$

By using the continuity of g_1 on $[0,L]\times[0,\beta_1^L]\times[-\rho,\rho]\times[-\rho,\rho]\times[-\rho,\rho]$ we have $\omega(\varepsilon)\to 0$ as $\varepsilon\to 0$ and by continuity of φ_1 we get $\varphi_1(\delta_1(\beta_1^L\omega(\varepsilon))\to 0$ as $\varepsilon\to 0$. So T is continuous on $C([0,L],h_p)\times C([0,L],h_p)\times C([0,L],h_p)$. Finally, we show that condition 1.3 is satisfied. Let $\varepsilon>0$ be arbitrary constants and let $\emptyset\neq X,Y,Z\subseteq D_\rho\times D_\rho\times D_\rho$ be bounded. Choose $(\upsilon,\nu,\omega)\in X\times Y\times Z$ then we get

$$\psi(\chi(T(\upsilon,\nu,\omega)(\wp)))$$
$$=\psi\Big(\lim_{n\to\infty}\sup_{\upsilon,\nu,\omega\in D_\rho}\Big(\sum_{k\geq n}\Big(|k\Delta\big(A_1(\wp)+h_1(\wp,\upsilon(\zeta_1(\wp)),\nu(\zeta_1(\wp)),\omega(\zeta_1(\wp)))$$
$$+f_1\Big(\wp,\upsilon(\zeta_1(\wp)),\nu(\zeta_1(\wp)),\omega(\zeta_1(\wp)),\phi\big(\int_0^{\beta_1(\wp)}g_1(\wp,\varsigma,\upsilon(\ell_1(\varsigma)),\nu(\ell_1(\varsigma)),\omega(\ell_1(\varsigma)))d\varsigma\big)\Big)|\Big)^p\Big)\Big)$$
$$\leq\psi\Big(4^p\Big(\lim_{n\to\infty}\sup_{\upsilon,\nu,\omega\in D_\rho}\Big(\sum_{k\geq n}\big(|k\Delta\big(A_1(\wp)\big)|^p\big)\Big)$$
$$+\lim_{n\to\infty}\sup_{\upsilon,\nu,\omega\in D_\rho}\Big(\sum_{k\geq n}|k\Delta(h_1(\wp,\upsilon(\zeta_1(\wp)),\nu(\zeta_1(\wp)),\omega(\zeta_1(\wp))))|^p\Big)$$
$$+\lim_{n\to\infty}\sup_{\upsilon,\nu,\omega\in D_\rho}\big(\sum_{k\geq n}|k\Delta\big(f_1\big(\wp,\upsilon(\zeta_1(\wp)),\nu(\zeta_1(\wp)),\omega(\zeta_1(\wp)),$$
$$\phi(\int_0^{\beta_1(\wp)}g_1(\wp,\varsigma,\upsilon(\ell_1(\varsigma)),\nu(\ell_1(\varsigma)),\omega(\ell_1(\varsigma))))\big)\big)d\varsigma|^p)\Big)\Big)$$
$$\leq\psi\Big(4^{2p}\Big(\lim_{n\to\infty}\sup_{\upsilon,\nu,\omega\in D_\rho}\Big(\sum_{k\geq n}|k\Delta(h_1(\wp,\upsilon(\zeta_1(\wp)),\nu(\zeta_1(\wp)),\omega(\zeta_1(\wp)))-h_1(\wp,0,0,0))|^p\Big)$$
$$+\lim_{n\to\infty}\sup_{\upsilon,\nu,\omega\in D_\rho}\Big(\sum_{k\geq n}|k\Delta(h_1(\wp,0,0,0))|^p\Big)$$
$$+\lim_{n\to\infty}\sup_{\upsilon,\nu,\omega\in D_\rho}\Big(\sum_{k\geq n}|k\Delta\Big(f_1\big(\wp,\upsilon(\zeta_1(\wp)),\nu(\zeta_1(\wp)),\omega(\zeta_1(\wp)),$$
$$\phi(\int_0^{\beta_1(\wp)}g_1(\wp,\varsigma,\upsilon(\ell_1(\varsigma)),\nu(\ell_1(\varsigma)),\omega(\ell_1(\varsigma))))-f_1(\wp,0,0,0,0)\big)\Big)d\varsigma|^p\Big)$$

