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Investigating Some Concepts related with Z – Essential Submodules

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ABSTRACT: In this paper, three new concepts are introduced namely Z – essential small submodules, Ze – Radical of a module and Ze – Socle of a module. Many propertied related with these concepts are discussed.

Key Words: Small submodules; essential submodules; ze- small submodule; Ze – Radical; Ze – Socle.

Contents

1. Introduction

Throughout this research, R is a commutative ring with $1 \neq 0$ and W is a nonzero left R – module. The concept small submodules and essential (large) submodules introduced by P. F. Flury and Coodearl at 1974. 1976 see [6,7]. Since that time many scientists studied these submodules and generalized them, see [2,3,4,10,8,13].

A submodule U of an R – module $W(U \leq W)$ is called small (briefly $U \ll W$) if $U + V \neq W$ for each proper submodule V of W (V¡ W). Equivalently $U \ll W$ if whenever $U + V = W, V \leq W$, then $V = W$, see [6,7,11]. A submodule U of M is called essential (large) (shortly, $U \underset{e}{\leq} W$) if $U \cap V \neq 0$ for any $(0) \neq V \leq W$. Equivalently $U \underset{e}{\leq} W$ if $U \cap V = (0)$, $V \leq W$ implies $V = (0)$, see [7,11] DX. Zhou and X. R. Zhang said that a submodule A of W is an essential small (denoted by A $\underset{e}{\ll} W$) if whenever $A + B = W$ and $B \underset{e}{\leq} W$, then $B = W$ [13].

Amina and Alaa in [4], referred to a submodule U of W by Z-small (briefly $U \underset{z}{\ll} M$) if whenever $U + V = W$ and $V \supseteq Z_2(M)$, then $V = W$ where $Z_2(W)$ is the second singular (or Goldi – torsion) submodule of W, and defined in [7] by $\frac{Z_2(W)}{Z(W)} = Z\left(\frac{W}{Z(W)}\right)$, where $Z(W) = \{x \in W : ann(x) \underset{e}{\leq} R), ann(x) = \{a \in R : ax = 0\}$ and $Z(W)$ is the singular submodule of W, W is called singular (nonsingular) if $Z(W) = W, (Z(W) = 0)$, W is named $Z_2 - torsion$ if $Z_2(W) = (0)$. Note that W is non - torsion if and only if W is nonsingular. If W is singular, then $Z_2(W) = W$. For more details about $Z_2 - torsion$ submodules, see [9,12].

In the work of Muntah, Yaha and Inaam [2], they presented the notion "Z-essential submodules" where a submodule A of W is called $Z - essential$ (shortly $A \underset{zes}{\leq} W$) if whenever $A \cap B = 0, A \subseteq Z_2(M)$ yields $B = (0)$. It is clear that every essential submodule is $Z - essential$, but the converse may not valid, see [2] for more information see [1,2].

These ideas motivate us to present new concepts namely Z-essential small submodules, ze-Radical of a module and ze-socle of a module.

In S.2 we referred to a submodule A of W by z-essential small submodule (briefly $A \underset{ze}{\ll} W$) if whenever $A + B = W$ and $B \underset{zes}{\leq} W$, then $B = W$. Since every essential submodule is $z - essential$, we deduced

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that every $A \underset{ze}{\ll} W$ implies $A \underset{e}{\ll} W$ but the converse may not be hold, see Remark 2 (1). Cleary $W \underset{ze}{\ll} W$ for any module $W \neq 0$. Many properties related with this type of submodules are introduced.

In S.3 we present and study new concepts of $ze-Rad(W)$ and $ze-Soc(W)$ where $ze-Rad(W) = \cap\{U : U$ is a maximal submodule and $U \underset{zes}{\leq} W\}$. See definition 3.1
Equivalently $ze-Rad(W) = \Sigma L$, where $L \underset{ze}{\ll} W$ (Theorem 3.1). Also $ze-soc(W) = \Sigma U,$ *and* U *is* $z-$*simple* submodule of W where U is a $z-$*simple* submodule of W if U is simple and $U \subseteq z_2(M)$, see definitions 3.2, 3.3, we notice that $ze-Soc(W) \subseteq \cap\{U : U \underset{zes}{\leq} W\}$.
Several properties of these notions are presented. Also many relationships with other concepts are established.

