Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–17.
ISSN-0037-8712
doi:10.5269/bspm.77269

Investigating the Properties and Diverse Applications of Special Polynomials Linked to Appell Sequences

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ABSTRACT: This paper presents a study that builds upon existing research by applying the monomiality principle to generate novel results. The study primarily focuses on the construction and analysis of tangent-based Appell polynomials, exploring their properties in detail, including their explicit and determinant forms, and their compliance with the monomiality principle. The study also investigates specific classes of Appell polynomials — namely, the tangent-based Bernoulli, Euler and Genocchi polynomials — and derives key outcomes for each. Additionally, the paper provides numerical and graphical representations of these polynomials, facilitating a deeper understanding of their characteristics. This research contributes to the broader field of special polynomials and their applications in various mathematical and scientific contexts.

Key Words: Hybrid Special Polynomials, generating function, series representation, determinant form, numerical and graphical representation, zeros.

Contents

1. Introduction

Tangent numbers play a fundamental role in the study of special polynomials and are typically expressed using the following formula:

$$\frac{2}{\exp(2w)+1} = \sum_{k=0}^{\infty} D_k \frac{w^k}{k!}, \qquad |w| < \frac{\pi}{2}.$$

This expression forms the basis for generating Tangent Polynomials, a class of polynomials that has received significant attention in mathematical research. First introduced by Ryoo, these polynomials have been extensively studied for their various properties and applications, as detailed in the works [10,11,12,13]. Studying tangent polynomials is particularly important in fields such as number theory, combinatorics and the theory of special functions, as their unique properties provide valuable insights into a variety of mathematical phenomena.
Throughout this article, we use the following notation conventions for number sets: the natural numbers are denoted by $\mathbb{N}$, which refers to the set of positive integers, represented by the set $\{1, 2, 3, \dots\}$. The

* Corresponding author.
Submitted June 09, 2025. Published August 24, 2025
2010 *Mathematics Subject Classification:* 33E20, 33C45, 33B10, 33E30, 11T23.

set of complex numbers is also denoted by $\mathbb{C}$, where $\mathbb{C}$ represents the field of all complex numbers of the form $a+bi$, with $a,b\in\mathbb{R}$ and $i=\sqrt{-1}$. Additionally, the set of non-negative integers is represented by $\mathbb{Z}_+$, which is equivalent to $\mathbb{N}_0=\{0,1,2,3,\dots\}$. These notational choices are standard and help to make the presentation of the results in the article more streamlined and easier to understand.

The expression representing the Tangent polynomials is as follows:

$$\frac{2}{\exp(2w)+1}\exp(\xi\ w)=\sum_{k=0}^{\infty}D_k(\xi)\frac{w^k}{k!},\quad |w|<\frac{\pi}{2}. \tag{1.1}$$

Tangent polynomials are defined as a family of polynomials by the following recursive formula:

$$D_k(\xi)=\xi^k-\sum_{s=1}^{k}2^{s-1}\binom{k}{s}D_{k-s}(\xi).$$

As some of the authors did not provide examples of the Appell-type polynomial families that we studied, it is useful to present some examples and graphs: (with the help of wxMaxima) to demonstrate the existence of the given polynomials (1.1):

$$\begin{aligned}
D_0(\xi) &= 1,\\
D_1(\xi) &= \xi-1,\\
D_2(\xi) &= \xi^2-2\xi,\\
D_3(\xi) &= \xi^3-3\xi^2+2,\\
D_4(\xi) &= \xi^4-4\xi^3.
\end{aligned}$$

Tangent polynomials exhibit several intriguing characteristics and have many notable applications, particularly in numerical analysis and approximation theory. Recently, numerous studies have been published on special functions and orthogonal polynomials that possess properties similar to those of the polynomials we are interested in studying. Therefore, I recommend citing the following works ([2,3,4,7,9]).

First introduced by Appell, Appell polynomials have a variety of applications in mathematics and physics, including algebraic geometry, differential equations, and quantum mechanics. They are closely related to other families of special functions, such as hypergeometric and Jacobi polynomials. A notable property of Appell polynomials is that combining two of them creates another Appell polynomial. This indicates that Appell polynomials form an abelian group when composed. This group property arises from the differential equation satisfied by Appell polynomials, which is a particular instance of the Heun equation. Furthermore, the Galois group associated with the Heun equation is known to be an abelian extension of the differential field generated by its solutions, providing additional clarification of the group property of Appell polynomials.

The Appell polynomial sequences form an abelian group when combined with the identity element, which is represented by the constant polynomial sequence $\mathfrak{A}_0(\xi)=1$. The commutative property of this group arises from the inherent symmetric characteristics of the Appell polynomials.

This group property has significant implications in various fields of mathematics and physics, particularly in the study of differential equations and integrable systems. For instance, using this group structure enables the derivation of recursion relations that govern the coefficients of Appell polynomials, facilitating the computation of their special values. Moreover, this property facilitates the creation of new families of Appell polynomials by combining existing ones, revealing novel and fascinating mathematical structures in the process.

In the 19th century, Appell introduced a family of polynomials known as the 'Appell polynomials', denoted by $\mathfrak{A}_k(\xi)$. These polynomials are characterised by their fulfilment of the following differential equation:

$$\frac{d}{d\xi}\mathfrak{A}_k(\xi)=k\ \mathfrak{A}_{k-1}(\xi),\quad k\in\mathbb{N}_0 \tag{1.2}$$

and generating relation expression:

$$\mathfrak{A}(w)\ e^{\xi w} = \sum_{k=0}^{\infty} \mathfrak{A}_k(\xi) \frac{w^k}{k!}, \tag{1.3}$$

where $\mathfrak{A}(w)$ is represented by

$$\mathfrak{A}(w) = \sum_{k=0}^{\infty} \mathfrak{A}_k \frac{w^k}{k!}, \qquad \mathfrak{A}_0 \neq 0 \tag{1.4}$$

on real line is convergent with Taylor expansion.

Named after the French mathematician Paul Appell, who introduced them during his studies of elliptic functions, these polynomials play a crucial role in mathematical discourse. The generating relation (1.3) provides a method for expressing the exponential function $e^{\xi w}$ as an infinite sum of the Appell polynomials $\mathfrak{A}_k(\xi)$ multiplied by powers of w. This relationship is a valuable tool for simplifying certain integrals and computing specific functions.

It is also interesting to note the emergence of hybrid special polynomials through the combination of the monomiality principle and operational rules. By applying these principles and operators, a powerful approach to solving various mathematical problems has been developed.

The concept of monomiality was first introduced by various authors in 1941 [5,8,15], who formulated the idea of poweroids in his work Stefpoweriod. Dattoli later refined this concept, establishing it as a key principle in the analysis of special polynomials. Monomiality involves expressing a set of polynomials in terms of monomials, which are the fundamental components of polynomials. This approach improves our understanding of the set of polynomials and their inherent properties.