$$
\begin{aligned}
&+\lim_{n\to\infty}\sup_{\upsilon,\nu,\omega\in D_\rho}\Big(\sum_{k\geq n}|k\Delta\big(f_1(\wp,0,0,0,0)\big)d\varsigma|^p\Big)\Big)\Big)\\
\leq\quad &\psi\Bigg(4^{2p}\Bigg(\frac{1}{3(4^{2p})}\theta\Bigg(\frac{1}{6}\Bigg(\lim_{n\to\infty}\sup_{\upsilon,\nu,\omega\in B_\rho}\Big(\sum_{k\geq n}|k\Delta\big(\upsilon(\zeta_1(\wp))\big)|^p\Big)+\Big(\sum_{k=1}^{\infty}|k\Delta\big(\nu(\zeta_1(\wp))\big)|^p\Big)\\
&+\Big(\sum_{k=1}^{\infty}|k\Delta\big(\omega(\zeta_1(\wp))\big)|^p\Big)+\lim_{n\to\infty}\sup_{\upsilon,\nu,\omega\in D_\rho}\Big(\sum_{k=1}^{\infty}|k\Delta\big(\upsilon(\zeta_1(\wp))\big)|^p\\
&+\Big(\sum_{k=1}^{\infty}|k\Delta\big(\nu(\zeta_1(\wp))\big)|^p\Big)+\Big(\sum_{k=1}^{\infty}|k\Delta\big(\omega(\zeta_1(\wp))\big)|^p\Big)\Big)\Bigg)\Bigg)\\
&+\varphi\Big(\lim_{n\to\infty}\sup_{\upsilon,\nu,\omega\in D_\rho}\Big(\sum_{k=1}^{\infty}|k\Delta\Big(\int_0^{\beta_1(\wp)}g_1(\wp,\varsigma,\upsilon(\ell_1(\varsigma)),\nu(\ell_1(\varsigma)),\omega(\ell_1(\varsigma)))\Big)d\varsigma|^p\Big)\Big)\Bigg)\\
\leq\quad &\psi\Bigg(\frac{1}{3}\Bigg(\theta\Bigg(\frac{1}{3}\lim_{n\to\infty}\sup_{\upsilon,\nu,\omega\in D_\rho}\Big(\Big(\sum_{k=1}^{\infty}|k\Delta\big(\upsilon(\zeta_1(\wp))\big)|^p\\
&+\sum_{k=1}^{\infty}|k\Delta\big(\nu(\zeta_1(\wp))\big)|^p+\sum_{k=1}^{\infty}|k\Delta\big(\omega(\zeta_1(\wp))\big)|^p\Big)\Big)\Bigg)\Bigg)\Bigg)\\
\leq\quad &\frac{1}{3}\theta\Bigg(\psi\Bigg(\frac{\chi(X)+\chi(Y)+\chi(Z)}{3}\Bigg)\Bigg).
\end{aligned}
$$

Now, Corollary 1.1 guarantees that T has a tripled fixed point in $D_\rho\times D_\rho\times D_\rho$. So, tripled system (1.1) have at least one solution in $C([0,L],h_p)\times C([0,L],h_p)\times C([0,L],h_p)$. □

