Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–15.
ISSN-0037-8712
doi:10.5269/bspm.77257

Crank Nicolson Approximation to Space-Time Fractional Stochastic Traveling Wave Equation with Additive White Noise

Priti Mojad*, Babasaheb Gavhane, Shrikisan Gaikwad and Kalyanrao Takale

ABSTRACT: This article describes a Crank-Nicolson finite difference approach for numerically predicting the solutions of space- time fractional stochastic traveling wave equation (STFSTWE) with additive white noise. The model employs a right-shifted Grunwald fractional derivative in the spatial dimension and a Caputo-type fractional derivative in the time dimension for discretization. The fractional orders for space and time ranges from $1 < \alpha \leq 2$ and $1 < \beta \leq 2$ respectively. The proposed numerical technique discretizes the governing equations into a nonlinear algebraic system at each time level with the coefficient matrix generated systematically using an automated procedure. Python gives graphical representations of the answers in numerous situations, establishing the method's efficiency. Furthermore, two numerical experiments are included to validate the numerical approach. Stability and convergence are also investigated using the mean square method.

Key Words: Additive noise, Crank Nicolson approximation, fractional derivatives, mean square method.

Contents

1. Introduction

In recent years, fractional differential equations have gained much attention because of their capacity to reproduce the memory and hereditary features observed in various natural and manufactured systems. Traditional integer-order models frequently fall short of representing such complicated dynamics, particularly in domains such as viscoelasticity, control theory, fluid dynamics, and biology [1]. Fractional calculus, which extends the concept of integer-order derivatives to non-integer orders, has provided a powerful foundation for more specifically clarifying these occurrences. This model's applicability is further improved by adding stochasticity, particularly when working with systems that experience random fluctuations. Modern stochastic modeling is based on stochastic processes. A more accurate depiction

* Corresponding author.
2010 *Mathematics Subject Classification:* 60G15, 26A33, 65M06.
Submitted June 09, 2025. Published October 30, 2025

of physical systems where noise and memory effects coexist is made possible by fractional stochastic differential equations (FSDEs), which combine fractional derivatives with stochastic processes. In one popular form, additive white noise is used to simulate random external influences that are not dependent on the system's state. Because of this, FSDEs are especially well-suited for simulating phenomena like financial markets [7], biological processes [2,11] and climate fluctuation [12] where randomness is a natural component of the environment.

Applications for FSDEs are many and span several fields. With lengthy memory and leaps, they are employed in finance to simulate asset prices and volatility, providing enhancements over traditional Black-Scholes models [13]. FSDEs aid in the simulation of systems with hereditary damping, anomalous transport, and viscoelastic materials in physics and engineering [1]. Moreover, FSDEs are used to describe porous media with random heterogeneities and irregular groundwater flows in hydrology [14]. Thus, these equations offer a strong foundation for modeling intricate systems in which memory and randomness are unavoidable. Since analytical solutions to FSDEs are sometimes unattainable, numerical approaches are essential. To account for both the fractional and stochastic component techniques including spectral approaches, Monte Carlo simulations, finite difference schemes, etc. have been modified.

The finite difference method (FDM) is notable among them due to its ease of use and efficiency. The Crank-Nicolson method included in the finite difference technique is especially well-suited for solving diffusion-type, hyperbolic and parabolic equations [6]. In this work, we use a Crank-Nicolson finite difference technique to investigate the numerical solution of STFSTWE driven by additive white noise.

The Space-Time Fractional Stochastic Traveling Wave Equation is given by ,

$$\frac{\partial^\alpha \theta(x,t)}{\partial t^\alpha} = K^2 \frac{\partial^\beta \theta(x,t)}{\partial x^\beta} + \sigma \dot{W}(t);\ (x,t) \in\ [0,L] \times [0,T], 1 < \alpha \leq 2, 1 < \beta \leq 2 \tag{1.1}$$

Subject to the initial Conditions :

$$\theta(x,0) = h_1(x), \quad 0 < x \leq\ L \tag{1.2}$$

$$\theta_t(x,0) = h_2(x), 0 < x \leq\ L \tag{1.3}$$

Boundary Conditions :

$$\theta(0,t) = 0, 0 < t \leq\ T \tag{1.4}$$

$$\theta(L,t) = 0, \quad 0 < t \leq\ T \tag{1.5}$$

where, $h_1(x), h_2(x)$ are known functions, K, σ , L , T are given constants & we want to approximate unknown function $\theta(x,t)$. Moreover, $\dot{W}(t)$ is the time white noise, where W(t) is one dimensional standard wiener process. $\frac{\partial^\alpha \theta(x,t)}{\partial t^\alpha}, \frac{\partial^\beta \theta(x,t)}{\partial x^\beta}$ be the fractional derivatives of order $1 < \alpha \leq 2$ & $1 < \beta \leq 2$ respectively.

This paper is arranged as Section (1) comes from the introduction. Section (2) contains key definitions and some lemma of fractional stochastic calculus. Section (3) covers the main technique. Section (4) discussed the stability. Section (5) shows convergence of the proposed method. Section (6) provides a particular numerical example in that we use Python to graphically depict findings to show the behavior of the solution, emphasizing how the dynamics of the system are affected by both fractional orders and random perturbations. Results and discussion covers in the section (7). The conclusion appears in section (8) .

2. Preliminaries

This section discuss several essential concepts and lemmas used in further study in fractional stochastic calculus.

Definition 2.1 *[4] Let $\alpha \geq 0$ and $k = \lceil \alpha \rceil \in \mathbb{N}$. The Caputo fractional derivative of order α of a sufficiently smooth function $f(x)$ is defined as*

$$D^\alpha f(x) := \begin{cases} \frac{1}{\Gamma(k-\alpha)} \int_0^x (x-t)^{k-\alpha-1} f^{(k)}(t)\, dt, & \text{if } k-1 < \alpha < k, \quad x > 0, \\ f^{(k)}(x), & \text{if } \alpha = k \in \mathbb{N}, \end{cases}$$

where $f^{(k)}(t)$ denotes the k-th order classical derivative of f, and the Gamma function $\Gamma(\cdot)$ is given by

$$\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}\,dt, \quad z > 0.$$

Definition 2.2 *[9] The right-shifted Grünwald-Letnikov approximation for the space-fractional derivative of order $\beta \in (1,2]$ is given by:*

$$\frac{\partial^\beta V}{\partial x^\beta}(x_i,t_n) \approx \frac{1}{h^\beta}\sum_{l=0}^{i+1} w_l V_{i-l+1}^n + \mathcal{O}(h),$$

where h is the spatial step size, and w_l are the normalized Grünwald weights, defined explicitly by:

$$w_l = \frac{\Gamma(l-\beta)}{\Gamma(l+1)\Gamma(-\beta)}, \quad l = 0,1,2,\ldots$$

Alternatively, the weights can be defined recursively by:

$$w_0 = 1, \quad w_l = w_{l-1}\left(1 - \frac{\beta+1}{l}\right), \quad l \geq 1.$$