2. Z−Essential Small Submodules

Definition 2.1 *A submodule A of a module W is named z-essential small submodule (briefly $A \underset{z}{\ll} W$) if $A + B = W, B \underset{zes}{\leq} W$, implies $B = W$.*

Remarks and Examples

1. $e-small$ submodules need not be ze-small, for example: consider $\mathbb{Z}_6$ *as* $\mathbb{Z}_6-module$. Since $\mathbb{Z}_6$ is the only essential submodule of itself, any submodule A of $\mathbb{Z}_6, A \underset{e}{\ll} W$. But all submodules of $\mathbb{Z}_6$ are z-essential. Take $A =<\overline{2}>, \overline{B} =<\overline{3}> \underset{zes}{\leq} \mathbb{Z}_6, A + B = \mathbb{Z}_6$ *and* $B \neq \mathbb{Z}_6;$ *ie* $A \underset{ze}{\not\ll} \mathbb{Z}_6$.

2. Obviously, small submodule is z-essential small. However the converse may be not true, for example:

   Example 2.1 *Consider the $Z-module$ $\mathbb{Z}_{24}$. Only $<\overline{4}>, <\overline{2}> \subseteq \mathbb{Z}_{24}$ are Z-essential in $\mathbb{Z}_{24}$. Note that $<\overline{2}> \underset{ze}{\ll} \mathbb{Z}_{24}$ since $<\overline{2}> + <\overline{2}> \neq \mathbb{Z}_{24}$, $<\overline{2}> + <\overline{4}> \neq \mathbb{Z}_{24}$, we have $<\overline{2}> \underset{ze}{\ll} \mathbb{Z}_{24}$. On the other hand $<\overline{2}> \not\ll \mathbb{Z}_{24}$.*

3. If $Z_2(W) = W, A \leq W$, then the following assertions are identical: $A \ll W, A \underset{e}{\ll} W$ *and* $A \underset{e}{\ll} W$.

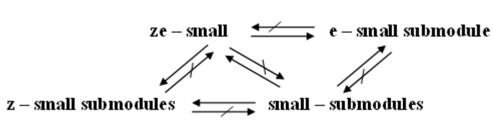
4. Let $A \leq W$ and W is an R − module if $A \underset{ze}{\ll} W,$ *then* $A \underset{z}{\ll} W$, but not conversely.

   Proof: Let $A + B = W, B \leq W$ *and* $B \supseteq Z_2(M)$. Claim $B \underset{zes}{\leq} W$. Let $B \cap C = 0$ *and* $C \subseteq Z_2(M)$. *As* $B \supseteq Z_2(M)$, *so* $B \subseteq C$. It follows that $C = B \cap C = 0$; *that is* $B \underset{zes}{\leq} W$. Since $A \underset{ze}{\ll} W$ we get $B = W$. Thus $A \underset{z}{\ll} W$. □

   Example 2.2 *Consider Z_{12} as Z – module. Since $Z_2(\mathbb{Z}_{12}) = \mathbb{Z}_{12}$, all submodules of Z_2 are z-small. But $<\overline{3}> + <\overline{2}> = \mathbb{Z}_{12}$ and $<\overline{2}> \underset{zes}{\leq} \mathbb{Z}_{12}$ (since $<\overline{2}> \underset{e}{\leq} \mathbb{Z}_{12}$) so that $<\overline{3}> \underset{ze}{\not\ll} \mathbb{Z}_{12}$. Note that $<\overline{3}> \underset{e}{\ll} \mathbb{Z}_{12}$.*

5. Let W be a z-uniform module (ie every nonzero submodule of W is z-essential) [1], $A \leq M.$ *Then* $A \ll W$ if and only if $A \underset{ze}{\ll} W$

We can summarize the above relations by the following diagram



6. Consider the $\mathbb{Z}$ – module $\mathbb{Z}_{p^\infty}$, every submodule is z-essential small.

The following results about z-essential small submodules are analogous with that of small and essential small submodules.

Theorem 2.1 *Let W be an R-module. The following assertions hold:*

(1). $A \leq B \leq W$ and $B \underset{ze}{\ll} W$, then $A \underset{ze}{\ll} W$.