The operators $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ play pivotal roles in the study of special polynomials. Acting as multiplication and differentiation operators for the set of polynomials $b_k(\xi)k \in \mathbb{N}$, these operators enable new polynomials to be created from existing ones. The equation

$$b_{k+1}(\xi) = \hat{\mathcal{M}}\{b_k(\xi)\}, \tag{1.5}$$

illustrates the multiplicative property of the operator $\hat{\mathcal{M}}$, generating a new polynomial $b_{k+1}(\xi)$ from its predecessor $b_k(\xi)$. Similarly, the equation

$$k\ b_{k-1}(\xi) = \hat{\mathcal{D}}\{b_k(\xi)\}, \tag{1.6}$$

demonstrates the derivative property of the operator $\hat{\mathcal{D}}$, producing a new polynomial $b_{k-1}(\xi)$ by differentiating $b_k(\xi)$ and multiplying it by the coefficient k.
Incorporating the monomiality principle and operational rules into the study of special polynomials has led to the development of hybrid special polynomials. These distinct polynomial families possess unique properties that provide valuable insights into solving a wide range of mathematical problems. The exploration of hybrid special polynomials continues to be a vibrant area of research, with far-reaching implications in various fields, including physics, engineering, and computer science.

The equations and properties discussed herein contribute to the broader theory of quasi-monomials and Weyl groups, which have applications in several mathematical and physical disciplines, such as representation theory, algebraic geometry, and quantum field theory. As a result, the polynomial set $\{b_k(\xi)\}_{m\in\mathbb{N}}$, defined by the operators in expressions (1.5) and (1.6), is classified as a quasi-monomial and adheres to the following formula:

$$[\hat{\mathcal{D}}, \hat{\mathcal{M}}] = \hat{\mathcal{D}}\hat{\mathcal{M}} - \hat{\mathcal{M}}\hat{\mathcal{D}} = \hat{1},$$

thus displays a Weyl group structure as a result.
In particular, $b_k(\xi)$ demonstrate the differential equation

$$\hat{\mathcal{M}}\hat{\mathcal{D}}\{b_k(\xi)\} = k\ b_k(\xi), \tag{1.7}$$

if $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ have differential realizations. The equation implies that the quasi-monomials are eigenfunctions of the operator $\hat{\mathcal{M}}\hat{\mathcal{D}}$, with eigenvalue k. This differential equation can be solved explicitly for certain choices of $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$, leading to explicit expressions for the quasi-monomials, represented by

$$b_k(\xi) = \hat{\mathcal{M}}^k \ \{1\}, \tag{1.8}$$

Given the initial condition $b_0(\xi) = 1$, the expression (1.8) provides a recursive method for computing the quasi-monomials. This involves iteratively applying the operator $\hat{\mathcal{M}}$ to the identity operator $\hat{I}$. Starting with $b_0(\xi) = 1$, the generating relation for $b_k(\xi)$ can be expressed in exponential form as follows:

$$e^{w\hat{\mathcal{M}}}\{1\} = \sum_{k=0}^{\infty} b_k(\xi)\frac{w^k}{k!}, \quad |w| < \infty \ \ , \tag{1.9}$$

by usage of identity (1.8).

The expression in (1.9) represents quasi-monomials as a series expansion in w, with the coefficients being derived from the quasi-monomials themselves. This approach makes precise computation easier and helps with the study of special functions.

The theory of quasi-monomials and Weyl groups is a powerful framework for analysing function families, with significant applications in mathematics and physics. Operational methods have been widely used in quantum mechanics and classical optics to simplify complex systems and enable deeper insights into quantum states and nonlinear optical phenomena. These techniques remain invaluable in advancing both theoretical and experimental physics. [16,17,18].

Therefore, considering (1.5) and (1.6), we have obtained the operators, commonly referred to as multiplicative and derivative operators for the Tangent polynomials. These operators are derived by taking the derivatives of expression (1.1) with respect to w and ξ respectively, and are expressed as follows:

$$D_{k+1}(\xi) = \hat{\mathcal{M}}_{\mathcal{D}}\{D_k(\xi)\} = \left(\xi - \frac{2}{e^{2w}+1}\right)\{D_k(\xi)\} \tag{1.10}$$

and

$$D_{k-1}(\xi) = \hat{\mathcal{D}_{\mathcal{D}}}\{D_k(\xi)\} = \frac{\partial}{\partial w}\{D_k(\xi)\}. \tag{1.11}$$

Also, because of (1.7), we derive the expression for the differential equation by making use of expressions (1.10) and (1.11) as:

$$\left(\xi\frac{\partial}{\partial \xi} - \frac{2}{e^{2w}+1}\frac{\partial}{\partial \xi} - k\right) = 0.$$

Motivated by the work of C. S. Ryoo, we introduce ${}_D\mathfrak{A}_k(\xi)$ Tangent-based Appell polynomials, which possess generating expressions of the form:

$$\mathfrak{A}(w)\frac{2}{e^{2w}+1}e^{\xi w} = \sum_{k=0}^{\infty} {}_D\mathfrak{A}_k(\xi)\frac{w^k}{k!}, \tag{1.12}$$

where

$$\mathfrak{A}(w) = \sum_{k=0}^{\infty}\mathfrak{A}_k\frac{w^k}{k!}, \qquad \mathfrak{A}_0 \ \neq \ 0.$$

Tangent polynomials have a wide range of applications in mathematics and physics. They provide an effective alternative to classical orthogonal polynomials, such as the Legendre and Chebyshev polynomials, for approximating functions within finite intervals. They are particularly well-suited to handling functions with singularities or discontinuities, where classical polynomials may struggle to converge. Furthermore, tangent polynomials frequently emerge as solutions to important differential equations such as the heat and Schrödinger equations, rendering them invaluable tools in quantum mechanics and statistical physics.

Tangent-based Appell polynomials have a wide range of applications and play a critical role in solving mathematical and physical problems. This manuscript is structured as follows: Section 2 introduces the Tangent-based Appell polynomials ${}_D\mathfrak{A}_k(\xi)$ and their key properties. Section 3 explores their quasi-monomial characteristics and determinant form. Section 4 examines the applications of these polynomials, focusing on selected members of the Tangent-based Bernoulli, Euler and Genocchi polynomial families, and presents relevant findings. Section 5,establishes the numerical and graphical representation of these selected members and represents their zeros graphically. Finally, some concluding remarks are presented.

