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Modified Results by Using New Generalized Definition of Fractional Derivative without Singular Kernel by Applying New Generalized Five Parameter Mittag-Leffler Function

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ABSTRACT: In a series of papers [43-46] reviewed all results and generalized the existing results by modifications. In this article, a new approach of the derivative of arbitrary order (FD) with the kernel of the smooth type that gains different depictions for the temporal and spatial variables has been given. It first applies to the time variables and hence it is fit to us transform of Laplace type (LT). Secondly, a definition is linked to the spatial type variables, by a global derivative of arbitrary order (FD), for which we will apply the transform of Fourier type (FT). The courtesy for this new methodology with a kernel of regular type was native from the vision that there is a period of global systems, which can designate the material heterogeneities and the fluctuations of unlike scales, which cannot be well described by traditional local theories or by arbitrary order models with the kernel of singular type. In this endeavour we are introducing a new generalized five parameter Mittag-Leffler function which is used in the definition of fractional derivative.

Key Words: Fractional derivative, Laplace transform, Fourier transform, Mittag-Leffler function, generalized Mittag-Leffler function, Prabhakar generalized Mittag-Leffler function, new generalized Mittag-Leffler function.

Contents

Introduction

The Fractional calculus (Arbitrary order Calculus) is a generalization of ordinary differentiation to non-integer cases The term of fractional calculus caught the attention of other great mathematicians, many of them directly or indirectly contributed to its development In the last 300 years the fractional calculus had an illustrious development as shown by the many mathematical volumes dedicated to it (e.g. Baleanu et al. [1], Caponetto [2], Caputo [3], Diethelm [4], Hilfer [5], Jiao et al [6], Kilbas et al. [7], Kyriakova [8], Mainardi [9], McBride [10], Miller and Ross [11], Petras [12], Samko et al [13], Podlubny [14], Sabatier et al. [15], Torres and Malinowska [16], Ying and Chen [17]) and by the distinguished diffusion as shown by the many meetings dedicated to it and the superfluous of articles appeared in mathematical (e.g. Kilbas and Marzan [18], Heinsalu et al [19], Luchko and Gorenflo [20]). The use of derivative of arbitrary order has also spread into many other fields of science as well mathematics and physics (e.g. Laskin [21], Naber [22], Baleanu et al. [23], Zavada [24], Baleanu et al. [25], Caputo and Fabrizio [26],[27]) such as biology (e.g. Cesarone et al. [28], Caputo and Cametti [29]), economy (e.g. Caputo [30]), demography (e.g. Jumarie [31]), geophysics (e.g. Iaffaldano [32]), medicine (e.g El

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Sahed [33]) and bioengineering (e.g. Magin [34]). The major developments in the fractional derivative which carried out by the scientists One can see also [35], [36], [37], [38], [39], [40] and [41]. However, some criticism has been made for the somewhat cumbersome mathematical expression of its definition and the consequent complications in the solutions of the fractional order differential equations. We consider an advanced definition of fractional derivative, which assumes two different representations for the temporal and spatial variable. The first works introduced on the time variable, where the real powers appearing and obtain the solutions of the usual fractional derivative will turn into integer powers, with some simplifications in these formulae and the computations. In this framework, it is properly used the Laplace transform. We are applying generalization of Mittag-Leffler function as a kernel in the definition of fractional derivative without Singular Kernel.

1. The New Generalization of Five Parameter Mittag-Leffler Function

In mathematics, the Mittag-Leffler functions are family of special functions They are complex-valued functions of a complex argument z and moreover depend on one or two complex parameters. The definition of the new generalization of Mittag-Leffler function in five parameters is given by authors first time here and can be defined by the Maclaurin series as:

$$E^{\delta,\varepsilon}_{\alpha,\beta,\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\delta)_k(\varepsilon)_k}{\Gamma(\alpha k+\beta+\gamma+1)} \frac{z^k}{k!k!} \tag{1.1}$$

where $\alpha, \beta, \gamma, \delta \in C, \Gamma(x)$ is the gamma function which is the most common extension of factorial function to complex numbers and defined for $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0, \mathrm{Re}(\varepsilon) > 0$ and, $\mathrm{Re}(\delta) > 0$.

In this case as $\alpha, \beta, \gamma, \varepsilon$ and δ are real and positive the series converges for all values of the argument z so the Mittag-Leffler function in five parameter is an entire function.This function will play the most important role in the theory of the fractional calculus because the solution of fractional differential equations are obtained in the form of Mittag-Leffler function.

Special Case I. If we put $\delta, \varepsilon = 1$ in equation no. (1) we get another new generalized three parameter Mittag-Leffler function is given as:

$$E^{1,1}_{\alpha,\beta,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta+\gamma+1)} \tag{1.2}$$

where $\alpha, \beta, \gamma \in C, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0$

Special Case II. If we put $\gamma = 0, \varepsilon = 1$ in equation no. (1) we get Prabhakar's [11] generalization of Mittag-Leffler function in 1971 is given as:

$$E^{\delta,1}_{\alpha,\beta,0}(z) = \sum_{k=0}^{\infty} \frac{(\delta)_k}{\Gamma(\alpha k+\beta+1)} \frac{z^k}{k!} \tag{1.3}$$

Where $\alpha, \beta, \delta \in C, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\delta) > 0$