(2). If $A \underset{ze}{\ll} W$ and $B \underset{ze}{\ll} W$, then $A + B \underset{ze}{\ll} W$ and conversely.

(3). If $h : W_1 \to W_2$ be an epimorphism and $A \underset{ze}{\ll} W_1$, then $h(A) \underset{ze}{\ll} W_2$.

(4). If $K \underset{ze}{\ll} L \leq W$, the $K \underset{ze}{\ll} W$. The converse hold if L is a direct Summand (d.s) of W.

(5). Let $A \leq B \leq W$. If $B \underset{ze}{\ll} W$ then $A \underset{ze}{\ll} W$ and $\frac{B}{A} \underset{ze}{\ll} \frac{W}{A}$. The converse hold if A is a z-closed of $W(A \underset{ze}{\leq} W)$ where $A \underset{ze}{\leq} W$ if A has no proper z-essential extension of A in W_2.

(6). Let W be a z-uniform module and $L \underset{ze}{\ll} W$. Then W is finitely generated (f.g) if and only if $\frac{W}{L}$ is f.g.

Proof: (1) and (2) and (3) follows directly
(4) ($\Longrightarrow$) It follows by part (3).
($\Longleftarrow$) To prove $K \underset{ze}{\ll} L$. Assume $K + X = L$ *and* $X \underset{zes}{\leq} L$. *As* L is a d.s of $W, W = L \oplus u$ for some $u \leq W$. *Then* $W = (K + X) \oplus u = K + (X \oplus u)$. But $X \underset{zes}{\leq} L$ implies $X \oplus u \underset{zes}{\leq} L \oplus u = W$. Since $K \underset{ze}{\ll} W$, we have $X \oplus u = W$ and hence $(X \oplus u) \cap L = L$.
Then by modular Law, $L = X + (u \cap L) = X$; *ie* $X = L$ *and* $K \underset{ze}{\ll} L$.
(5) ($\Longrightarrow$) It follows by parts (1) and (3).
($\Longleftarrow$) If $A \underset{ze}{\ll} W$ *and* $\frac{B}{L} \underset{ze}{\ll} \frac{W}{A}$. Assume $B + X = W$ *and* $X \underset{zes}{\leq} W$. Then $\frac{B}{A} + \frac{X+A}{A} = \frac{W}{A}$. Since $A \underset{ze}{\leq} W$, so by [1], Theorem 3.4.8 (1) ($\Longrightarrow$)(4), $\frac{X+A}{A} \underset{zes}{\leq} \frac{W}{A}$. But $\frac{B}{A} \underset{ze}{\ll} \frac{W}{A}$, we deduce that $\frac{X+A}{A} = \frac{W}{A}$; that is $X + A = W$. Also $A \underset{ze}{\ll} W$ *and* $X \underset{zes}{\leq} W$ *imply* $X = W$. Thus $B \underset{ze}{\ll} W$.
(6) By Remarks and Examples 2.1 (5), L $\underset{ze}{\ll}$ W if and only if $L \ll W$. Hence the result follows by [9, Proposition 3.2.4 (6), P.54] □

Proposition 2.1 *For a module W and $a \in W$. $Ra \underset{ze}{\not\ll} M$ if and only if there exists a maximal submodule u of W with $u \underset{zes}{\leq} W$ and $a \notin u$.*

Proof: ($\Longrightarrow$) Suppose $Ra \underset{ze}{\not\ll} W$. Then there exists $A \underset{zes}{\leq} W, A \neq W$ *with* $Ra + A = W$. Consider $C = A : A \underset{zes}{\leq} W,\ Ra + A = W, C \neq \phi$, and so by Zorn's Lemma C has a maximal element say A_0. It can be Checked that A_0 is a maximal submodule, with $a \notin A_0$.
($\Longleftarrow$) Assume there is a maximal submodule A with $A \underset{zes}{\leq} W$ *and* $a \notin A$. *Then* $Ra + A = W$ and so $Ra \underset{ze}{\ } W$. □

Proposition 2.2 *Let W_1 and W_2 be R – module, $A_1 \leq W_1, A_2 \leq W_2$. Then $A_1 \underset{ze}{\ll} W_1$ and $A_2 \underset{ze}{\ll} W_2$ if and only if $A_1 \oplus A_2 \underset{ze}{\ll} W_1 \oplus W_2$*