2. Tangent-based Appell polynomials ${}_D\mathfrak{A}_k(\xi)$

This section introduces general methods for determining the ${}_D\mathfrak{A}_k(\xi)$ sequences. A polynomial is classified as an Appell type of degree k if it satisfies expression (1.2). The following result is established:

Theorem 2.1 *Since, we observe ${}_D\mathfrak{A}_k(\xi)$ polynomials are given by (1.12), therefore we have*

$$\frac{\partial}{\partial \xi}\{{}_D\mathfrak{A}_k(\xi)\} = k\ {}_D\mathfrak{A}_{k-1}(\xi). \tag{2.1}$$

Proof: Differentiate both sides of equation (1.12),with respect to w, and replace k by $k-1$ in the resulting equation. Furthermore, comparing the coefficients of the same exponents of w, we get assertion (2.1). □

Moreover, we will prove the following results to introduce these Appell-based Tangent polynomials:

Theorem 2.2 *For the polynomials ${}_D\mathfrak{A}_k(\xi)$, the succeeding generating expression holds true:*

$$\mathfrak{A}(w)\frac{2}{e^{2w}+1}e^{\xi w} = \sum_{k=0}^{\infty} {}_D\mathfrak{A}_k(\xi)\frac{w^k}{k!}. \tag{2.2}$$

Proof: Replacing ξ in (1.1) by multiplicative operator of Appell polynomials [1] given by $\hat{\mathcal{M}} = \left(\xi + \frac{\mathfrak{A}'(w)}{\mathfrak{A}(w)} = \sum\limits_{k=0}^{\infty}\beta_k\frac{w^k}{k!}\right)$, it follows that

$$\frac{2}{e^{2w}+1}e^{\left(\xi+\frac{\mathfrak{A}'(w)}{\mathfrak{A}(w)}\right)w} = \sum_{k=0}^{\infty} D_k\left(\xi + \frac{\mathfrak{A}'(w)}{\mathfrak{A}(w)}\right)\frac{w^k}{k!},$$

In view of the Cromford identity, the left-hand side of the previous equation yields

$$\mathfrak{A}(w)\ \frac{2}{e^{2w}+1}e^{\xi w} = \sum_{k=0}^{\infty} D_k\left(\xi + \frac{\mathfrak{A}'(w)}{\mathfrak{A}(w)}\right)\frac{w^k}{k!}.$$

Denoting $D_k\left(\xi + \frac{\mathfrak{A}'(w)}{\mathfrak{A}(w)}\right)$ in the right-hand side of above equation by ${}_D\mathfrak{A}_k(\xi)$, we are lead to assertion (2.2). □

Next, we establish the explicit forms of Tangent-based Appell polynomials ${}_D\mathfrak{A}_k(\xi)$ by proving the results listed as:

Theorem 2.3 *For,$\xi \in \mathbb{C}$ and $k \in \mathbb{Z}^+$, the Tangent-based Appell polynomials ${}_D\mathfrak{A}_k(\xi)$ satisfy the listed explicit forms:*

$$\textbf{(a)}\quad {}_D\mathfrak{A}_k(\xi) = \sum_{m=0}^{k}\binom{m}{k}D_m\ \mathfrak{A}_{k-m}(\xi). \tag{2.3}$$

$$\textbf{(b)}\quad {}_D\mathfrak{A}_k(\xi) = \sum_{m=0}^{k}\binom{m}{k}D_m(\xi)\ \mathfrak{A}_{k-m}. \tag{2.4}$$

Proof: **(a)** Generating equation (1.12) can be written as

$$\frac{2}{e^{2w}+1}\,\mathfrak{A}(w)e^{\xi w}=\sum_{m=0}^{\infty}D_k\frac{w^m}{m!}\sum_{k=0}^{\infty}\mathfrak{A}_k(\xi)\frac{w^k}{k!}.$$

Applying the Cauchy product rule and replacing m by $m-k$ in the resulting equation and comparing the coefficients of the same exponents of w, we get assertion (2.3).

(b) Generating equation (1.12) can be written as

$$\frac{2}{e^{2w}+1}\,\mathfrak{A}(w)e^{\xi w}=\sum_{m=0}^{\infty}D_m(\xi)\frac{w^m}{m!}\sum_{k=0}^{\infty}\mathfrak{A}_k\frac{w^k}{k!}. \tag{2.5}$$

Applying the Cauchy product rule and replacing m by $m-k$ in the resultant equation and then comparing the coefficients of the same exponents of w, we get assertion (2.4). □

Next, we establish an identity for the the Tangent-based Appell polynomials ${}_D\mathfrak{A}_k(\xi)$ by proving the result listed as:

Remark 2.1

$$_D\mathfrak{A}_k(\xi_1+\xi_2)=\sum_{m=0}^{k}\binom{k}{m}(-\xi_2)^m\ {}_D\mathfrak{A}_k(\xi_1).$$

Proof: Taking $\xi\to\xi_1+\xi_2$ in the l.h.s. of expression (2.5), we have

$$\frac{2}{e^{2w}+1}\,\mathfrak{A}(w)e^{(\xi_1+\xi_2)w}=e^{-\xi_2 w}\frac{2}{e^{2w}+1}\,\mathfrak{A}(w)e^{(\xi_1)w}.$$

Using the right-hand side of the expression (2.1) on both sides of the previous expression yields the following:

$$\sum_{k=0}^{\infty}\mathfrak{A}_k(\xi_1+\xi_2)\frac{w^k}{k!}=\sum_{m=0}^{\infty}(-\xi_2)^m\ \frac{w^m}{m!}\sum_{k=0}^{\infty}\mathfrak{A}_k(\xi_1)\frac{w^k}{k!},$$

using Cauchy product rule in the right-hand side of previous expression by replacing k by $k-m$ and equate the coefficient's of same exponents of w, in the resultant expression, we get the desired result. □

3. Quasi-monomiality principle and determinant form

The main purpose of the monomiality principle is to identify the multiplicative and derivative operators. Furthermore, it is used to formulate Tangent-based Appell polynomials. ${}_D\mathfrak{A}_k(\xi)$ within the context of the monomiality principle, we prove the following results:

Theorem 3.1 *The Tangent-based Appell polynomials ${}_D\mathfrak{A}_k(\xi)$, satisfy the succeeding multiplicative and derivative operators:*

$$\hat{M}({}_D\mathfrak{A})=\xi-\frac{2}{e^{2w}+1}+\frac{\mathfrak{A}'(w)}{\mathfrak{A}(w)} \tag{3.1}$$

or

$$\hat{M}({}_D\mathfrak{A})=\xi-\frac{2}{e^{2w}+1}+\sum_{m=0}^{\infty}\beta_m\ \frac{w^m}{m!},$$

respectively, where $\sum_{m=0}^{\infty}\beta_m\frac{w^m}{m!}=\frac{\mathfrak{A}'(w)}{\mathfrak{A}(w)}$,
and

$$\hat{D}({}_D\mathfrak{A})=\frac{\partial}{\partial\xi}. \tag{3.2}$$

Proof: Taking derivatives of generating relation (2.2), with respect to w on b/s, it follows that

$$\frac{\partial}{\partial w}\left\{\mathfrak{A}(w)\frac{2}{e^{2w}+1}e^{\xi w}\right\} = \frac{\partial}{\partial w}\left[\sum_{k=0}^{\infty} {}_D\mathfrak{A}_k(\xi)\frac{w^k}{k!}\right],$$

Therefore, we have

$$\left(\xi - \frac{2}{e^{2w}+1} + \frac{\mathfrak{A}'(w)}{\mathfrak{A}(w)}\right)\left[\mathfrak{A}(w)\frac{2}{e^{2w}+1}e^{\xi w}\right] = \left[\sum_{k=0}^{\infty} k\ {}_D\mathfrak{A}_k(\xi)\frac{w^{k-1}}{k!}\right],$$

using the right-hand side of expression (2.1)left-hand side of the previous equation, it follows that

$$\left(\xi - \frac{2}{e^{2w}+1} + \frac{\mathfrak{A}'(w)}{\mathfrak{A}(w)}\right)\left[\sum_{k=0}^{\infty} {}_D\mathfrak{A}_k(\xi)\frac{w^k}{k!}\right] = \left[\sum_{k=0}^{\infty} k\ {}_D\mathfrak{A}_k(\xi)\frac{w^{k-1}}{k!}\right].$$

Replacing k by $k+1$ in the right-hand side of above expression and then given (1.5) in the resultant equation, the assertion (3.1) is proved.