Example 3.1 *Consider the equation*

$$
\begin{cases}
\upsilon(\wp)=\frac{1}{(k(k+1))}e^{-(\wp+1)^2}+\frac{1}{5k^2(k+1)}e^{-(\wp)^3}+\frac{1}{4^4(\wp^2+4)}\arctan(\upsilon(\wp))+\frac{1}{4^4(t^2+8)}\ln(1+\nu(\wp))\\
+\frac{1}{4^4(\wp^2+3)}\sin(\omega(\wp))+\frac{1}{7k^2(k+1)}e^{-\wp^2}+\frac{1}{4^4(\wp^2+3)}\sin(\upsilon(\wp))+\frac{1}{4^4(\wp+6)}\sin(\nu(\wp))+\frac{1}{4^4(\wp^4+9)}\arctan(\omega(\wp))\\
+\frac{1}{4^4(\wp^3+5)}\int_0^{\wp^2}\frac{\sin(\upsilon(\wp))\cos(\nu(\wp))\arctan(\omega(\wp))}{k^2e^{\wp}(1+\upsilon(\varsigma))(1+\nu(\varsigma))(1+\omega(\varsigma))}d\varsigma\\
\nu(\wp)=\frac{2}{(k(k+1))}e^{-(\wp+2)^2}+\frac{2}{7k^2(k+1)}e^{-(\wp)^3}+\frac{2}{4^4(\wp^3+6)}\sin(\nu(\wp))+\frac{1}{4^4(\wp^3+9)}\arctan(\omega(\wp))\\
+\frac{1}{4^4(\wp^3+8)}\ln(1+\upsilon(\wp))+\frac{2}{15k^2(k+1)}e^{-\wp^3}+\frac{2}{4^4(\wp^3+5)}\sin(\nu(\wp))+\frac{2}{4^4(\wp+8)}\arctan(\omega(\wp))+\frac{2}{4^4(\wp^4+7)}\sin(\upsilon(\wp))\\
+\frac{1}{4^4(\wp^3+6)}\int_0^{\wp^3}\frac{\arctan(\nu(\wp))+\sin(\omega(\wp))\cos(\upsilon(\wp))}{k^2e^{2\wp}(2+\nu(\varsigma))(2+\omega(\varsigma))(2+\upsilon(\varsigma))}d\varsigma\\
\omega(\wp)=\frac{3}{(k(k+1))}e^{-(\wp+3)^2}+\frac{3}{10k^2(k+1)}e^{-(\wp)^3}+\frac{3}{4^4(\wp^4+9)}\ln(1+\omega(\wp))+\frac{3}{4^4(\wp^3+12)}\arctan(\upsilon(\wp))\\
+\frac{3}{4^4(\wp^3+5)}\sin(\nu(\wp))+\frac{3}{16k^2(k+1)}e^{-\wp^4}+\frac{3}{4^4(\wp^4+4)}\sin(\omega(\wp))+\frac{3}{4^4(\wp+3)}\sin(\upsilon(\wp))+\frac{3}{4^4(\wp+11)}\arctan(\nu(\wp))\\
+\frac{3}{4^4(\wp^3+3)}\int_0^{\wp^4}\frac{\sin(\omega(\wp))\cos(\upsilon(\wp))+\arctan(\nu(\wp))}{k^2e^{3\wp}(3+\omega(\varsigma))(3+\upsilon(\varsigma))(3+\nu(\varsigma))}d\varsigma,
\end{cases}
\tag{3.1}
$$

where
$\zeta_1(\wp)=\ell_1(\wp)=\varphi_1(\wp)=\wp,\ \beta_1(\wp)=\wp^2,\ \theta(\wp)=6\wp,\ A_1(\wp)=\frac{1}{(k(k+1))}e^{-(\wp+1)^2},\ M_1'=\frac{1}{7},\ M_1''=\frac{1}{5},$

$$
\begin{aligned}
h_1(\wp,\upsilon(\zeta_1(\wp)),\nu(\zeta_1(\wp)),\omega(\zeta_1(\wp)))&=\frac{1}{5k^2(k+1)}e^{-(\wp)^3}+\frac{1}{4^4(\wp^2+4)}\arctan(\upsilon(\wp))\\
&+\frac{1}{4^4(\wp^2+8)}\ln(1+\nu(\wp))+\frac{1}{4^4(\wp^2+3)}\sin(\omega(\wp)),\\
f_1(\wp,\upsilon(\zeta_1(\wp)),\nu(\zeta_1(\wp)),\omega(\zeta_1(\wp)),l_1(\ell_1(\varsigma)))&=\frac{1}{7k^2(k+1)}e^{-\wp^2}+\frac{1}{4^4(\wp^2+3)}\sin(\upsilon(\wp))\\
&+\frac{1}{4^4(\wp+6)}\sin(\nu(\wp))+\frac{1}{4^4(\wp^4+9)}\arctan(\omega(\wp))+\frac{1}{4^4(\wp^3+5)}(l_1(\wp)),\\
g_1(\wp,\varsigma,\upsilon(\ell_1(\varsigma)),\nu(\ell_1(\varsigma)),\omega(\ell_1(\varsigma)))&=\frac{\sin(\upsilon(\wp))\cos(\nu(\wp))+\arctan(\omega(\wp))}{e^{\wp}(1+\upsilon(\varsigma))(1+\nu(\varsigma))(1+\omega(\varsigma))}.
\end{aligned}
$$