Proof: ($\Longrightarrow$) Let i, j be the inclusion mappings, $i : W_1 \to W_1 \oplus W_2$ *and* $j : W_2 \to W_1 \oplus W_2$. Then $i(A_1) = A_1 \oplus (0) \underset{ze}{\ll} W_1 \oplus W_2$ *and* $j(A_2) = (0) + A_2 \underset{ze}{\ll} W_1 \oplus W_2$ by Remarks and Examples 2.1 (3)
Hence $A_1 \oplus A_2 = (A_1 \oplus (0)) \oplus ((0) \oplus A_2) \underset{ze}{\ll} W_1 \oplus W_2$ by Remarks and Examples 2.1 (2).
($\Longleftarrow$) Let $\rho_1,\ \rho_2$ be the natural projections, $\rho_1 : W_1 \oplus W_2 \to W_1$ *and* $\rho_2 : W_1 \oplus W_2 \to W_2$. *Then* $\rho_1(A_1 \oplus A_2) = A_1 \ll W_1$ *and* $\rho_2(A_1 \oplus A_2) = A_2 \underset{ze}{\ll} W_2$ by Remarks and Examples 2.1 (3) □

3. Ze -Radical and Z − Socle of a module

In this section we introduce and study two concepts Ze -Radical of a module and Z − socle of a module. It is known that the Jacobson Radical of a module W (denoted by $Rad(W)$ *or* $J(M)$) is the intersection of all maximal submodules. Note $Rad(W) = W$ if W has no maximal submodules. Also $Rad(W) = \sum_{U \ll W} U$ [11].
We define the following:

Definition 3.1 *For an R – module W. Let* $ze{-}Rad(W) = \cap\{K : K$ *is a maximal submodule and* $K \underset{zes}{\leq} W\}$.

Theorem 3.1 *For a module* $W, ze - Rad(W) = \Sigma L; L \underset{ze}{\ll} W$.

Proof: Let $L \underset{ze}{\ll} W, K$ is a maximal submodule of W and $K \underset{zes}{\leq} W.\ If\ L \nsubseteq K$, then $L + K = W$. But $L \underset{ze}{\ll} W$ *and* $K \underset{zes}{\leq} W$ imply $K = W$ which is a contradiction. Thus $L \subseteq K$ *and so* $L \subseteq \cap\{K :$ $K$is a maximal submodule and $K \underset{zes}{\leq} W\}$. Hence

$$\sum_{L \underset{ze}{\ll} W} L \subseteq Ze - Rad(W) \tag{3.1}$$

Let $x \in Ze - Rad(W)$. Suppose $Rx \underset{ze}{\not\ll} W$, then by Proposition 2.4, there is a maximal submodule V of W with $V \underset{zes}{\leq} W$ *and* $x \notin V$. That is $x \notin Ze - Rad(W)$ which is a contradiction. Thus $Rx \underset{ze}{\ll} W$ and

$$Ze - Rad(W) \subseteq \sum_{L \underset{ze}{\ll} W} L. \tag{3.2}$$

Therefore $Ze - Rad(W) = \sum_{L \underset{ze}{\ll} W} L$ (by 3.1 and 3.2). □

Remark 3.1 Let W be an R-module. Then:

(1) Clearly $Rad(W) \subseteq Ze - Rad(W)$, but the reverse inclusion may be not hold, for example: Consider $\mathbb{Z}_{24}$ as $\mathbb{Z} - module, Rad\mathbb{Z}_{24} =< \overline{6} >$ and $Ze - Rad(W) =< \overline{2} >$.

(2) $\underset{z}{Rad}(W) = \{U : U$ is a maximal submodule,$u \supseteq Z_2(W)\}$, see [5] since $U \supseteq Z_2(W)$ implies $U \underset{zes}{\leq} W$ by the proof Remarks 2.1 (4), hence $\underset{z}{Rad}(W) \supseteq Ze - Rad(W)$.