Further, taking derivatives of expression (2.2) with respect to ξ on b/s, it follows that

$$\frac{\partial}{\partial \xi}\left\{\mathfrak{A}(w)\frac{2}{e^{2w}+1}e^{\xi w}\right\} = w\left\{\mathfrak{A}(w)\frac{2}{e^{2w}+1}e^{\xi w}\right\},$$

which further can be written as:

$$\frac{\partial}{\partial \xi}\left[\sum_{k=0}^{\infty} {}_D\mathfrak{A}_k(\xi)\frac{w^k}{k!}\right] = \left[\sum_{k=0}^{\infty} {}_D\mathfrak{A}_k(\xi)\frac{w^{k+1}}{k!}\right].$$

Replacing k by $k-1$ in the right-hand side of above equation and then in view of expression (1.6), the assertion (3.2) follows. □

Next, we find the differential equation satisfied by these polynomials:

Theorem 3.2 *The Tangent-based Appell polynomials ${}_D\mathfrak{A}_k(\xi)$, satisfy the succeeding differential equation:*

$$\left[\xi\frac{\partial}{\partial \xi} - \frac{2}{e^{2w}+1}\frac{\partial}{\partial \xi} + \frac{\mathfrak{A}'(w)}{\mathfrak{A}(w)}\frac{\partial}{\partial \xi} - k\right] = 0. \tag{3.3}$$

Proof: Inserting the expression (3.1) and (3.2) in the expression (1.7), we get the assertion (3.3). □

Further, we give the determinant representation to these Tangent-based Appell polynomials ${}_D\mathfrak{A}_k(\xi)$ by proving the result:

Theorem 3.3 *The Tangent-based Appell polynomials ${}_D\mathfrak{A}_k(\xi)$ give rise to determinant in the following form:*

$${}_D\mathfrak{A}_k(\xi) = \frac{(-1)^k}{(\gamma_0)^{k+1}}\begin{vmatrix} 1 & \mathfrak{A}_1(\xi) & \mathfrak{A}_2(\xi) & \cdots & \mathfrak{A}_{k-1}(\xi) & \mathfrak{A}_k(\xi) \\ \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{k-1} & \gamma_k \\ 0 & \gamma_0 & \binom{2}{1}\gamma_1 & \cdots & \binom{k-1}{1}\gamma_{k-2} & \binom{k}{1}\gamma_{k-1} \\ 0 & 0 & \gamma_0 & \cdots & \binom{k-1}{2}\gamma_{k-3} & \binom{k}{2}\gamma_{k-2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \gamma_0 & \binom{k}{k-1}\gamma_1 \end{vmatrix},$$

(3.4)

where

$$\gamma_k,\ k=0,1,\cdots \text{ are the coefficients of Maclaurins series of } \frac{1}{\mathfrak{A}(w)}.$$

Proof: Multiplying both sides of equation (2.2) by $\frac{1}{\mathfrak{A}(w)}=\sum_{m=0}^{\infty}\gamma_m\frac{w^m}{m!}$, we find

$$\sum_{k=0}^{\infty}\mathfrak{A}_k(\xi)\frac{w^k}{k!}=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\ \gamma_m\frac{w^m}{m!}\ {}_D\mathfrak{A}_k(\xi)\frac{w^k}{k!},$$

which on using Cauchy product rule becomes

$$\mathfrak{A}_k(w)(\xi)=\sum_{m=0}^{k}\binom{k}{m}\ \gamma_m\ {}_D\mathfrak{A}_{k-m}(\xi).$$

To obtain the explicit expression for the sequence ${}_D\mathfrak{A}_{k-m}(\xi)$, we derive a system consisting of m-linear equations, where each equation corresponds to a coefficient comparison arising from the expansion of the generating function in powers of the polynomial basis. The unknowns in this system are the coefficients ${}_D\mathfrak{A}_{k-m}(\xi)$, indexed for $k=0,1,2,\ldots$.

This system has a clearly defined structure: the associated coefficient matrix is **lower triangular**, composed of repeated entries that are governed by the parameter γ_0. Thanks to its triangular nature, the determinant of the system matrix is easily calculated as $(\gamma_0)^{k+1}$, which significantly simplifies the inversion process.

We then employ Cramer's Rule to solve for each unknown coefficient. However, to streamline the computation process and clarify the structure of the solution more transparent, we perform a **strategic transformation** on the numerators: specifically, we consider the transpose of the numerator matrix and implement a row-shifting operation. For indices $i=1,2,\ldots,k-1$, the i-th row is mapped to the $(i+1)$-th position. This modified arrangement enables us to capture the determinantal representation of each coefficient in an algebraically elegant and computationally efficient form.

□

4. Applications

A key feature of the Appell polynomial framework is its remarkable adaptability. This allows diverse polynomial families to be generated through the selection of an appropriate underlying function, denoted by the symbol $\mathfrak{A}(w)$, in the generating function formalism. This function acts as a central building block: by modifying its form, a wide variety of Appell-type polynomials can be constructed, each exhibiting unique structural and algebraic characteristics. This flexibility not only unifies many classical polynomials, such as Bernoulli, Euler, and Genocchi, and provides a natural pathway for developing new families with properties tailored to specific mathematical or applied contexts. The analytical depth of this approach lies in how a simple change in $\mathfrak{A}(w)$ can give rise to entirely new operational rules, recurrence relations, and orthogonality behaviours.

The Bernoulli polynomials form a fundamental class within the Appell family and are defined through the generating function [6]:

$$\left(\frac{w}{e^w-1}\right)e^{\xi w}=\sum_{k=0}^{\infty}\mathfrak{B}_k(\xi)\frac{w^k}{k!}.$$

Here, the function $\mathfrak{A}(w)=\frac{w}{e^w-1}$ serves as the characteristic kernel. The Bernoulli polynomials $\mathfrak{B}_k(\xi)$ can also be expressed in terms of Bernoulli numbers $\mathfrak{B}_k:=\mathfrak{B}_k(0)$ via the series expansion

$$\mathfrak{B}_k(\xi)=\sum_{m=0}^{k}\binom{k}{m}\mathfrak{B}_m\xi^{k-m},$$

which highlights their Appell structure. These polynomials are closely linked to number theory, particularly with the Riemann zeta function and the Euler–Maclaurin summation formula.