Definition 2.3 *[15] A stochastic difference scheme is said to be stable with respect to the mean square norm if there exist positive constants ϵ and δ, and non-negative constants λ and v such that the following inequality holds:*

$$E\left\|u^{n+1}\right\|^2 \leq \lambda e^{vt} E\left\|u^0\right\|^2,$$

for all $t = (n+1)\Delta t$, where the time step Δt and spatial step Δx satisfy the conditions:

$$0 \leq \Delta x \leq \epsilon, \quad 0 \leq \Delta t \leq \delta.$$

Definition 2.4 *[15] A difference scheme is called unconditionally stable if no restrictions on the relationship between Δx and Δt are needed unless it is called conditionally stable.*

Definition 2.5 *[5] A stochastic difference scheme $L_i^m S_i^m = G_i^m$, which approximates a stochastic differential equation (SDE) $Lv = G$ at time $t = (m+1)\Delta t$ is said to be convergent in the mean square sense if*

$$E\left|u_i^m - u\right|^2 \to 0 \quad as \quad (ih, m\Delta t) \to (x,t), \quad h \to 0, \quad \Delta t \to 0.$$

Lemma 2.1 *[10] Let $b_j = (j+1)^{2-\alpha} - j^{2-\alpha}$ for $j = 1,2,\ldots$ and $\alpha > 0$. Then:*

1. *$b_j > 0$ (positivity)*
2. *$b_j > b_{j+1}$ (monotonically decreasing)*
3. *$b_{j+1} - b_j < b_j - b_{j-1}$ (convex sequence)*

Lemma 2.2 *[10] Let w_l denote the Grünwald–Letnikov weights for a fractional derivative of order $\beta > 0$. Then the coefficients w_l satisfy:*

1. *Initial values: $w_0 = 1, \quad w_1 = -\beta, \quad w_2 = \frac{\beta(\beta-1)}{2}$*
2. *Monotonic non-increasing and non-negative tail: $1 \geq w_2 \geq w_3 \geq \cdots \geq 0$*
3. *Bounded cumulative sum: $\sum_{l=0}^n w_l \leq n, \quad$ for $n = 0,1,2,\ldots$*

3. Numerical Scheme

The technique for solving STFSTWE through Crank Nicolson approximation is detailed here. In order to develop the numerical technique for the proposed model (1.1) - (1.5), we utilized the notations below.

Let us define $\mathcal{A} = \{t_n = n\Delta t | \Delta t = \frac{T}{M}, n = 0, 1, 2, ..., M\}$ be a set of uniformly distributed points on interval [0,T]. likewise, consider $\mathcal{B} = \{x_i = ih | h = \frac{L}{N}, i = 0, 1, 2, ..., N\}$ be a uniformly distributed points on interval [0,L] and $\theta_i^n = \theta(x_i, t_n)$ represent the numerical approximation at the mesh point (x_i, t_n) to the exact solution $\theta(x, t)$.

3.1. Approximation in time

According to definition (2.1), $\frac{\partial^\alpha \theta(x,t)}{\partial t^\alpha}$ can be written as,

$$\frac{\partial^\alpha \theta(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \, \frac{\partial^2 \theta(x,s)}{\partial s^2} \, ds, & 1 < \alpha < 2, \, x > 0 \\ \\ \frac{\partial^2 \theta(x,t)}{\partial t^2}, & \alpha = 2 \end{cases}$$

At mesh points (x_i, t_{n+1}) , i = 0,1,...,N , n = 0,1,...,M-1 time fractional derivative $\frac{\partial^\alpha \theta(x,t)}{\partial t^\alpha}$ for $1 < \alpha \leq 2$ approximated as follows :

$$\begin{aligned}
\frac{\partial^\alpha \theta(x_i, t_{n+1})}{\partial t^\alpha} &= \frac{1}{\Gamma(2-\alpha)} \int_0^{t_{n+1}} (t_{n+1} - s)^{1-\alpha} \, \frac{\partial^2 \theta(x_i, s)}{\partial s^2} \, ds \\
&= \frac{1}{\Gamma(2-\alpha)} \Big[\sum_{j=0}^{j=n} \int\limits_{j\Delta t}^{(j+1)\Delta t} \frac{\partial^2 \theta(x_i, t_{n+1} - \eta)}{\partial \eta^2} \, \mathrm{d}\eta \Big] \Big[\sum_{j=0}^{j=n} \int\limits_{j\Delta t}^{(j+1)\Delta t} \eta^{1-\alpha} \, \mathrm{d}\eta \Big] \\
&= \frac{\Delta t^{2-\alpha}}{(2-\alpha)\Gamma(2-\alpha)} \Big[\sum_{j=0}^{j=n} \frac{\theta_i^{n+1-j} + \theta_i^{n-1-j} - 2\theta_i^{n-j}}{\Delta t^2} + o(\Delta t^2) \Big] \Big[\sum_{j=0}^{j=n} (j+1)^{2-\alpha} - (j)^{2-\alpha} \Big] \\
&= \frac{\Delta t^{-\alpha}}{\Gamma(3-\alpha)} \Big[\sum_{j=0}^{j=n} (\theta_i^{n+1-j} + \theta_i^{n-1-j} - 2\theta_i^{n-j}) \big((j+1)^{2-\alpha} - (j)^{2-\alpha} \big) \Big] \\
&\qquad + \frac{(\Delta t)^{2-\alpha} \, o(\Delta t^2)}{\Gamma(3-\alpha)} \Big[\sum_{j=0}^{j=n} (j+1)^{2-\alpha} - (j)^{2-\alpha} \Big] \\
&= \frac{\Delta t^{-\alpha}}{\Gamma(3-\alpha)} \Big[\sum_{j=0}^{j=n} (\theta_i^{n+1-j} + \theta_i^{n-1-j} - 2\theta_i^{n-j}) \big((j+1)^{2-\alpha} - (j)^{2-\alpha} \big) \Big] \\
&\qquad + \frac{[(\Delta t)(n+1)]^{2-\alpha} o(\Delta t^2)}{\Gamma(3-\alpha)}
\end{aligned}$$

Obviously, if $(n+1)(\Delta t) < \infty$ then with $\frac{[(\Delta t)(n+1)]^{2-\alpha} o(\Delta t^2)}{\Gamma(3-\alpha)} \approx o(\Delta t^2)$ we have

$$\frac{\partial^\alpha \theta(x_i, t_{n+1})}{\partial t^\alpha} = \frac{\Delta t^{-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{j=n} \Big[(\theta_i^{n+1-j} + \theta_i^{n-1-j} - 2\theta_i^{n-j}) \big((j+1)^{2-\alpha} - (j)^{2-\alpha} \big) \Big] + o(\Delta t^2)$$