(3) Since every ze-small submodule is $e{-}small$, so that $Ze{-}Rad(W) \subseteq \underset{e}{Rad}(W),\ where\ \underset{e}{Rad}(W) = \Sigma_{U \underset{ze}{\ll} W} U$. Note that $Rad(W) \subseteq \underset{e}{Rad}(W)$.

(4) $Ze - Rad(\mathbb{Z}) = \underset{z}{Rad}\ \mathbb{Z} = Rad\ \mathbb{Z} = (0)$.

(5) $Ze - Rad(Q) = 0, Ze - Rad\ \mathbb{Z}_6 = (0)$.

(6) $Ze - Rad(\mathbb{Z}_{P^\infty}) = \mathbb{Z}_{P^\infty}$.

Proposition 3.1

(1) Let W_1 and W_2 be $R-$modules. Then if $\Psi : W_1 \to W_2$ be an $R-$homomorphism, the $\Psi(Ze-Rad(W_1)) \subseteq Ze-Rad(W_2)$.

(2) If every proper submodule U of W with $U \underset{zes}{\leq} W$ is contained in a maximal submodule of W then $Ze-Rad(W)$ is the largest $Ze-small$ submodule of W_1.

Proof: (1) It is valid by Theorem 3.1 and Theorem 2.1 (3).
(2) Suppose that $Ze-Rad(W_1) \underset{ze}{\not\ll} W$. Hence $Ze-Rad(W)+u=W$ for some $U<W$ *and* $U \underset{zes}{\leq} W$. By hypothesis, $\exists$ a maximal submodule V of W with $u \subseteq V$. *Hence* $V \underset{zes}{\leq} W$. Also $Ze-RadW \subseteq V$, so $Ze-RadW+V=W$ *implies* $V=W$ *which is a contradiction. Thus* $ze-Rad(W) \underset{ze}{\ll} W$. Now Let $H \underset{ze}{\ll} W$. Hence $H \subseteq ze-Rad(W)$ *and so* $ze-Rad(W)$ is the largest ze-small submodule of W. □

Corollary 3.1 *Let W be a f.g or multiplication module. Then ze-Rad(W) is the largest ze-small submodule of W.*

Corollary 3.2 *Let W be a Z-uniform module. Then W is f.g if and only if* $(I) Ze-RadW \underset{ze}{\ll} W$ *and* $(II) \frac{W}{ze-Rad(W)}$ *is f.g.*

Proof: $\Longrightarrow$ Since W is f.g, $\frac{W}{Ze-Rad(W)}$ is f.g and by Corollary 3.5, $Ze-Rad(W) \underset{ze}{\ll} W$.
$\Longleftarrow$ Assume $x_1+ze-Rad(W), x_2+ze-Rad(W), \ldots, x_n+Ze-Rad(W)$ generate $\frac{W}{Ze-Rad(W)}$. Then for any $w \in W, w+Ze-Rad(W) = r_1(x_1+Ze-Rad(W))+\ldots+r_n(x_n+Ze-Rad(W))$. so that $w-r_1x_1-\ldots-r_nx_n \in Ze-Rad(W)$, *hence* $w=(r_1x_1+\ldots+r_nx_n)+(w-r_1x_1-\ldots-,r_nx_n) \in <x_1,\ldots,x_n> +ze-Rad(W)$. But $Ze-Rad(W) \underset{ze}{\ll} W$ *and* $<x_1,\ldots,x_n> \underset{zes}{\leq} W$, *which implies* $W=<x_1,\ldots,x_n>$; ie W is f.g. □

Proposition 3.2 *Let* W *be an* $R-module, a \in W$. *Then* $Ra \underset{ze}{\ll} W$ *if and only if* $a \in Ze-Rad(W)$.

Proof: It follows directly by Proposition 2.4. □

Corollary 3.3 *An arbitrary sum of ze-small submodules of a module W is ze-small if and only if ze-Rad(W)* $\underset{ze}{\ll} W$.

Proposition 3.3 *Let* W *be an* $R-module$. *Then* $Ze-Rad(W)=W$ *if and only if all f.g submodules are Ze-small in W.*

Proof: $\Longrightarrow$ It follows by Theorem 2.3 (2).
$(\Longleftarrow)$ It is clear. □

Proposition 3.4 *For an* $R-moduleW, (Ze-Rad(R))W \subseteq Ze-Rad(W)$.