The Euler polynomials constitute another important subclass of Appell polynomials, generated by the function [6]:

$$\left(\frac{2}{e^{w}+1}\right)e^{\xi w}=\sum_{k=0}^{\infty}\mathfrak{E}_k(\xi)\frac{w^k}{k!}.$$

Here, $\mathfrak{A}(w)=\frac{2}{e^w+1}$ governs the structure of the generating function. These polynomials are related to the classical Euler numbers, which can be obtained by evaluating $\xi=\frac{1}{2}$, namely

$$\mathfrak{E}_k:=2^k\mathfrak{E}_k\left(\tfrac{1}{2}\right).$$

An explicit representation of the polynomials is given by

$$\mathfrak{E}_k(\xi)=\sum_{m=0}^{k}\binom{k}{m}\frac{\mathfrak{E}_m}{2^m}\left(\xi-\tfrac{1}{2}\right)^{k-m},$$

which underlines their symmetric properties and relevance in approximation theory and Fourier analysis.

The Genocchi polynomials arise from a generating function involving a slight modification of the Euler kernel [14]:

$$\left(\frac{2w}{e^{w}+1}\right)e^{\xi w}=\sum_{k=0}^{\infty}\mathfrak{G}_k(\xi)\frac{w^k}{k!},$$

with the characteristic function $\mathfrak{A}(w)=\frac{2w}{e^w+1}$. These polynomials are similarly defined through the expansion

$$\mathfrak{G}_k(\xi)=\sum_{m=0}^{k}\binom{k}{m}\mathfrak{G}_m\xi^{k-m},$$

where $\mathfrak{G}_k:=\mathfrak{G}_k(0)$ denotes the Genocchi numbers. Notably, Genocchi polynomials appear in various number-theoretic identities and are known for their alternating sign behaviour and relations to tangent numbers and Bernoulli polynomials of higher order.

By taking Bernoulli, Euler and Genocchi polynomial's as members of Appell family, we obtain different member's of Tangent-based Appell family as Tangent-based Bernoulli polynomials ${}_D\mathfrak{B}_k(\xi)$, Tangent-based Euler polynomials ${}_D\mathfrak{E}_k(\xi)$ and Tangent-based Genocchi polynomials ${}_D\mathfrak{G}_k(\xi)$. These polynomials are given by generating expression as listed:

$$\frac{2w}{(e^{w}-1)(e^{2w}+1)}e^{\xi w}=\sum_{k=0}^{\infty}{}_D\mathfrak{B}_k(\xi)\frac{w^k}{k!},\tag{4.1}$$

$$\frac{4}{(e^{w}+1)(e^{2w}+1)}e^{\xi w}=\sum_{k=0}^{\infty}{}_D\mathfrak{E}_k(\xi)\frac{w^k}{k!}\tag{4.2}$$

and

$$\frac{4w^2}{(e^{w}+1)^2}e^{\xi w}=\sum_{k=0}^{\infty}{}_D\mathfrak{G}_k(\xi)\frac{w^k}{k!},$$

respectively.

As an example, the tangent-based Bernoulli polynomials and tangent-based Euler polynomials are (with the help of MATLAB) as follows.

The tangent-based Bernoulli polynomials

$$\begin{aligned}
{}_D\mathfrak{B}_0(\xi) &= 1,\\
{}_D\mathfrak{B}_1(\xi) &= \xi - \frac{3}{2},\\
{}_D\mathfrak{B}_2(\xi) &= \xi^2 - 3\xi + \frac{7}{6},\\
{}_D\mathfrak{B}_3(\xi) &= \xi^3 - \frac{9}{2}\xi^2 + \frac{7}{2}\xi + \frac{3}{2}.\\
{}_D\mathfrak{B}_4(\xi) &= \xi^4 - 6\xi^3 + 7\xi^2 + 6\xi - \frac{121}{30}.
\end{aligned}$$

The tangent-based Euler polynomials

$$\begin{aligned}
{}_D\mathfrak{E}_0(\xi) &= 1,\\
{}_D\mathfrak{E}_1(\xi) &= \xi - \frac{3}{2},\\
{}_D\mathfrak{E}_2(\xi) &= \xi^2 - 3\xi + 1,\\
{}_D\mathfrak{E}_3(\xi) &= \xi^3 - \frac{9}{2}\xi^2 + 3x + \frac{9}{4}.\\
{}_D\mathfrak{E}_4(\xi) &= \xi^4 - 6\xi^3 + 6\xi^2 + 9x - 5.
\end{aligned}$$

Further, in view of expressions (2.3) and (2.4), these Tangent-based Bernoulli polynomials ${}_D\mathfrak{B}_k(\xi)$, Tangent-based Euler polynomials ${}_D\mathfrak{E}_k(\xi)$ and Tangent-based Genocchi polynomials ${}_D\mathfrak{G}_k(\xi)$, satify the listed series representations:

Corollary 4.1 *For, $\xi \in \mathbb{C}$ and $k \in \mathbb{Z}^+$, the Tangent-based Bernoulli polynomials ${}_D\mathfrak{B}_k(\xi)$ satisfy the listed explicit forms:*

$$\textbf{(a)}\quad {}_D\mathfrak{B}_k(\xi) = \sum_{m=0}^{k} \binom{m}{k} D_m\ \mathfrak{B}_{k-m}(\xi).$$

$$\textbf{(b)}\quad {}_D\mathfrak{B}_k(\xi) = \sum_{m=0}^{k} \binom{m}{k} D_m(\xi)\ \mathfrak{B}_{k-m}.$$

Corollary 4.2 *For, $\xi \in \mathbb{C}$ and $k \in \mathbb{Z}^+$, the Tangent-based Euler polynomials ${}_D\mathfrak{E}_k(\xi)$ satisfy the listed explicit forms:*

$$\textbf{(a)}\quad {}_D\mathfrak{E}_k(\xi) = \sum_{m=0}^{k} \binom{m}{k} D_m\ \mathfrak{E}_{k-m}(\xi).$$

$$\textbf{(b)}\quad {}_D\mathfrak{E}_k(\xi) = \sum_{m=0}^{k} \binom{m}{k} D_m(\xi)\ \mathfrak{E}_{k-m}.$$

Corollary 4.3 *For, $\xi \in \mathbb{C}$ and $k \in \mathbb{Z}^+$, the Tangent-based Genocchi polynomials ${}_D\mathfrak{G}_k(\xi)$ satisfy the listed explicit forms:*

$$\textbf{(a)}\quad {}_D\mathfrak{G}_k(\xi) = \sum_{m=0}^{k} \binom{m}{k} D_m\ \mathfrak{G}_{k-m}(\xi).$$

$$\textbf{(b)}\quad {}_D\mathfrak{G}_k(\xi) = \sum_{m=0}^{k} \binom{m}{k} D_m(\xi)\ \mathfrak{G}_{k-m}.$$

Consequently, in view of Remark (2.1), we obtain the following identity satisfied by these polynomials: The Tangent-based Bernoulli polynomials ${}_D\mathfrak{B}_k(\xi)$, Tangent-based Euler polynomials ${}_D\mathfrak{E}_k(\xi)$ and Tangent-based Genocchi polynomials ${}_D\mathfrak{G}_k(\xi)$ hold the identity listed as:

Remark 4.1

$$ {}_D\mathfrak{A}_k(\xi_1+\xi_2)=\sum_{m=0}^{k}\binom{k}{m}(-\xi_2)^m\ {}_D\mathfrak{A}_k(\xi_1), $$

$$ {}_D\mathfrak{E}_k(\xi_1+\xi_2)=\sum_{m=0}^{k}\binom{k}{m}(-\xi_2)^m\ {}_D\mathfrak{E}_k(\xi_1) $$

and

$$ {}_D\mathfrak{G}_k(\xi_1+\xi_2)=\sum_{m=0}^{k}\binom{k}{m}(-\xi_2)^m\ {}_D\mathfrak{G}_k(\xi_1), $$

respectively.