Therefore, Caputo type fractional derivative of fractional order $1 < \alpha \leq 2$ approximated as,

$$\frac{\partial^\alpha \theta(x_i, t_{n+1})}{\partial t^\alpha} = \frac{\Delta t^{-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{j=n} b_j (\theta_i^{n+1-j} + \theta_i^{n-1-j} - 2\theta_i^{n-j}) \tag{3.1}$$

where, $b_j = (j+1)^{2-\alpha} - (j)^{2-\alpha}$

3.2. Approximation in space

According to definition (2.2) , At mesh points (x_i, t_{n+1}) , i = 0,1,...,N , n = 0,1,...,M-1 space fractional derivative of order $1 < \beta \leq 2$ is given by ,

$$\frac{\partial^\beta \theta(x_i, t_{n+1})}{\partial t^\beta} = \frac{1}{h^\beta} \sum_{l=0}^{i+1} w_l \theta(x_i, t_{n+1}) + O(h) \tag{3.2}$$

3.3. Approximation of noise term

[6,11] A stochastic process $\{W(t) : t \geq 0\}$ is called a Wiener process (or standard Brownian motion) if it satisfies the following properties:

1. Initial Condition: W(0) = 1.
2. Gaussian Increments: For all $0 \leq s < t$, the increment $W(t) - W(s)$ is normally distributed with mean zero and variance $t - s$, that is, W(t) - W(s)$\sim \sqrt{t-s} \cdot \mathcal{N}(0,1)$,
3. Independent Increments: For all $0 \leq s < t < u$, the increments $W(t) - W(s)$ and $W(u) - W(t)$ are independent.

The white noise process $\dot{W}(t)$ is formally defined as the generalized derivative of a standard Brownian motion $W(t)$, i.e., $\dot{W}(t) = \frac{dW(t)}{dt}$ with zero mean and unit variance.

Finite difference approximation gives, $\dot{W}(t) = \frac{W((n+1)\Delta t) - W(n\Delta t))}{\Delta t} = \frac{X_n}{\Delta t}$
where,

$$X_n = W((n+1)\Delta t) - W(n\Delta t)) \approx \sqrt{\Delta t} \mathcal{N}(0,1);\ n = 0, 1, ..., M-1 \tag{3.3}$$

$\mathcal{N}(0,1)$ represents a normal distribution with a mean of 0 and a variance of 1.

3.4. Scheme formulation for the given model

Now, grouping Equations (3.1), (3.2) and (3.3), the numerical approximation of space-time fractional stochastic traveling wave equation as follows :

$$\sum_{j=0}^{j=n} b_j(\theta_i^{n+1-j} + \theta_i^{n-1-j} - 2\theta_i^{n-j}) = \frac{K^2 \Gamma(3-\alpha)(\Delta t)^\alpha}{2h^\beta} \Big[\sum_{l=0}^{i+1} w_l \theta_{i-l+1}^{n+1} + \sum_{l=0}^{i+1} w_l \theta_{i-l+1}^{n} \Big] + \sigma \Gamma(3-\alpha)(\Delta t)^{\alpha-1} X_n$$

$$b_0(\theta_i^{n+1} + \theta_i^{n-1} - 2\theta_i^n) + \sum_{j=1}^{j=n} b_j(\theta_i^{n+1-j} + \theta_i^{n-1-j} - 2\theta_i^{n-j}) = a \Big[\sum_{l=0}^{i+1} w_l \theta_{i-l+1}^{n+1} + \sum_{l=0}^{i+1} w_l \theta_{i-l+1}^{n} \Big] + c X_n$$

$$\theta_i^{n+1} - a \Big[\sum_{l=0}^{i+1} w_l \theta_{i-l+1}^{n+1} + \sum_{l=0}^{i+1} w_l \theta_{i-l+1}^{n} \Big] = 2\theta_i^n - \theta_i^{n-1} - \sum_{j=1}^{j=n} b_j(\theta_i^{n+1-j} + \theta_i^{n-1-j} - 2\theta_i^{n-j}) + c X_n \tag{3.4}$$

Where,

$a = \frac{K^2 \Gamma(3-\alpha)(\Delta t)^\alpha}{2h^\beta}$
$c = \sigma \Gamma(3-\alpha)(\Delta t)^{\alpha-1}$
$b_j = (j+1)^{2-\alpha} - (j)^{2-\alpha}; j = 0, 1, ..., n$
$X_n = W_{n+1} - W_n \approx \sqrt{\Delta t} \mathcal{N}(0,1)$
$b_0 = 1$

The initial conditions (1.2) & (1.3) approximated as,

$$\theta_i^o = h_1(x_i);\ i = 1, 2, ..., N-1 \tag{3.5}$$

$$\theta_i^{-1} = \theta_i^1 - 2\Delta t\ (h_2(x_i));\ i = 1, 2, ..., N-1 \tag{3.6}$$

The boundary conditions (1.4) & (1.5) approximated as,

$$\theta_0^n = 0;\ n = 1, 2, ..., M-1 \tag{3.7}$$

$$\theta_N^n = 0;\ n = 1, 2, ..., M-1 \tag{3.8}$$

Using approximated initial conditions (3.5) & (3.6) , for n=0 equation (3.4) becomes ,

$$\theta_i^1 - \frac{a}{2}\Big[\sum_{l=0}^{i+1} w_l \theta_{i-l+1}^1 + \sum_{l=0}^{i+1} w_l \theta_{i-l+1}^0\Big] = \theta_i^0 + (\Delta t)(h_2(x_i))$$

Similarly , Using initial conditions (3.5) & (3.6) , for n=1,2,...,M-1 equation (3.4) becomes ,

$$\begin{aligned}\theta_i^{n+1} - a\Big[\sum_{l=0}^{i+1} w_l \theta_{i-l+1}^{n+1} + \sum_{l=0}^{i+1} w_l \theta_{i-l+1}^n\Big] &= 2\theta_i^n - \theta_i^{n-1} - \sum_{j=1}^{j=n-1} b_j(\theta_i^{n+1-j} + \theta_i^{n-1-j} - 2\theta_i^{n-j}) \\ &\quad - 2b_n(\theta_i^1 - (\Delta t)(h_2(x_i)) - \theta_i^0) + cX_n\end{aligned}$$

Therefore, the complete Crank-Nicolson discretization of space-time fractional stochastic traveling wave equation is given by,
For n=0 ,

$$\theta_i^1 - \frac{a}{2}\Big[\sum_{l=0}^{i+1} w_l \theta_{i-l+1}^1 + \sum_{l=0}^{i+1} w_l \theta_{i-l+1}^0\Big] = \theta_i^0 + (\Delta t)(h_2(x_i)) \tag{3.9}$$