Also
$A_2(\wp)=\frac{2}{(k(k+1))}e^{-(\wp+1)^2},\ \zeta_2(\wp)=\ell_2(\wp)=\varphi_2(\wp)=\wp,\ \beta_2(\wp)=\wp^3,\ M_2'=\frac{2}{15},\ M_2''=\frac{2}{7},$

$$
\begin{aligned}
h_2(\wp,\nu(\zeta_2(\wp)),\omega(\zeta_1(\wp)),\upsilon(\zeta_1(\wp)))&=\frac{2}{7k^2(k+1)}e^{-(\wp)^3}+\frac{2}{4^4(\wp^3+6)}\sin(\nu(\wp))\\
&+\frac{1}{4^4(\wp^3+9)}\arctan(\omega(\wp))+\frac{1}{4^4(\wp^3+8)}\ln(1+\upsilon(\wp)),
\end{aligned}
$$

$$f_2(\wp,\nu(\zeta_2(\wp)),\omega(\zeta_2(\wp)),\upsilon(\zeta_2(\wp)),l_2(\ell_2(\varsigma))) = \frac{2}{15k^2(k+1)}e^{-\wp^3} + \frac{2}{4^4(\wp^3+5)}\sin(\nu(\wp)) + \frac{2}{4^4(\wp+8)}\arctan(\omega(\wp)) + \frac{2}{4^4(\wp^4+7)}\sin(\upsilon(\wp)) + \frac{1}{4^4(\wp^3+6)}(l_2(\wp)),$$

$$g_2(\wp,\varsigma,\nu(\ell_2(\varsigma)),\omega(\ell_2(\varsigma)),\upsilon(\ell_2(\varsigma))) = \frac{\arctan(\nu(\wp))+\sin(\omega(\wp))\cos(\upsilon(\wp))}{e^{2\wp}(2+\nu(\varsigma))(2+\omega(\varsigma))(2+\upsilon(\varsigma))},$$

and
$A_3(\wp) = \frac{3}{(k(k+1))}e^{-(\wp+1)^2}$, $\zeta_3(\wp)=\ell_3(\wp)=\varphi_3(\wp)=\wp$, $\beta_3(\wp)=\wp^4$, $M_3'=\frac{3}{16}$, $M_3''=\frac{3}{10}$,

$$h_3(\wp,\omega(\zeta_3(\wp)),\upsilon(\zeta_3(\wp)),\nu(\zeta_3(\wp))) = \frac{3}{10k^2(k+1)}e^{-(\wp)^3} + \frac{3}{4^4(\wp^4+9)}\ln(1+\omega(\wp)) + \frac{3}{4^4(\wp^3+12)}\arctan(\upsilon(\wp)) + \frac{3}{4^4(\wp^3+5)}\sin(\nu(\wp)),$$

$$f_3(\wp,\omega(\zeta_3(\wp)),\upsilon(\zeta_3(\wp)),\nu(\zeta_3(\wp)),l_3(\ell_3(\varsigma))) = \frac{3}{16k^2(k+1)}e^{-\wp^4} + \frac{3}{4^4(\wp^4+4)}\sin(\omega(\wp)) + \frac{3}{4^4(\wp+3)}\sin(\upsilon(\wp)) + \frac{3}{4^4(\wp+11)}\arctan(\nu(\wp)) + \frac{3}{4^4(\wp^3+3)}(l_3(\wp)),$$

$$g_3(\wp,\varsigma,\omega(\ell_3(\varsigma)),\upsilon(\ell_3(\varsigma)),\nu(\ell_3(\varsigma))) = \frac{\sin(\omega(\wp))\cos(\upsilon(\wp))+\arctan(\nu(\wp))}{e^{3\wp}(3+\omega(\varsigma))(3+\upsilon(\varsigma))(3+\nu(\varsigma))}.$$

Now, we consider the conditions of Theorem (3.1)
(i) The function $A_1(\wp) = \frac{1}{(k(k+1))^{\frac{1}{p}}}e^{-(\wp+1)^2}$, *is bounded and continuous and* $M_1 = \sup\{A_1(\wp):\ \wp\in[0,L]\} = 1$.
(ii) The mappings $\zeta_1(\wp)=\ell_1(\wp)=\varphi_1(\wp)=\wp, \beta_1(\wp)=\wp^2 : [0,L]\to[0,\infty)$ *are continuous and* $\lim_{\wp\to\infty}\zeta_1(\wp) = \lim_{\wp\to\infty}\wp = 0$.
(iii) The mapping $\phi(\wp)=\wp$ *is continuous and* $\exists$ *constant* 1 *so that*

$$|\phi(\wp_1)-\phi(\wp_2)|^p = |\wp_1-\wp_2|^p \le |\wp_1-\wp_2|^p.$$

(iv) The mapping f_1, h_1 *and* $\varphi_1(\wp)=\wp$ *are continuous and* $\varphi_1(0)=0$ $\theta=6\wp$ *is* $L-$*function and* $\psi(\wp)=\wp$, $\theta o\psi=\psi o\theta$.