Proof: Let $x \in Ze-Rad(W)$. Define $\psi_x : R \to W$ *by* $\psi_x(r)=rx$ *for all* $r \in R$. Then $\psi_x(Ze-Rad(R)) \subseteq Ze-Rad(W)$ *i.e* $(Ze-Rad(R))W \subseteq Ze-Rad(W)$. □

Proposition 3.5 *Let* W *be an* $R-module$. *Then:*

1. $Ze-Rad\left(\frac{W}{Ze-Rad(W)}\right)=0$.

2. $Ze-Rad(W)=K$ *if and only if* $K \subseteq Ze-Rad(W)$ *and* $Ze-Rad\frac{W}{K}=0$ *(and so* $Rad\left(\frac{W}{K}\right)=0$*).*

Proof:

$$1.\ Ze-Rad\left(\frac{W}{Ze-Rad(W)}\right)=\cap\left\{\frac{A}{Ze-Rad(W)}:\ \frac{A}{Ze-Rad(W)}\text{ is maximal and }\frac{A}{Ze-Rad(W)}\underset{zes}{\leq}\frac{W}{Ze-Rad(W)}\right.$$

$$\subseteq\cap\left\{\frac{A}{Ze-Rad(W)}:A\ is\ a\ maximal\ submodule\ in\ W\ and\ A\underset{zes}{\leq}W\right\}$$

$$=\left\{\frac{\cap A}{Ze-Rad(W)}:A\ is\ a\ maximal\ submodule\ in\ W\ and\ A\underset{zes}{\leq}W\right\}$$

$$=\frac{Ze-RadW}{Ze-RadW}=0.\ Thus\ Ze-Rad\left(\frac{W}{Ze-Rad(W)}\right)=0.$$

2. $(\Longleftarrow)$ Let $\pi: W\to\frac{W}{K}$ be the natural projection. Then $\pi(Ze-Rad(W))\subseteq Ze-Rad\left(\frac{W}{K}\right)=0$. Hence $\frac{Ze-Rad(W)+K}{K}=0$; that is $Ze-Rad(W)\subseteq K$. But $K\subseteq ze-Rad(W)$. *Thus* $ze-Rad(W)=K$.

$$(\Longrightarrow)Cleary\ K\subseteq Ze-Rad(W).\ Now\ Ze-Rad\left(\frac{W}{K}\right)$$

$$=\cap\left\{\frac{U}{K}:\frac{U}{K}\quad is\ a\ maximal\ submodule\ in\frac{W}{K}\ and\quad\frac{u}{K}\underset{zes}{\leq}\frac{W}{K}\right\}$$

$$\subseteq\cap\left\{\frac{U}{K}:U\ \text{ is a maximal submodule in}W\ and\ U\ \underset{zes}{\leq}w\right\}$$

$$=\frac{\cap\left\{U:U\quad is\ a\ maximal\ in\ W\ and\ u\underset{zes}{\leq}W\right\}}{K}$$

$$=\frac{Ze-Rad(W)}{K}$$

But $Ze-Rad(W)=K$, so that $\frac{Ze-Rad(W)}{K}=0$ *and* $Ze-Rad\left(\frac{W}{K}\right)=0$. *As* $Rad\left(\frac{W}{K}\right)\subseteq Ze-Rad\left(\frac{W}{K}\right)$ we have $Rad\left(\frac{W}{K}\right)=0$. □

Next we prove the following:

Proposition 3.6 *Let* $W=\underset{i\in I}{\oplus}W_i$, *where* W_i *is an* $R-module,\forall i\in I$. *Then:*

1. $Ze-Rad(W)=\underset{i\in I}{\oplus}(Ze-Rad(W_i))$. *and*

2. $\frac{W}{Ze-Rad(W)}=\underset{i\in I}{\oplus}\frac{W_i}{Ze-Rad(W_i)}$.