For n $\geq$ 1 ,

$$\begin{aligned}\theta_i^{n+1} - a\Big[\sum_{l=0}^{i+1} w_l \theta_{i-l+1}^{n+1} + \sum_{l=0}^{i+1} w_l \theta_{i-l+1}^n\Big] &= 2\theta_i^n - \theta_i^{n-1} - \sum_{j=1}^{j=n-1} b_j(\theta_i^{n+1-j} + \theta_i^{n-1-j} - 2\theta_i^{n-j}) \\ &\quad - 2b_n(\theta_i^1 - (\Delta t)(h_2(x_i)) - \theta_i^0) + cX_n\end{aligned} \tag{3.10}$$

Initial conditions :

$$\theta_i^o = h_1(x_i); i = 1, 2, ..., N-1 \tag{3.11}$$

Boundary conditions:

$$\theta_0^n = 0; n = 1, 2, ..., M-1 \tag{3.12}$$

$$\theta_N^n = 0; n = 1, 2, ..., M-1 \tag{3.13}$$

Where,

$a = \frac{K^2\Gamma(3-\alpha)(\Delta t)^\alpha}{2h^\beta}$, $c = \sigma\Gamma(3-\alpha)(\Delta t)^{\alpha-1}$, $b_j = (j+1)^{2-\alpha} - (j)^{2-\alpha}; j = 0, 1, ..., n$ and $X_n = W_{n+1} - W_n \approx \sqrt{\Delta t}\mathcal{N}(0, 1)$.

3.5. Matrix representation of the derived scheme

$$\text{Let, } \theta_n = \begin{bmatrix} \theta_1^n \\ \theta_2^n \\ \vdots \\ \vdots \\ \theta_{N-1}^n \end{bmatrix}_{(N-1)\times 1}, \quad G = \begin{bmatrix} (\Delta t)(h_2(x_1)) \\ (\Delta t)(h_2(x_2)) \\ \vdots \\ \vdots \\ (\Delta t)(h_2(x_{N-1})) \end{bmatrix}_{(N-1)\times 1}$$

and I is the identity matrix of order (N-1) $\times$ (N-1).

Therefore, Matrix form of complete Crank-Nicolson discretized space-time fractional stochastic traveling wave equation as given below,
For n=0 ,

$$V\theta^1 = (2I - V)\theta^0 + G \tag{3.14}$$

For $n \geq 1$,

$$(2V - I)\theta^{n+1} = [(4 - b_1)I - 2V]\theta^n + \sum_{j=1}^{n-1}(2b_j - b_{j-1} - b_{j+1})\theta^{n-j} - b_n\theta^1 + (2b_n - b_{n-1})\theta^0 + 2b_n(\Delta t)(h_2(x_i))I + cX_n \tag{3.15}$$

Initial conditions :

$$\theta_i^o = h_1(x_i); i = 1, 2, ..., N-1 \tag{3.16}$$

Boundary conditions :

$$\theta_0^n = 0, n = 1, 2, ..., M-1 \tag{3.17}$$

$$\theta_N^n = 0; n = 1, 2, ..., M-1 \tag{3.18}$$

Where, V is a matrix of order (N-1) $\times$ (N-1) given by ,

$$V = (v_{ij})_{(N-1)\times(N-1)} = \begin{cases} 1 - \frac{a}{2}w_1 & , i = j \\ -\frac{a}{2}w_{i-j+1} & , j \leq i-1 \\ -\frac{a}{2}w_0 & , j = i+1 \\ 0 & , j \geq i+2 \end{cases}$$

4. Stability

In this section, we prove stability of proposed numerical technique.

Theorem 4.1 *For discretized equations (3.9) - (3.13) of STFSTWE (1.1) - (1.5) there exist non-negative constant λ independent of h such that it satisfies the condition $E|\theta^n|^2 \leq \lambda E|\theta^0|^2$, $n \,\epsilon N$ under Crank-Nicolson finite difference scheme, then the proposed scheme is said to be unconditionally stable in mean square sense with respect to the $|\ .\ |_\infty = \sqrt[2]{sup_i|\ .\ |^2}$ norm.*

Proof: Define,

$$E|\theta^n|^2 = \sup_{1\leq i\leq N-1} E|\theta_i^n|^2$$

where, $\theta^n = (\theta_1^n, \theta_2^n, ..., \theta_{N-1}^n)^T;\ n = 0, 1, 2, ..., M.$

From equation (3.9) we get,

$$\begin{aligned} E|\theta_i^1|^2 &= E|\theta_i^0 + (\Delta t)(h_2(x_i)) + \frac{a}{2}\Big[\sum_{l=0}^{i+1} w_l\theta_{i-l+1}^1 + \sum_{l=0}^{i+1} w_l\theta_{i-l+1}^0\Big]|^2 \\ &= E|\theta_i^0 + (\Delta t)(\frac{\theta_i^1 - \theta_i^{-1}}{2\Delta t}) + \frac{a}{2}\Big[\sum_{l=0}^{i+1} w_l\theta_{i-l+1}^1 + \sum_{l=0}^{i+1} w_l\theta_{i-l+1}^0\Big]|^2 \\ &\leq \Big[1 + \frac{1}{2}(1-1) + \frac{a}{2}(2\sum_{l=0}^{i+1} w_l)\Big]^2 \sup_{1\leq i\leq N-1} E|\theta_i^0|^2 \\ &\leq \Big[1 + a\sum_{l=0}^{i+1} w_l\Big]^2 E|\theta^0|^2 \end{aligned}$$

By using lemma (2.2) , $a\sum_{l=0}^{i+1} w_l < 0$

$$E|\theta^1|^2 \leq \lambda E|\theta^0|^2 \quad ; \quad \lambda = 1$$

So the result is true for n=1,
Suppose the result is true for n $\leq$ k

$$E|\theta^k|^2 \leq \lambda E|\theta^0|^2$$

From equation (3.10) we get,

$$\begin{aligned}
E|\theta_i^{k+1}|^2 &= E|2\theta_i^k - \theta_i^{k-1} - \sum_{j=1}^{k-1} b_j\Big(\theta_i^{k+1-j} + \theta_i^{k-1-j} - 2\theta_i^{k-j}\Big) + a\Big[\sum_{l=0}^{i+1} w_l\theta_{i-l+1}^k + \sum_{l=0}^{i+1} w_l\theta_{i-l+1}^{k+1}\Big] \\
&\qquad - 2b_k\Big(\theta_i^1 - (\Delta t)\Big(\frac{\theta_i^1 - \theta_i^{-1}}{2\Delta t}\Big) - \theta_i^0\Big) + cX_n|^2 \\
&= E|(2-b_1)\theta_i^k + (-1-b_2+2b_1)\theta_i^{k-1} - \sum_{j=2}^{k-1}(b_{j+1}+b_{j-1}-2b_j)\theta_i^{k-j} - b_k\theta_i^1 - b_k\theta_i^{-1} \\
&\qquad + 2b_k\theta_i^0 + a\Big[\sum_{l=0}^{i+1} w_l\theta_{i-l+1}^k + \sum_{l=0}^{i+1} w_l\theta_{i-l+1}^{k+1}\Big] + cX_n|^2 \\
&\leq |(2-b_1) + (-1-b_2+2b_1) - \sum_{j=2}^{k-1}(b_{j+1}+b_{j-1}-2b_j) - b_k - b_k + 2b_k \\
&\qquad + \ 2a\sum_{l=0}^{i+1} w_l|^2 \sup_{1\leq i\leq N-1} E|\theta_i^k|^2 + \ c^2 \sup_{1\leq i\leq N-1} E|\theta_i^k|^2 \\
&\leq \Big\{|(2-b_1) + (-1-b_2+2b_1) - \sum_{j=2}^{k-1}(b_{j+1}+b_{j-1}-2b_j) - b_k - b_k + 2b_k + 2a\sum_{l=0}^{i+1} w_l|^2 + \ c^2\Big\}E|\theta^k|^2 \\
&\leq \Big\{|1 + b_k - b_{k-1} + 2a\sum_{l=0}^{i+1} w_l|^2 + \ c^2\Big\}E|\theta^0|^2
\end{aligned}$$