Also, in view of the expressions (3.4) , these Tangent-based Bernoulli polynomials ${}_D\mathfrak{B}_k(\xi)$, Tangent-based Euler polynomials ${}_D\mathfrak{E}_k(\xi)$ and Tangent-based Genocchi polynomials ${}_D\mathfrak{G}_k(\xi)$, satisfy the listed determinant representations:

Corollary 4.4 *The Tangent-based Bernoulli polynomials ${}_D\mathfrak{B}_k(\xi)$ give rise to determinant in the following form:*

$$ {}_D\mathfrak{B}_k(\xi)=\frac{(-1)^k}{(\gamma_0)^{k+1}}\begin{vmatrix} 1 & \mathfrak{B}_1(\xi) & \mathfrak{B}_2(\xi) & \cdots & \mathfrak{B}_{k-1}(\xi) & \mathfrak{B}_k(\xi) \\ \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{k-1} & \gamma_k \\ 0 & \gamma_0 & \binom{2}{1}\gamma_1 & \cdots & \binom{k-1}{1}\gamma_{k-2} & \binom{k}{1}\gamma_{k-1} \\ 0 & 0 & \gamma_0 & \cdots & \binom{k-1}{2}\gamma_{k-3} & \binom{k}{2}\gamma_{k-2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \gamma_0 & \binom{k}{k-1}\gamma_1 \end{vmatrix}. $$

Corollary 4.5 *The Tangent-based Euler polynomials ${}_D\mathfrak{E}_k(\xi)$ give rise to determinant in the following form:*

$$ {}_D\mathfrak{E}_k(\xi)=\frac{(-1)^k}{(\gamma_0)^{k+1}}\begin{vmatrix} 1 & \mathfrak{E}_1(\xi) & \mathfrak{E}_2(\xi) & \cdots & \mathfrak{E}_{k-1}(\xi) & \mathfrak{E}_k(\xi) \\ \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{k-1} & \gamma_k \\ 0 & \gamma_0 & \binom{2}{1}\gamma_1 & \cdots & \binom{k-1}{1}\gamma_{k-2} & \binom{k}{1}\gamma_{k-1} \\ 0 & 0 & \gamma_0 & \cdots & \binom{k-1}{2}\gamma_{k-3} & \binom{k}{2}\gamma_{k-2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \gamma_0 & \binom{k}{k-1}\gamma_1 \end{vmatrix}. $$

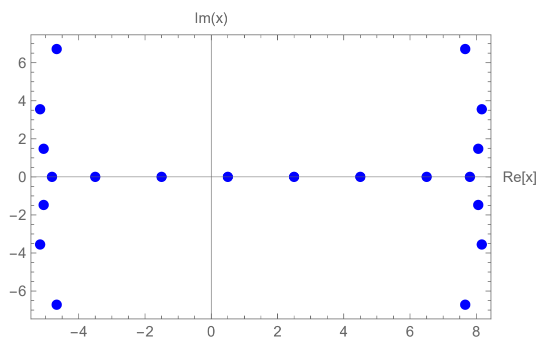
Corollary 4.6 *The Tangent-based Genocchi polynomials ${}_D\mathfrak{G}_k(\xi)$ give rise to determinant in the following* *form:*

$$
{}_D\mathfrak{G}_k(\xi) = \frac{(-1)^k}{(\gamma_0)^{k+1}} \begin{vmatrix} 1 & \mathfrak{G}_1(\xi) & \mathfrak{G}_2(\xi) & \cdots & \mathfrak{G}_{k-1}(\xi) & \mathfrak{G}_k(\xi) \\ \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{k-1} & \gamma_k \\ 0 & \gamma_0 & \binom{2}{1}\gamma_1 & \cdots & \binom{k-1}{1}\gamma_{k-2} & \binom{k}{1}\gamma_{k-1} \\ 0 & 0 & \gamma_0 & \cdots & \binom{k-1}{2}\gamma_{k-3} & \binom{k}{2}\gamma_{k-2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \gamma_0 & \binom{k}{k-1}\gamma_1 \end{vmatrix}.
$$

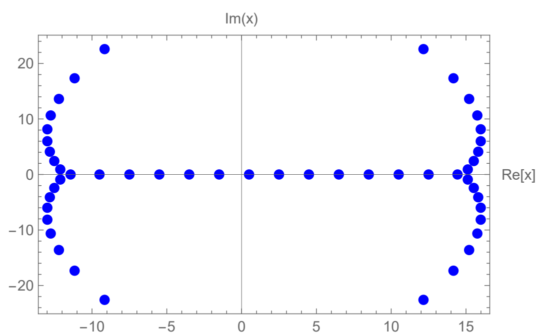
5. Some values with graphical representations and zeros of the ${}_D\mathfrak{B}_k(\xi)$, ${}_D\mathfrak{E}_k(\xi)$ and ${}_D\mathfrak{G}_k(\xi)$

In this section, zero distribution patterns are shown for Tangent-based Bernoulli polynomials ${}_D\mathfrak{B}_k(\xi) = 0$, Tangent-based Euler polynomials ${}_E\mathfrak{B}_k(\xi) = 0$ and Tangent-based Genocchi polynomials ${}_G\mathfrak{B}_k(\xi) = 0$. Graphs for unique values of the indices were plotted using Mathematica Wolfram.