Now, let $\wp\in[0,\infty)$ *and* $x,y,z,l,u,v,w,q\in\mathbb{R}$ *with* $|u|<|x|$, $|v|<|y|$ *and* $|w|<|z|$. *Then by using the function* $\theta(\wp)=6\wp$ *and* $\varphi_1(\wp)=\wp$ *we can get following results*

$$\begin{aligned}
&|f_1(\wp,x,y,z,l)-f_1(\wp,u,v,w,q)|^p\\
&= \Big|\frac{1}{4^4(\wp^3+3)}(\sin(x(\wp))-\sin(u(\wp)) + \frac{1}{4^4(\wp+6)}(\sin(y(\wp))-\sin(v(\wp))\\
&\quad + \frac{1}{4^4(\wp^4+9)}(\arctan(z(\wp))-\arctan(w(\wp)) + + \frac{1}{4^4(\wp^3+5)}(l(\wp)-q(\wp))\Big|^p\\
&\le \frac{1}{3}4^{2p}(\frac{1}{4^{4p}}(|\sin(x(\wp))-\sin(u(\wp))|^p + |\sin(y(\wp))-\sin(v(\wp))|^p\\
&\quad + |\arctan(z(\wp))-\arctan(w(\wp))|^p + |l(\wp)-q(\wp)|^p)\\
&\le \frac{1}{3(4^{2p})}(|x-u|^p+|y-v|^p+|z-w|^p+|l-q|^p)\\
&= \frac{1}{3(4^{2p})}(\theta(\frac{1}{6}(|x-u|^p+|y-v|^p+|z-w|^p)) + \varphi_1(|l-q|^p)),
\end{aligned}$$

and

$$\begin{aligned}
&|h_1(\wp,x,y,z)-h_1(\wp,u,v,w)|^p\\
&= \Big|\frac{1}{4^4(\wp^4+4)}(\arctan(x(\wp))-\arctan(u(\wp)) + \frac{1}{4^4(\wp^2+8)}(\ln(1+|y(\wp)|)-\ln(1+|v(\wp)|)\\
&\quad + \frac{1}{4^4(\wp^2+3)}(\sin(z(\wp))-\sin(w(\wp))\Big|^p\\
&\le \frac{1}{3}4^{2p}(\frac{1}{4^{4p}}(|x-u|^p+|y-v|^p+|z-w|^p)\\
&= \frac{1}{3}(\frac{1}{4^{2p}}(\theta(\frac{1}{6}(|x-u|^p+|y-v|^p+|z-w|^p).
\end{aligned}$$

(v) The mappings $\wp\to|h_i(\wp,0,0,0)|$ *and* $\wp\to|f_i(\wp,0,0,0,0)|$ *are bounded on* $[0,L]$; *i.e.*

$$M_1' = \sup\{\sum_{k=1}^{\infty}|k\Delta f_i(\wp,0,0,0,0)|^p,\ \wp\in[0,\infty)\} = \sup\{\sum_{k=1}^{\infty}|k\Delta\frac{1}{7k^2(k+1)}e^{-\wp^2}|^p,\ \wp\in[0,L]\} = \frac{1}{7} < \infty,$$

$$M_1^{''}=\sup\{\sum_{k=1}^{\infty}|k\Delta h_i(\wp,0,0,0)|^p,\ \wp\in[0,\infty)\}=\sup\{\sum_{k=1}^{\infty}|k\Delta(\frac{1}{5k^2(k+1)e^{-\wp^3}})|^p,\ \wp\in[0,L]\}=\frac{1}{5}<\infty,$$