By using lemma (2.2) , $2a\sum\limits_{l=0}^{i+1} w_l < 0$

$$E|\theta^{k+1}|^2 \leq \lambda E|\theta^0|^2 \ ; \ \lambda = (1 + b_k - b_{k-1})^2 + \ c^2$$

So the result is true for n= k+1,

By mathematical induction, $E|\theta^n|^2 \leq \lambda E|\theta^0|^2$, where λ is a non-negative constant independent of h, According to definitions (2.3) and (2.4) , The proposed Crank-Nicolson finite difference scheme is unconditionally stable. □

5. Convergence

In this section, the convergence of discretized form (3.9) - (3.13) of STFSTWE (1.1) - (1.5) under crank-nicolson finite difference scheme is discussed.

Let $\widehat{\theta}_i^n$ and θ_i^n are exact and approximate solutions of the STFSTWE (1.1)- (1.5) at the mesh point (x_i, t_n) respectively.

Let T_i^n represent the local truncation error at time level n , for $1 \leq i \leq N$, then based on equations (3.9) - (3.13) we have,
for n=0,

$$T_i^1 = \widehat{\theta}_i^1 \ - \frac{a}{2}\sum_{l=0}^{i+1} w_l\widehat{\theta}_{i-l+1}^1 - \widehat{\theta}_i^0 - (\Delta t)(h_2(x_i)) = o(h + (\Delta t^2)) \tag{5.1}$$

for $n \geq 1$,

$$
\begin{aligned}
T_i^{n+1} = \widehat{\theta}_i^{n+1} - a\sum_{l=0}^{i+1} w_l \widehat{\theta}_{i-l+1}^{n+1} - a\sum_{l=0}^{i+1} w_l \widehat{\theta}_{i-l+1}^{n} + \sum_{j=1}^{j=n-1} b_j(\widehat{\theta}_i^{n+1-j} + \widehat{\theta}_i^{n-1-j} - 2\widehat{\theta}_i^{n-j}) \\
- 2\widehat{\theta}_i^{n} + \widehat{\theta}_i^{n-1} + 2b_n(\widehat{\theta}_i^{1} - (\Delta t)(h_2(x_i)) - \widehat{\theta}_i^{0}) - cX_n = o(h + (\Delta t^2))
\end{aligned}
\tag{5.2}
$$

Theorem 5.1 *Crank-Nicolson finite difference scheme designed to solve STFSTWE (1.1) - (1.5) is unconditionally convergent .*

Proof: Let,

$$E|e^n|^2 = \max_{1\leq i\leq N-1} E|e_i^n|^2$$

where, $e^n = (e_1^n, e_2^n, ..., e_{N-1}^n)^T$ be the error vector for $1 \leq n \leq M$ and $|e_i^n| = |\widehat{\theta}_i^n - \theta_i^n|$ be the error at each mesh point (x_i, t_n).
Moreover, let

$$T^n = \max_{1\leq i\leq N-1} |T_i^n|$$

We have to prove that, $E|e^n|^2 \leq \zeta E|e^0|^2$. From equations (3.9) & (5.1) we get,

$$
\begin{aligned}
E|e_i^1|^2 &= E|e_i^0 + (\Delta t)(h_2(x_i)) + \frac{a}{2}\sum_{l=0}^{i+1} w_l e_{i-l+1}^1|^2 + T_i^1 \\
&= E|e_i^0 + (\Delta t)(\frac{e_i^1 - e_i^{-1}}{2\Delta t}) + \frac{a}{2}\sum_{l=0}^{i+1} w_l e_{i-l+1}^1 + \frac{a}{2}\sum_{l=0}^{i+1} w_l e_{i-l+1}^0|^2 + T_i^1 \\
&\leq \Big[1 + a\sum_{l=0}^{i+1} w_l\Big]^2 \max_{1\leq i\leq N-1} E|e_i^0|^2 + \max_{1\leq i\leq N-1} |T_i^1| \\
&\leq \Big[1 + a\sum_{l=0}^{i+1} w_l\Big]^2 E|e^0|^2 + T^1
\end{aligned}
$$

By using lemma (2.2) , $a\sum_{l=0}^{i+1} w_l < 0$

$$E|e^1|^2 \leq \zeta E|e^0|^2 + o(h + (\Delta t)^2); \zeta = 1$$

So the result is true for n=1. Suppose, the result is true for n $\leq$ k

$$E|e^k|^2 \leq \zeta E|e^0|^2$$

From equation (3.10) & (5.2) we get,

$$
\begin{aligned}
E|e^{k+1}|^2 = E|2e_i^k - e_i^{k-1} - \sum_{j=1}^{k-1} b_j\left(e_i^{k+1-j} + e_i^{k-1-j} - 2e_i^{k-j}\right) + a\sum_{l=0}^{i+1} w_l e_{i-l+1}^{k+1} + a\sum_{l=0}^{i+1} w_l e_{i-l+1}^{k} \\
- 2b_k\left(e_i^1 - (\Delta t)\Big(\frac{e_i^1 - e_i^{-1}}{2\Delta t}\Big) - e_i^0\right) + cX_k|^2 + T_i^{k+1} \\
= E|(2 - b_1)e_i^k + (-1 - b_2 + 2b_1)e_i^{k-1} - \sum_{j=2}^{k-1}(b_{j+1} + b_{j-1} - 2b_j)e_i^{k-j} - b_k e_i^1
\end{aligned}
$$

$$- b_k e_i^{-1} + 2b_k e_i^0 + a\sum_{l=0}^{i+1} w_l e_{i-l+1}^{k+1} + a\sum_{l=0}^{i+1} w_l e_{i-l+1}^{k} \ + cX_k|^2 + T_i^{k+1}$$

$$\leq |(2-b_1) + (-1-b_2+2b_1) - \sum_{j=2}^{k-1}(b_{j+1}+b_{j-1}-2b_j) - b_k - b_k + 2b_k$$

$$+ 2a\sum_{l=0}^{i+1} w_l|^2 \max_{1\leq i\leq N-1} E|e_i^k|^2 + c^2 \max_{1\leq i\leq N-1} E|e_i^k|^2 + \max_{1\leq i\leq N-1} |T_i^{k+1}|$$

$$\leq \Big\{|(2-b_1) + (-1-b_2+2b_1) - \sum_{j=2}^{k-1}(b_{j+1}+b_{j-1}-2b_j) - b_k - b_k + 2b_k + 2a\sum_{l=0}^{i+1} w_l|^2 + c^2\Big\}E|e^k|^2 + T^{k+1}$$

$$\leq \Big\{|1 + b_k - b_{k-1} + 2a\sum_{l=0}^{i+1} w_l|^2 + \ c^2\Big\}E|e^0|^2 + T^{k+1}$$