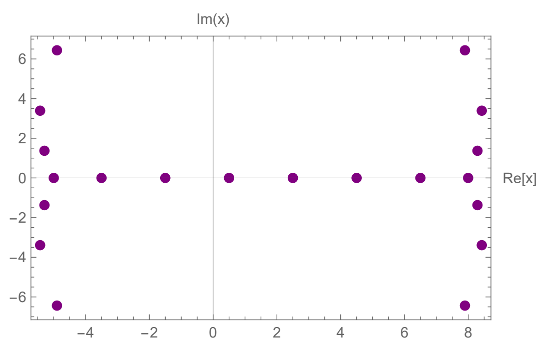
In Figure 2, we set the zero distribution in blue for Tangent-based Bernoulli polynomials ${}_D\mathfrak{B}_k(\xi) = 0$, then, we set the zero distribution in purple for Tangent-based Euler polynomials ${}_D\mathfrak{E}_k(\xi) = 0$, finally, we set the zero distribution in orange for Tangent-based Genocchi polynomials ${}_D\mathfrak{G}_k(\xi) = 0$



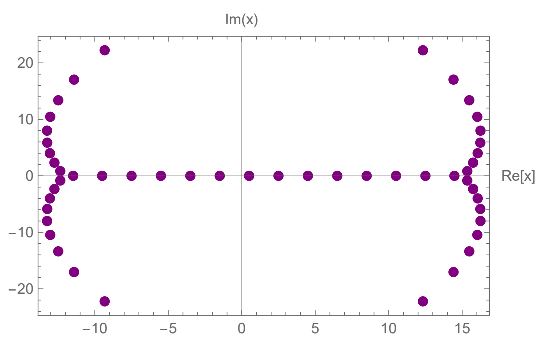
(a) Zeros of ${}_D\mathfrak{B}_{30}(\xi) = 0$



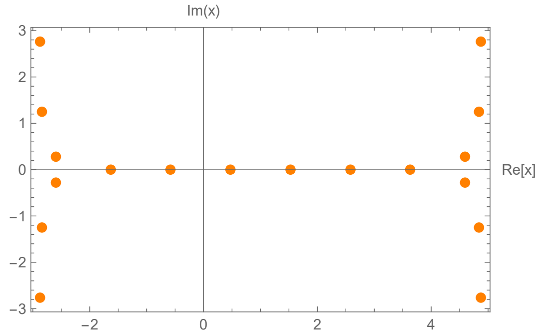
(b) Zeros of ${}_D\mathfrak{B}_{50}(\xi) = 0$



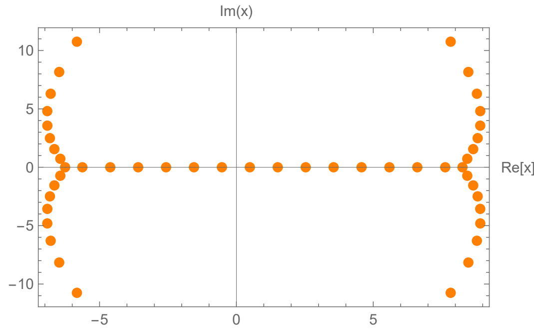
(c) Zeros of ${}_D\mathfrak{E}_{30}(\xi) = 0$



(d) Zeros of ${}_D\mathfrak{E}_{50}(\xi) = 0$



(a) Zeros of ${}_D\mathfrak{G}_{30}(\xi) = 0$

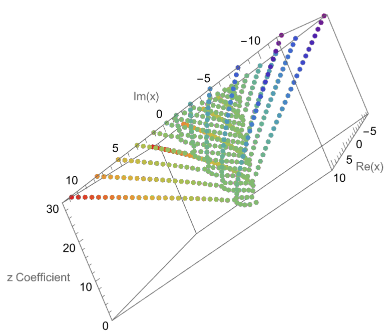


(b) Zeros of ${}_D\mathfrak{G}_{50}(\xi) = 0$

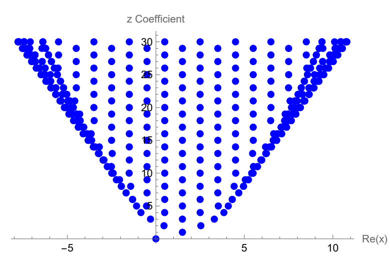
Figure 2: Zero distribution of the Tangent-based Bernoulli polynomials ${}_D\mathfrak{B}_k(\xi)$, Tangent-based Euler polynomials ${}_D\mathfrak{E}_k(\xi)$ and Tangent-based Genocchi polynomials ${}_D\mathfrak{G}_k(\xi)$ respectively, for n = 30, 50.

Also, we set different views for the stack of zeros of the Tangent-based Bernoulli, Tangent-based Euler and Tangent-based Genocchi polynomials as shown in Figures 3, 4 and 5.

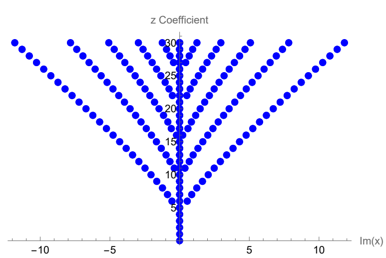
In Figure 3 (a) the data visualization of the zeros, in (b) the real zeros, in (c) the imaginary zeros and (d) the imaginary and real zeros of Tangent-based Bernoulli ${}_D\mathfrak{B}_{30}(\xi)$ are



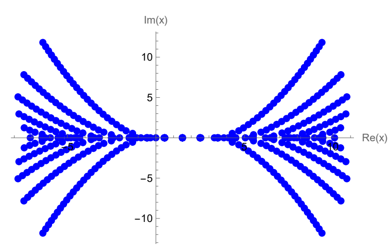
(a) Data visualization of zeros of ${}_D\mathfrak{B}_{30}(\xi) = 0$, in 3D



(b) Real zeros of ${}_D\mathfrak{B}_{30}(\xi) = 0$



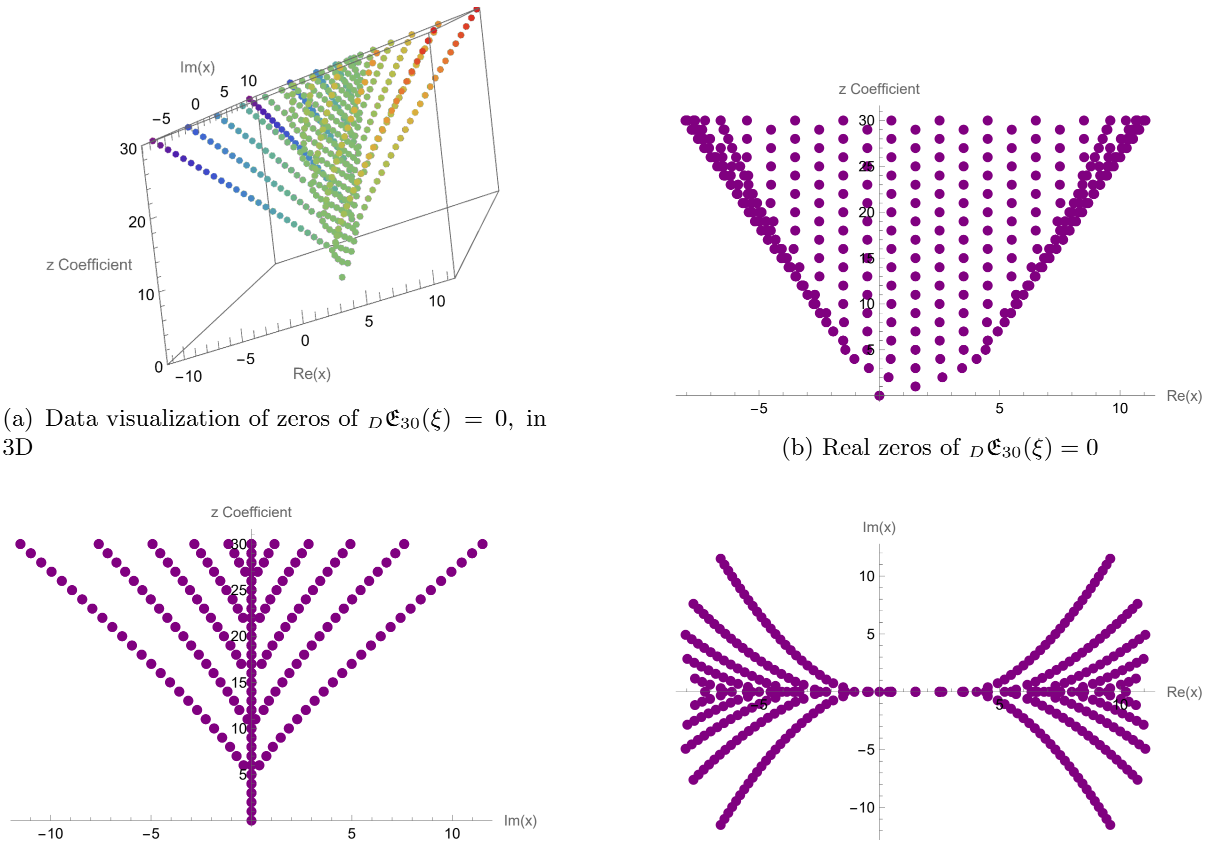
(c) Imaginary zeros of ${}_D\mathfrak{B}_{30}(\xi) = 0$