(vi) Obviously, g_1 is continuous and for each $\wp,\varsigma\in[0,L]$ and $u,v,w,x,y,z\in\mathbb{R}$ we get

$$\begin{aligned}
&|g_1(\wp,\varsigma,x,y,z)-g_1(\wp,\varsigma,u,v,w)|^p\\
&\quad= |\frac{1}{k^2}(\frac{\sin(x(\wp))\cos(y(\wp))+\arctan(z(\wp))}{e^{\wp}(1+x(\varsigma))(1+y(\varsigma))(1+z(\varsigma))}-\frac{\sin(u(\wp))\cos(v(\wp))+\arctan(w(\wp))}{e^{t}(1+u(\varsigma))(1+v(\varsigma))(1+w(\varsigma))})|^p\\
&\quad\le \frac{2^p}{k^{2p}}(|\frac{\sin(x(\wp))\cos(y(\wp))+\arctan(z(\wp))}{e^{\wp}(1+x(\varsigma))(1+y(\varsigma))(1+z(\varsigma))}|^p+|\frac{\sin(u(\wp))\cos(v(\wp))+\arctan(w(\wp))}{e^{\wp}(1+u(\varsigma))(1+v(\varsigma))(1+w(\varsigma))}|^p\\
&\quad\le \frac{2^p}{k^{2p}}(2^p(|\frac{\sin(x(\wp))\cos(y(\wp))}{e^{\wp}(1+x(\varsigma))(1+y(\varsigma))(1+z(\varsigma))}|^p+|\frac{\arctan(z(\wp))}{e^{\wp}(1+x(\varsigma))(1+y(\varsigma))(1+z(\varsigma))}|^p\\
&\qquad+|\frac{\sin(u(\wp))\cos(v(\wp))}{e^{\wp}(1+u(\varsigma))(1+v(\varsigma))(1+w(\varsigma))}|^p+|\frac{\arctan(w(\wp))}{e^{\wp}(1+u(\varsigma))(1+v(\varsigma))(1+w(\varsigma))}|^p))\\
&\quad\le \frac{2^{2p+2}}{k^{2p}}|\frac{1}{e^{\wp}}|^p.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\lim_{k\to\infty}\sum_{k=1}^{\infty}|k\Delta(\int_0^{\beta_i(\wp)}g_i(\wp,\varsigma,x(\ell(\varsigma)),y(\ell(\varsigma)),z(\ell(\varsigma)))-g_i(\wp,\varsigma,u(\ell(\varsigma)),v(\ell(\varsigma)),w(\ell(\varsigma))))|^p d\varsigma\\
&\qquad=\lim_{k\to\infty}\sum_{k=1}^{\infty}k^p\Delta\int_0^{\wp^2}\frac{2^{2p+2}}{k^{2p}}\frac{1}{e^{p\wp}}d\varsigma=\lim_{k\to\infty}\frac{2^{2p+2}}{k^p}\frac{\varsigma}{e^{p\wp}}\Big|_0^{\wp^2}=0,
\end{aligned}$$

uniformly to $u,v,w,x,y,z,\in\mathbb{R}$.

Furthermore, we get

$$\begin{aligned}
&|\int_0^{\wp^2}g_1(\wp,\varsigma,\upsilon,\nu,\omega)|^p\\
&\quad= |\int_0^{\wp^2}\frac{\sin(\upsilon(\wp))\cos(\nu(\wp))+\arctan(\omega(\wp))}{e^{\wp}(1+\upsilon(\varsigma))(1+\nu(\varsigma))(1+\omega(\varsigma))}|^p\\
&\quad\le \int_0^{\wp^2}|\frac{\sin(\upsilon(\wp))\cos(\nu(\wp))}{e^{\wp}(1+\upsilon(\varsigma))(1+\nu(\varsigma))(1+\omega(\varsigma))}|^p+\int_0^{\wp^2}|\frac{\arctan(\omega(\wp))}{e^{\wp}(1+\upsilon(\varsigma))(1+\nu(\varsigma))(1+\omega(\varsigma))}|^p\\
&\quad\le \int_0^{\wp^2}\frac{2}{e^{p\wp}}d\varsigma=\frac{2\wp^2}{e^{p\wp}},
\end{aligned}$$

for any $\wp,\varsigma\in[0,L]$ and $\upsilon,\nu,\omega\in\mathbb{R}$.