By using lemma (2.2) , $a\sum_{l=0}^{i+1} w_l < 0$

$$E|e^{k+1}|^2 \leq \zeta E|e^0|^2 + o(h+(\Delta t)^2); \zeta = (1+b_k-b_{k-1})^2 + \ c^2$$

So the result is true for n= k+1,

By mathematical induction, $E|e^n|^2 \leq \zeta E|e^0|^2 + o(h+(\Delta t)^2)$, where ζ is a non-negative constant independent of h,

According to definition (2.5) , $E|\widehat{\theta}_i^n - \theta_i^n|^2 \to 0$ as $(ih, n\Delta t) \to (x,t), h \to 0, \Delta t \to 0$. Therefore, the discretized form (3.9) - (3.13) under the Crank-Nicolson finite difference scheme is unconditionally convergent. □

6. Numerical Experiment

In this section, the numerical results of the proposed method to solve equations of STFSTWE are presented. All computations are done using Python programming language.

Example 6.1 *The space-time fractional stochastic traveling wave equation with particular initial and boundary conditions.*

$$\frac{\partial^\alpha \theta(x,t)}{\partial t^\alpha} = K^2\frac{\partial^\beta \theta(x,t)}{\partial x^\beta} \ + \sigma \dot{W}(t);\ (x,t) \in \ [0,1]\times[0,1],\ 1<\alpha\leq 2, \quad 1<\beta\leq 2$$

Subject to,
Initial Conditions :

$$\theta(x,0) = 0, \quad 0 < x \leq \ 1$$

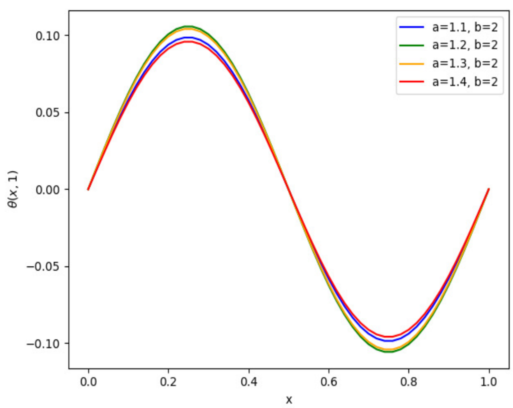
$$\theta_t(x,0) = 2\pi K sin(2\pi x),\ 0 < x \leq \ 1$$

Boundary Conditions :

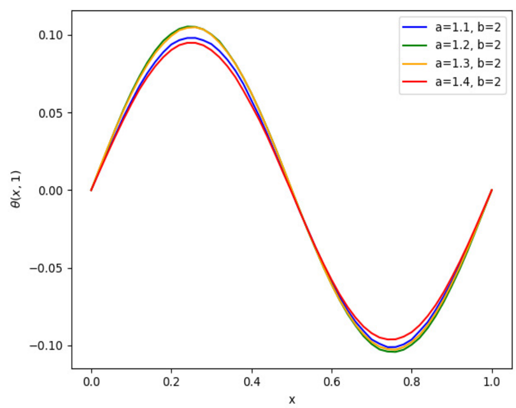
$$\theta(0,t) = 0,\ 0 < t \leq \ 1$$

$$\theta(1,t) = 0, \quad 0 < t \leq \ 1$$

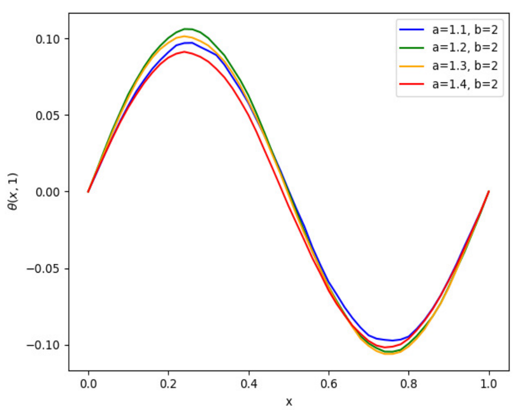
When we take $\alpha = 2$, β= 2 and K=1 , then exact solution to this problem is given by, $\theta(x,t) = sin(2\pi x).sin(2K\pi t)$.