Proof: 1. Let $\rho_i\to W_i$ be the i.th projection and $y=\sum_{i\in I}y_i\in Ze-Rad(W)$. *Then* $\rho_i(y)=y_i\in ze-Rad(W)$ and so $y\in\underset{i\in I}{\oplus}Ze-Rad(W_i)$. Hence $Ze-Rad(W)\subseteq\underset{i\in I}{\oplus}Ze-Rad(W_i)$. Now $\forall i\in I$, *let* $j_i: W_i\to W$ be i.th injection mapping, so $j_i(Ze-Rad(W_i))\subseteq Ze-Rad(W)$. Thus $\underset{i\in I}{\oplus}Ze-Rad(W_i)\subseteq Ze-Rad(W)$.

2. $\frac{w}{Ze-Rad(W)}=\frac{\underset{i\in I}{\oplus}W_i}{\underset{i\in I}{\oplus}Ze-Rad(W_i)}\cong\underset{i\in I}{\oplus}\frac{W_i}{Ze-Rad(W_i)}$ □

Recall that for any $R-module\ W,\Sigma U$, where U is a simple submodule of W is called socle of W (soc(W)). Also $soc(W)=\cap\{U:U\underset{e}{\leq}W\}$ [11].

Definition 3.2 *A submodule U of a module W is called $Z-simple$ if U is simple and $U \subseteq Z_2(M)$.*

Definition 3.3 *For any $R-$module W, let $\Sigma U : U$ is $Z-simple$ be called $Ze-socle$ of W.(shortly $Ze-soc(W)$).*

It is clear that

1. $Ze - soc(W) \subseteq soc(W)$.
2. $Ze - soc(W)$ is a semisimple (since soc(W) is the largest semisimple submodule of W).
3. $Ze - soc(W) \subseteq Z_2(W)$.
4. $Ze - soc(W)$ is the largest $Z - semisimple\ submodule\ of\ W$, where a submodule U is called $Ze - semisimple$ if it is sum of all $Z - simple\ submodules\ of\ W$.

Theorem 3.2 *For any $R-module$ W, let $H = \cap\{U : U \underset{zes}{\leq} W\}$. Then $Ze - soc(W) \subseteq H$.*

Proof: Let T be a simple submodule and $U \subseteq Z_2(W)$. For any $L \underset{zes}{\leq} W, T \cap L \neq 0$ (because if $T \cap L = (0)$ then $T = 0$ since $T \subseteq Z_2(W)$ *and* $L \underset{zes}{\leq} W$. But $T \neq 0$ since it is simple. Hence $T \cap L = T$; that is $T \subseteq L$ *and so* $T \subseteq \cap\{L : L \underset{zes}{\leq} W\}$. Thus $Ze - soc(W) \subseteq H$. □

Example 3.1

1. *Consider $\mathbb{Z}_6$ as $\mathbb{Z}_6 - module$, since every submodule of $\mathbb{Z}_6$ is $\mathbb{Z} - essential$, then $\cap\{U : U \underset{zes}{\leq} \mathbb{Z}_6 = (0)$ and so $ze - soc(W) = (0)$. On the other hand $soc(Z_6) = Z_6$.*
2. *If $Z_2(M) = M$, then $soc(W) = ze - soc(W)$.*

Proof: It follows directly since simple and $Z - simple$ submodule are identical. □

3. For $\mathbb{Z} - module \mathbb{Z}_6, soc(\mathbb{Z}_6) = ze - soc(\mathbb{Z}_6) = \mathbb{Z}_6$.

Proposition 3.7 *For any homomorphism $f : W \to N, f(Ze - soc(W)) \subseteq Ze - soc(W)$.*

Proof: For any simple submodule S of W with $U \subseteq Z_2(M), f(S)$ is either (0) of $f(S)$ is a simple submodule of N, and $f(U) \subseteq f(Z_2(W)) \subseteq Z_2(f(W)) \subseteq Z_2(N)$, and so $f(Ze - soc(W)) \subseteq Ze - soc(N)$. □

Theorem 3.3 *Let $f : M \to N$ be a monomorphism and $Im(f) \underset{zes}{\leq} N$. Then $f(Ze - soc(W)) = Ze - soc(N)$ and $f^{-1}(Ze - socN) = Ze - soc(W)$.*