So

$$\begin{aligned}
M_1^{'''}&=\sup\{\sum_{k=1}^{\infty}|k\Delta(\phi(\int_0^{\beta_1(\wp)}|g_i(\wp,\varsigma,\upsilon(\ell(\varsigma)),\nu(\ell(\varsigma)),\omega(\ell(\varsigma)))))|^p,\ \wp,\varsigma\in[0,L],\upsilon,\nu,\omega\in\mathbb{R}\}\\
&\le\sup\{\sum_{k=1}^{\infty}k^p\Delta\frac{2\wp^2}{k^{2p}e^{p\wp}}:\ \wp\in[0,L]\}=r_0<\infty.
\end{aligned}$$

Also

$$\lim_{k\to\infty}\sum_{k=1}^{\infty}|k\Delta f_i(\wp,0,0,0,0)|^p=\lim_{k\to\infty}\sum_{k=1}^{\infty}|k\Delta(\frac{1}{7k^2(k+1)}e^{-\wp^2}|^p=0,$$

$$\lim_{k\to\infty}\sum_{k=1}^{\infty}|k\Delta h_i(\wp,0,0,0)|^p=\lim_{k\to\infty}\sum_{k=1}^{\infty}|k\Delta(\frac{1}{5k^2(k+1)}e^{-\wp^3}|^p=0,$$

and

$$\lim_{k\to\infty}\sum_{k=1}^{\infty}|k\Delta A_1(\wp)|^p=\lim_{k\to\infty}\sum_{k=1}^{\infty}|k\Delta(\frac{1}{(k(k+1))}e^{-(\wp+1)^2}|^p=0.$$

Thus, all hypothesis of Th 3.1 are fulfils. So, the E.q (3.1) has at least one solution in $C(I,h_p)\times C(I,h_p)\times C(I,h_p)$.

4. Conclusion

In this paper, we introduce the Hausdorff measure of noncompactness in p-Hahn sequence space and we obtain an extension of Darboś fixed point theorem. By applying the technique of measure of noncompactness and extended of Darboś theorem, we investigate the existence of solution of a class of tripled system of nonlinear integral equations in the p-Hahn sequence space. Finally, we present one example to verify the usefulness of main results.

References

1. A. Aghajani, J. Banaś. and Y., Jalilian, *Existence of solution for a class of nonlinear Volterra singular integral equation.* Comput. Math. Appl., 62 (2011) 1215–1227.

2. A. Aghajani. and N. Sabzali, *Existence of coupled fixed points via measure of noncompactness and applications.* J. Nonlinear Convex Anal. 15(5) (2014) 941–952.

3. R. R. Akhmerov, M.I. Kamenski, A.S. Potapov, A. E. Rodkina. and B. N. Sadovskii, *Measures Of Noncompactness And Condensing Operators.* Birkhauser Verlag. Basel 1992.

4. H. Amiri Kayvanloo, M. Khanehgir. and R. Allahyari, *A generalization of Darbo's theorem with application to the solvability of systems of integral diffrential equations in Sobolev spaces.* Int. J. Non. Anal. Appl, (2021) 12 (1), 287-300.

5. J. Banaś. and K. Goebel, *Measures of noncompactness in Banach spaces, Lecture notes in pure and applied mathematics.* 60 Marcel Dekker, New York, 1980.

6. J. Banaś. and M. Lecko, *Solvability of infinite systems of differential equations in Banach sequence spaces.* J. Comput. Appl. Math. **137** (2001) 363–375.

7. J. Banaś. and M. Mursaleen, *Sequence spaces and measure of noncompactness with applications to differential and integral equation.* Springer, India 2014.

8. V. Berinde. and M. Borcut, *Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces.* Nonlinear Anal. Theory Methods Appl. 2011, 74, 4889–4897.

9. A. Das, M. Rabbani, S.A. Mohiuddine. and B.C. Deuri, *Iterative algorithm and theoretical treatment of existence of solution for (k, z)-Riemann-Liouville fractional integral equations.* J. Pseudo-Differ. Oper. Appl. (2022) 13:39.

10. A. Das, S.A. Mohiuddine, A. Alotaibi and B.C. Deuri, *Generalization of Darbo-type theorem and application on existence of implicit fractional integral equations in tempered sequence spaces.* Alexandria Eng. J. 61 (2022) 2010-2015.

11. G. Darbo, *Punti uniti in trasformazioni a codominio non compatto.* Rend. Sem. Mat. Uni. Padova., 24 (1955) 84-92.

12. L.S. Goldenstein, L.T. Gohberg. and A.S. Murkus, *Investigations of some properties of bounded linear operators with their $q-$norms.* Ucen. Zap. Kishinevsk. Uni. 29 (1957) 29–36.