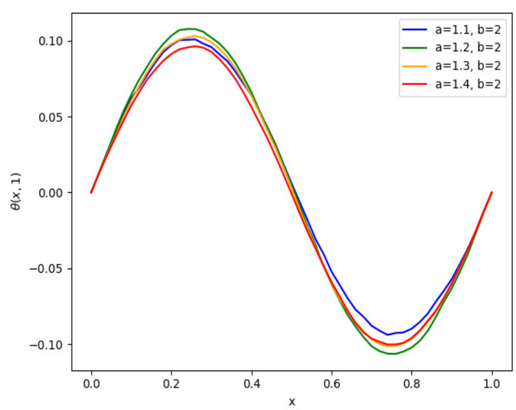
(a) $\sigma = 0$



(b) $\sigma = 1$

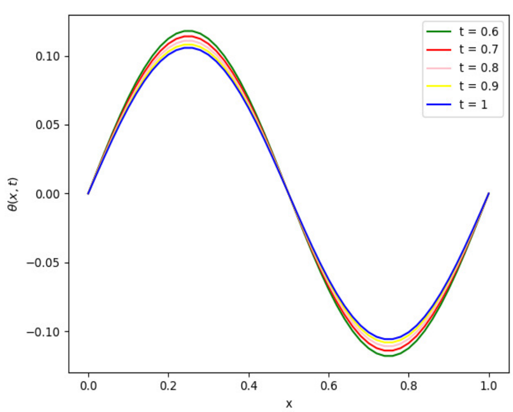


(c) $\sigma = 2$

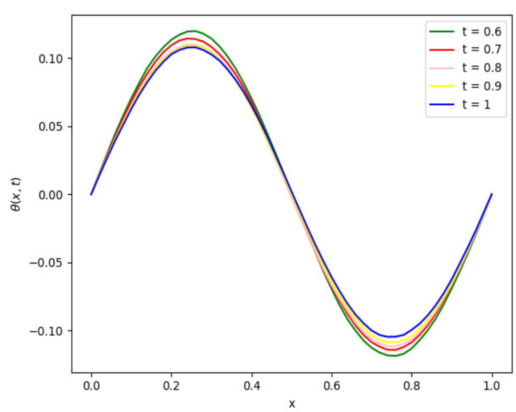


(d) $\sigma = 3$

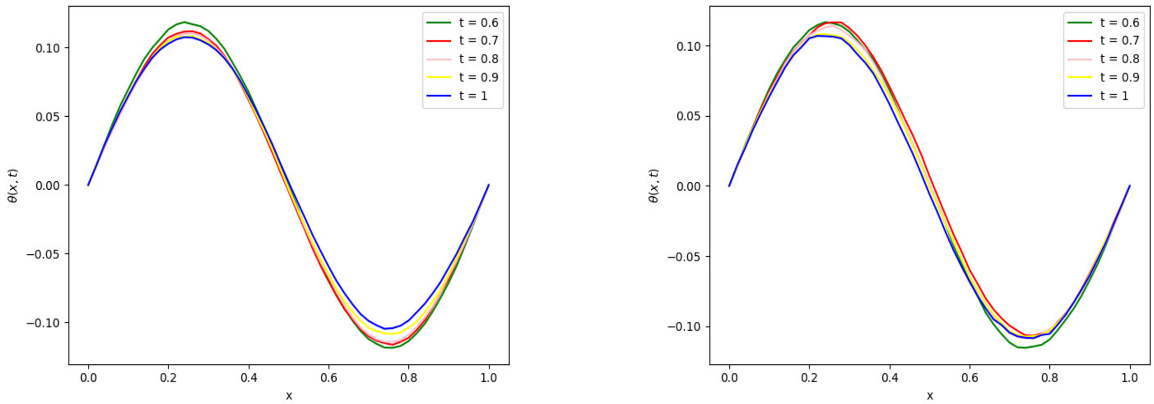
Figure 1: The approximate 2D mean solutions of STFSTWE at time $t = 1$.



(a) $\sigma = 0$

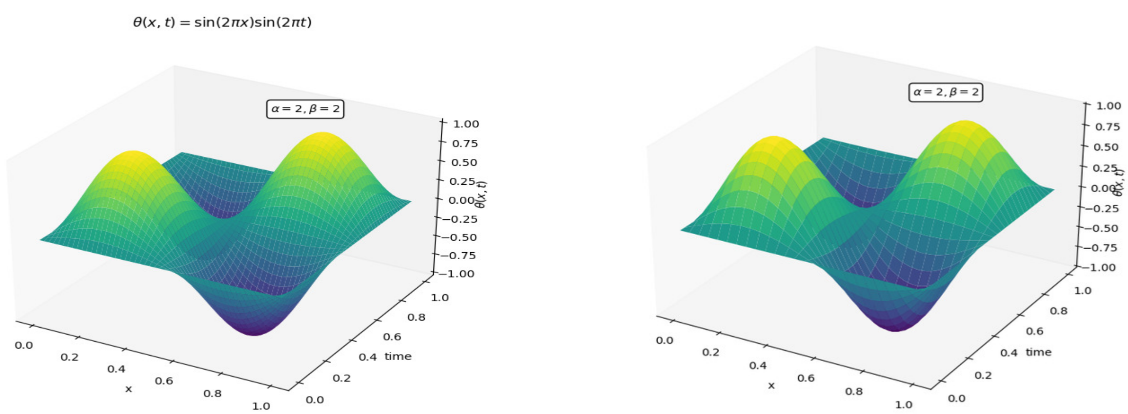


(b) $\sigma = 1$



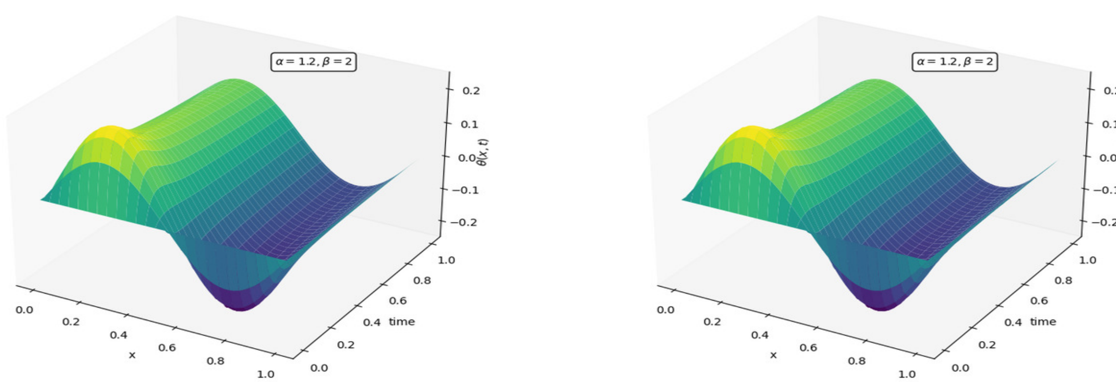
(c) $\sigma = 2$ (d) $\sigma = 3$

Figure 2: The approximate 2D mean solutions of STFSTWE with $\alpha = 1.2$ and $\beta = 2$.

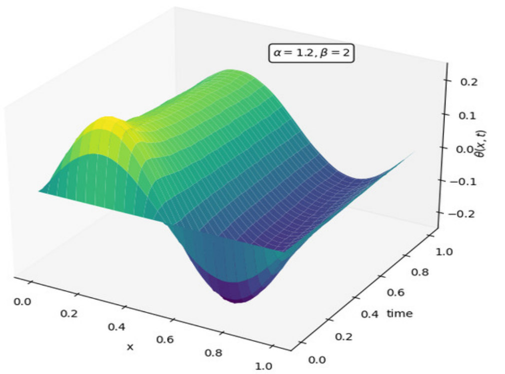


(a) Exact solution (b) Numerical solution

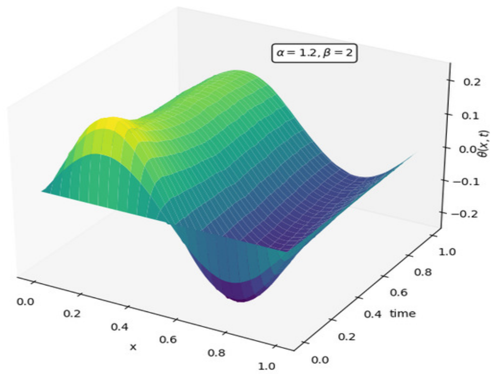
Figure 3: Comparison of numerical solution with exact solution of STFSTWE having $\alpha = 2$, $\beta = 2$, $\sigma = 0$



(a) $\sigma = 0$ (b) $\sigma = 1$



(c) $\sigma = 2$



(d) $\sigma = 3$

Figure 4: The approximate 3D mean solutions of STFSTWE with $\alpha = 1.2$ and $\beta = 2$

7. Results and Discussion

The strength of stochastic fluctuations has a significant impact on the behavior of the approximate solution to the Example 6.1 . These fluctuations are influenced by the system-specific parameter c , which is dependent on model coefficients like α and Δt, in addition to the noise intensity parameter σ. The resultant value $c\sigma$ controls the fluctuation amplitude and behaves as an effective noise intensity. The effective noise is too weak when $\sigma < 1$, causing negligible fluctuations and an inability to properly capture the stochastic behavior. On the other hand, the system shows appropriate stochastic dynamics with noticeable variations when $\sigma \geq 1$.