Proof: $f(Ze - soc(W)) \subseteq Ze - soc(N)$ is obtained by Proposition 3.16. Let S be a simple submodule of N *and* $S \subseteq Z_2(N)$. Since $Im(f) \underset{zes}{\leq} N, Im(f) \cap S \neq 0$ because if $Imf \cap S = 0$ then $S = (0)$ which is a contradiction. Thus $S \subseteq Im(f)$. To show $f^{-1}(S)$ is simple in W and $f^{-1}(S) \subseteq Z_2(W)$, *for* $U \subseteq f^{-1}(S)$, then $f(U) \subseteq ff^{-1}(S) \subseteq S$ *and hence* $f(U) = S$ which implies $U = f^{-1}(S)$ since f is monomorphism. Thus $f^{-1}(S)$ is simple. Also since $S \subseteq Z_2(N)$ and f is monomorphism, then $f^{-1}(S) \subseteq f^{-1}(Z_2(W)) \subseteq Z_2 f^{-1}(N) = Z_2(W)$. Thus $f^{-1}(S) \subseteq Ze - soc(W)$ which implies $ff^{-1}(S) \subseteq f(Ze - soc(W))$ and so that $S \subseteq f(Ze - soc(W))$. Thus $ze - soc(N) \subseteq f(Ze - soc(W))$ and hence $f(Ze - soc(W)) = Ze - soc(N)$ *and* $Ze - soc(W) = f^{-1}(Ze - soc(N))$ □

Proposition 3.8 *Let W be an $R-module$ and $F \leq W$. Then $Ze - soc(F) = F \cap Ze - soc(W)$.*

Proof: Let $i : F \to W$ be the inclusion mapping. Then $Ze - soc(F) = i(Ze - soc(F)) \leq Ze - soc(W)$. But $Ze - soc(F) \leq F$. Hence $Ze - soc(F) \subseteq F \cap (Ze - soc(W))$. Consider the following sequence:

$$0 \to F \cap (Ze - soc(W)) \xrightarrow{i} Ze - soc(W) \xrightarrow{\pi} \frac{Ze - soc(W)}{F_n(Ze - soc(W))} \to 0$$

This sequence is short exact, so it splits and $F \cap Ze - soc(W)$ is $Z - semisimple\ in\ F$. To establish that $F \cap (Ze - soc(W)) \subseteq Ze - soc(F)$. Let $D \leq F \cap (Ze - soc(W))$, so $D \leq Ze - soc(W)$. As $Ze - soc(W)\ is\ Z - semisimple, D$ is a d.s of $Ze - soc(W)$, that is $Ze - soc(W) = D \oplus A$ for some $A \leq Ze - soc(W). Hence\ F \cap (Ze - soc(W)) = D \oplus (A \cap F),\ and\ so\ D\ is\ a\ d.s\ of\ F \cap (Ze - soc(W))$, which implies that $F \cap (Ze - soc(W))\ is\ Z - semisimple,\ but\ F \cap (Ze - soc(W)) \leq F$, so that $F \cap Ze - soc(W) \subseteq Ze - soc(F)(since\ Ze - soc(F)$ is the largest $Z - semisimpleinF$.
Thus $F \cap (Ze - soc(W)) = Ze - soc(F)$. □

Proposition 3.9 *Let* $W = \underset{i \in I}{\oplus} W_i$ *be an* $R - module$*. Then* $Ze - soc(W) = \underset{i \in I}{\oplus}(Ze - soc(W_i))$.

Proof: For any $i \in I, j_i : W_i ßW$ be inclusion mapping. Then by Proposition 3.4 (11),

$$Ze - soc(W_i) \subseteq Ze - soc(W) \tag{3.3}$$

Let $m = \Sigma m_i \in Ze - soc(W)\ and\ \rho_i : W \to W_i$ be the i-th projection. Then $\rho_i(m) = \oplus_i \in Ze - soc(W_i)$ by Proposition 3.6 (1), and so $m \in \underset{i \in I}{\oplus} Ze - soc(W_i)$. Hence

$$Ze - soc(W) \subseteq \underset{i \in I}{\oplus} Ze - soc(W_i) \tag{3.4}$$

By 3.3 and 3.4, we get $Ze - soc(W) = \underset{i \in I}{\oplus} Ze - soc(W_i)$. □

4. Conclusion

1. Several properties of small, e – small submodules generalized to Z – e – small submodule.
2. Several results related with Ze -Radical and Ze – Socle are analogous to that of Radical and Socle.

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