(d) Imaginary and Real zeros of ${}_D\mathfrak{B}_{50}(\xi) = 0$

Figure 3: Stacking structure zeros of the Tangent-based Bernoulli polynomials ${}_D\mathfrak{B}_k(\xi)$, for n = 30.

In Figure 4 (a) the data visualization of the zeros, in (b) the real zeros, in (c) the imaginary zeros and (d) the imaginary and real zeros of Tangent-based Euler polynomials ${}_D\mathfrak{E}_{30}(\xi)$ are



(a) Data visualization of zeros of ${}_D\mathfrak{E}_{30}(\xi) = 0$, in 3D

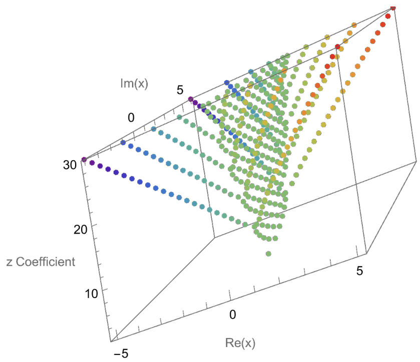
(b) Real zeros of ${}_D\mathfrak{E}_{30}(\xi) = 0$

(c) Imaginary zeros of ${}_D\mathfrak{E}_{30}(\xi) = 0$

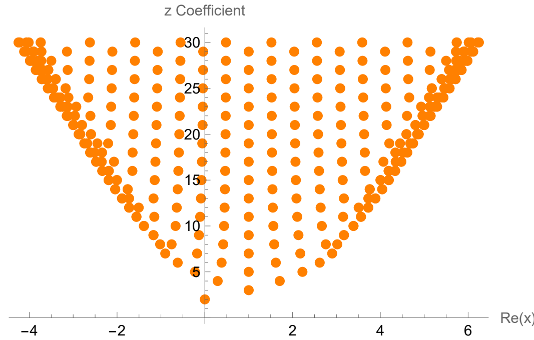
(d) Imaginary and real zeros of ${}_D\mathfrak{E}_{30}(\xi) = 0$

Figure 4: Stacking structure zeros of the Tangent-based Euler polynomials ${}_D\mathfrak{E}_k(\xi)$, for n = 30.

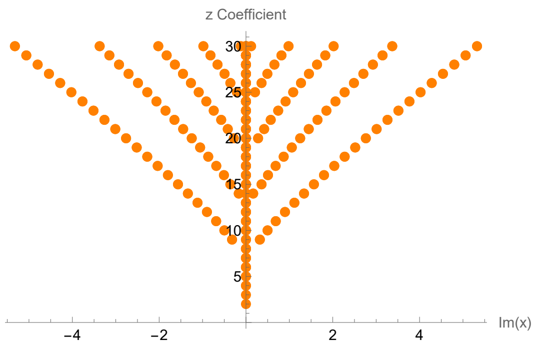
In Figure 5 (a) the data visualization of the zeros, in (b) the real zeros, in (c) the imaginary zeros and (d) the imaginary and real zeros of Tangent-based Genocchi polynomials ${}_D\mathfrak{G}_{30}(\xi)$ are



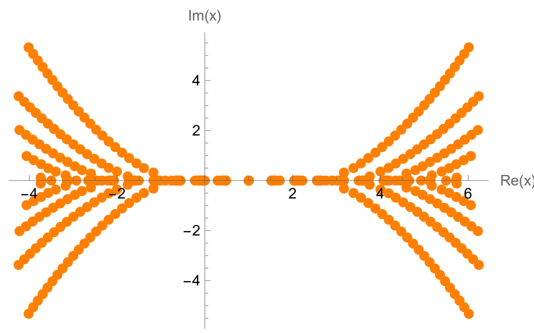
(a) Data visualization of zeros of ${}_D\mathfrak{G}_{30}(\xi) = 0$, in 3D



(b) Real zeros of ${}_D\mathfrak{G}_{30}(\xi) = 0$



(c) Imaginary zeros of ${}_D\mathfrak{G}_{30}(\xi) = 0$



(d) Imaginary and real zeros of ${}_D\mathfrak{G}_{30}(\xi) = 0$

Figure 5: Stacking structure zeros of the Tangent-based Genocchi polynomials ${}_D\mathfrak{G}_k(\xi)$, for n = 30.

6. Conclusion

This paper provides a significant extension of previous research by exploring the generation of Tangent-based Appell polynomials by applying the monomiality principle. The study highlights key properties of these polynomials, including their explicit and determinant forms, while demonstrating how they adhere to the monomiality principle. In particular, the research focuses on specific families of Appell polynomials, such as Tangent-based Bernoulli, Euler, and Genocchi polynomials, deriving important outcomes for each.

Additionally, numerical and graphical representations of the Tangent-based Bernoulli, Euler, and Genocchi polynomials have been presented, offering valuable insights into their behaviours and applications. These findings not only deepen our understanding of the structural properties of these polynomials but also provide a foundation for their use in various mathematical and scientific domains.

Looking forward, future research can expand on this work by exploring further generalizations and potential applications of Tangent-based Appell polynomials in areas such as number theory, combinatorics, and physics. Investigating more complex forms of these polynomials, as well as their interactions with other special functions, could reveal additional insights into their theoretical and practical significance. Furthermore, computational methods for evaluating and visualizing these polynomials could be refined to support more advanced applications in engineering, signal processing, and other scientific disciplines.

7. Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

8. Conflict of interest

The authors have no relevant financial or non-financial interests to disclose.

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