The approximate 2D mean solution to the STFSTWE with fixed parameters α=1.2, β=2 at t = 1 under various noise intensities σ = 0, 1, 2, 3 are illustrated in Figure 1. The 2D approximate mean solutions for parameters α = 1.2, β=2 at time steps t = 0.6, 0.7, 0.8, 0.9, 1 for values of the noise strength parameter σ = 0, 1,2,3 are displayed in Figure 2. As time increases, noise influence becomes more noticeable, increasing the mean solution's fluctuation levels. Figure 3 compares the exact and numerical solution of STFSTWE derived by the proposed method when α=2 , $\beta = 2$ and $\sigma = 0$. Figure 4 present the 3D approximate mean solutions to the STFSTWE for α =1.2, β=2 , t=1. We restrict the limit of the z-axis, which results in a better visualizing impact of noise intensity σ = 0,1,2,3. As σ increases, the solutions show growing instability and higher amplitude fluctuations, reflecting the enhanced impact of stochasticity on system behavior.

8. Conclusion

In this paper, we created and executed a Crank-Nicolson finite difference method for solving the space-time fractional stochastic traveling wave equation (STFSTWE). A detailed theoretical study was carried out to ensure the proposed method's stability and consistency in the mean square sense. The results demonstrate that the scheme is unconditionally stable and convergent, making it suitable for a wide range of fractional stochastic circumstances. Furthermore, the research indicates an important dependency of the solution behavior on the fractional order parameters α and β . In deterministic and stochastic circumstances, as the time fractional derivative order α increases, the amplitude of the solution decreases, particularly for fixed values of the space fractional derivative β .This pattern illustrates the crucial role of the fractional order in directing the system's dynamics. The stochastic term σ generates visible disturbances on the surface of the mean solution. This underlines the role of stochasticity in affecting the system's behavior and variability. The graphical findings provide additional assurance of these effects on both deterministic and random factors. In conclusion, the proposed Crank-Nicolson finite difference approach is a reliable, precise and efficient numerical tool for solving STFSTWE.

Acknowledgments

The author gratefully acknowledges the financial support provided by the Chhatrapati Shahu Maharaj Research, Training and Human Development Institute (SARTHI), Government of India, for funding this research work.

Declaration of Interest

The authors declare that they have no conflict of interest.

References

1. Djaouti, A. M., And Liaqat,M. I., *Qualitative Analysis for the Solutions of Fractional Stochastic Differential Equations* , axioms. MDPI, 13, 438, (2024).
2. Charpentier, B.C., *Stochastic Modeling of Plant Virus Propagation with Biological Control*, Mathematics. MDPI, 9, 456, (2021).
3. Oksendal, B., *Stochastic Differential Equations:An Introduction with Applications*, 6th Edition, Springer Berlin, Heidelberg, (2003).
4. Mirzaee, F., And Samadyar, N.,*Combination of finite diference method and meshless method based on radial basis functions to solve fractional stochastic advection–difusion equations*, Engineering with Computers. Springer-Verlag London Ltd., (2019).
5. Mirzaee, F., Sayevand, K.,Rezaei, S., And Samadyar, N., *Finite Difference and Spline Approximation for Solving Fractional Stochastic Advection-Diffusion Equation* , Iran J. Sci. Technol. Trans. Sci.. (2020).
6. Smith, G. D., *Numerical solution of Partial Differential equations: Finite Difference Methods*, 3rd Edition, Clarendon Press, Oxford, (1985).
7. Ali, I., And Khan,S.U.,*A Dynamic Competition Analysis of Stochastic Fractional Differential Equation Arising in Finance via Pseudospectral Method* , Mathematics. MDPI, 11, 1328, (2023).
8. Batiha, I. M., Abubaker, A. A., Jebril, I. H., Shaikh, S.B., And Matarneh, K., *A Numerical Approach of Handling Fractional Stochastic Differential Equations*, MDPI. 12, 388,(2023).
9. Podlubny, I., *Fractional differential equations*, Mathematics in Science and Engineering, 1st Edition, San Diego, Academic Press, (1998).
10. Ghode, K., Takale, K., Gaikwad, S., And Bondar, K., *Python: powerful tool for solving space-time fractional traveling wave equation*, Advances and Applications in Mathematical Sciences. Mili Publications, India, 22, 503-526, (2022).
11. Lima,L.S., *Fractional Stochastic Differential Equation Approach for Spreading of Diseases*, entropy. MDPI, 24, 719, (2022).
12. Ghil, M., And Lucarini, V., *The physics of climate variability and climate change*, Reviews of modern physics. American Physical Society, 92, (2020).
13. Marie, N., *On a constrained Fractional Stochastic Volatility Model* ,(2016).
14. Houenafa, S., Ronoh, E., Johnson, o., And Moore, S., *Levy-induced Stochastic Differential Equation Models in Rainfall-Runoff Systems for Assessing Extreme Hydrological Event Risks*, Research Square, (2024).
15. Angew, Z. Z., *Difference Methods for Stochastic Partial Differential Equations*, Math. Mech.. Wiley-Vch Verlag GmbH & Co. KGaA, Weinheim, 82, 11–12, (2002).

Priti Mojad,
Department of Mathematics,
New Arts Commerce and Science College, Ahmednagar-City,
Affiliated to Savitribai Phule Pune University, India.
Orcid: https://orcid.org/0009-0005-9333-9277
E-mail address: pritimojad1020@gmail.com.

and

Babasaheb Gavhane,
Department of Mathematics,
K. J. Somaiya College of Arts, Commerce and Science, Kopargaon- City,
Affiliated to Savitribai Phule Pune University, India.

Orcid: https://orcid.org/0009-0007-3206-9829
E-mail address: bdgavhane2605@gmail.com.

and

Shrikisan Gaikwad,
Department of Mathematics,
New Arts Commerce and Science College, Ahmednagar-City,
Affiliated to Savitribai Phule Pune University, India.
Orcid: https://orcid.org/0000-0001-8394-0329
E-mail address: sbgmathsnagar@gmail.com

and

Kalyanrao Takale,
Department of Mathematics,
RNC Arts JDB Commerce and NSC Science College, Nashik-City,
Affiliated to Savitribai Phule Pune University, India.
Orcid: https://orcid.org/0000-0002-7277-684X
E-mail address: kalyanraotakale1@gmail.com