13. L.S. Goldenstein. and A.S. Murkus, *On a meausure of noncompactness of bounded sets and linear operators.* Studies in Algebra and Math. Anal. kishinev. (1965) 45–54.

14. H. Hahn, *Über Folgen linearer operationen.* Monatshefte für Mathematik und Physik, 32(1) (1922) 3-88.

15. B. Hazarika, A. Das, R. Arab. and M. Mursaleen, *Solvability of the infinite system of integral equations in two variables in the sequence spaces c_0 and l_1.* J. Comput. Appl. Math., 326 (2012) 183–192.

16. H.A. Hammad. and M. De la Sen, *A technique of tripled coincidence points for solving a system of nonlinear integral equations in POCML spaces.* J. Inequal. Appl. 2020, 2020, 211.

17. H.A. Hammad. and M. De la Sen, *A tripled fixed point technique for solving a tripled-system of integral equations and Markov process in CCbMS.* Adv. Differ. Equ. 2020, 2020, 567.

18. V. Karakaya, N.E.H. Bouzara, K. Dogan. and Y. Atalan, *Existence of tripled fixed points for a class of condensing operators in Banach spaces.* Sci. World J. 2014, 2014, 541862.

19. M. Kirisci, *$p-$Hahn sequence space.* Far East Journal of Mathematical Sciences, vol. 90, no. 1, pp. 45–63, 2014.

20. K. Kuratowski, *Sur les espaces complets.* Fund. Math., 15 (1930) 301-309.

21. P. Malaki, S. Kumar. and M. Mursaleen, *Solvability of infinite systems of third order differential equations in a sequence space $n(\phi)$ via measures of non-compactness.* Universal J. Math. Appl., 8 (1) (2025) 30–40.

22. E. Malkowsky, V. Rakočević. and V.O. Tuğ, *Compact operators on the Hahn space.* Monatshefte für Mathematik, 196(3) (2021) 519-551.

23. H. Mehravaran. and H. Amiri Kayvanloo, *Solvability of infinite system of nonlinear convolution type integral equations in the tempered sequence space $m^{\beta}(\varphi, p)$).* Asian-European Journal of Mathematics , (2023) 16 (1), 2350004 (17 pages).

24. H. Mehravaran, H. Amiri Kayvanloo. and R. Allahyari, *Solvability of infinite systems of fractional differential equations in the space of tempered sequence space $m^{\beta}(\phi)$.* International Journal of Nonlinear Analysis and Applications, (2022) 13(1), 1023-1034.

25. H. Mehravaran, H. Amiri Kayvanloo. and M. Mursaleen, *Solvability of infinite systems of fractional differential equations in the double sequence space $2^c(\Delta)$.* Fract Calc Appl Anal 25, 2298–2312 (2022).

26. S.A. Mohiuddine, A. Das. and A. Alotaibi, *Existence of solutions for nonlinear integral equations in tempered sequence spaces via generalized Darbo-type theorem.* J. Funct. Spaces 2022 (2022) 4527439, pp8.

27. S.A. Mohiuddine, A. Das. and A. Alotaibi, *Existence of solutions for infinite system of nonlinear $q-$fractional boundary value problem in Banach spaces.* Filomat 37(30) (2023) 10171-10180.

28. M. Mursaleen, *Application of measure of noncompactness to infinite system of differential equations.* Can. Math. Bull., 56 (2013) 388–394.

29. M. Mursaleen, *Some geometric properties of a sequence space related to l_p,*. Bull. Aust. Math. Soc., 67(2) (2003) 343–347.

30. M. Mursaleen, B. Bilalov. and S.M.H. Rizvi, *Applications of measures of noncompactness to infinite system of fractional differential equations*. Filomat, 31(11) (2017) 3421–3432.

31. M. Mursaleen. and S.A. Mohiuddine, *Applications of measures of noncompactness to the infinite system of differential equations in l_p spaces*. Nonlinear Anal., 75 (2012) 2111–2115.

32. M. Mursaleen. and S.M.H. Rizvi, *Solvability of infinite system of second order differential equations in c_0 and l_1 by Meir–Keeler condensing operator*. Proc. Am. Math. Soc., 144(10) (2016) 4279–4289.

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