Special Case III. If we put $\delta = 1, \varepsilon = 1$ and $\gamma = 0$ in equation no. (1) we get Wiman's [3] generalization of Mittag-Leffler function in 1905 is given as:

$$E^{1,1}_{\alpha,\beta,0}(z) = E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta+1)} \tag{1.4}$$

Where $\alpha, \beta \in C, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0,$

Special Case IV. If we put $\delta = 1,\ \varepsilon = 1,\ \gamma = 0$ and $\beta = 0$ in equation no. (1) we get Mittag-Leffler function [5] which is discovered by a Swedish Mathematician G. M. Mittag-Leffler in 1903.

$$E^{1,1}_{\alpha,0,0}(z) = E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} \tag{1.5}$$

Where $\alpha \in C, \mathrm{Re}(\alpha) > 0$

Special Case V. If we put $\delta = 1,\ \varepsilon = 1,\ \gamma = 0, \alpha = 1$ and $\beta = 0$ in equation no. (1) we get exponential function

$$E^{1,1}_{1,0,0}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z \tag{1.6}$$

2. A New Fractional Time Derivative by Using Generalized Five Parameter Mittag-Leffler Function

Let us recall the usual Caputo fractional time derivative (UFD_t) of order α, given by

$$D_t^{(\alpha)} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(\tau)}{(t-\tau)} d\tau \tag{2.1}$$

with $\alpha \in [0,1]$ and $a \in [-\infty, t), f \in H^1(a,b), b > a$. By changing the kernel $(t-\tau)^{-\alpha}$ with the function $\exp\left(-\frac{\alpha}{1-\alpha}t\right)$ and $\frac{1}{\Gamma(1-\alpha)}$ with $\frac{M(a)}{1-a}$, and we replace exponential function by Mittag-Leffler function we obtain the following new definition of fractional time derivative NFD

$$D_t^{(\alpha)} f(t) = \frac{M(\alpha)}{(1-\alpha)} \int_{-\infty}^t f'(\tau) E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma} \left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau \tag{2.2}$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$. According to the definition (7), the NFD_t is zero when $f(t)$ is constant, as in the NFD_t, but, contrary to the UFD_t, the kernel does not have singularity for $t = \tau$. The new NFDt can also be applied to functions that do not belong to $H^1(a,b)$. Indeed, the definition (7) can be formulated also for $f \in L^1(-\infty, b)$ and for any $\alpha \in [0,1]$ as

$$D_t^{(\alpha)} f(t) = \frac{\alpha M(\alpha)}{(1-\alpha)} \int_{-\infty}^t (f(t) - f(\tau)) E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma} \left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau \tag{2.3}$$

Now, it is worth observing that if we put $\sigma = \frac{1-\alpha}{\alpha} \in [0,\infty], \alpha = \frac{1}{1+\sigma} \in [0,1]$ the definition (2.2) of NFD_t assumes the form

$$D_t^{(\alpha)} f(t) = \frac{N(\sigma)}{\sigma} \int_a^t f'(\tau) E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma} \left[-\frac{(t-\tau)}{\sigma}\right] d\tau \tag{2.4}$$

where $\sigma \in [0,\infty]$ and $N(\sigma)$ is the corresponding normalization term of $M(\alpha)$, such that $N(0) = N(\infty) = 1$. Moreover, because

$$\lim_{\sigma \to 0, \beta \to 1} \frac{1}{\sigma} E^{\delta}_{\vartheta,\beta,\gamma} \left[-\frac{(t-\tau)}{\sigma}\right] = \delta(t-\tau) \tag{2.5}$$

and for $\alpha \to 1, \vartheta = 1, \delta = 1, \beta = 1, \gamma = 1$ we have $\sigma \to 0$.

Then (see [35] and [36])

$$\lim_{\substack{\alpha \to 1 \\ k \to 1}} D_t^{(\alpha)} f(t) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t f'(\tau) E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma} \left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau$$

$$\lim_{\substack{\alpha \to 1 \\ k \to 1}} D_t^{(\alpha)} f(t) = \lim_{\sigma \to 0} \frac{N(\sigma)}{\sigma} \int_a^t f'(\tau) E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma} \left[-\frac{(t-\tau)}{\sigma}\right] d\tau$$

Otherwise, when $\alpha \to 0, \vartheta = 1, \delta = 1, \beta = 1, \gamma = 1, k \to 1$ then $\sigma \to +\infty$. Hence

$$\begin{aligned} \lim_{\alpha \to 0} D_t^{(\alpha)} f(t) &= \lim_{\alpha \to 0} \frac{M(\alpha)}{(1-\alpha)} \int_a^t f'(\tau) \left\{ E^{\delta}_{\vartheta,\beta,\gamma} \left[-\left(\frac{\alpha(t-\tau)}{1-\alpha}\right)\right]\right\} d\tau \\ \lim_{\alpha \to 1} D_t^{(\alpha)} f(t) &= \lim_{\sigma \to \infty} \frac{N(\sigma)}{\sigma} \int_a^t f'(\tau) E^{\delta}_{\vartheta,\beta,\gamma} \left[-\frac{(t-\tau)}{\sigma}\right] d\tau \\ &= f(t) - f(a) \end{aligned} \tag{2.6}$$

Theorem 2.1 *For NFD_t, if the function $f(t)$ is such that $f^{(s)}(a)=0$, $s=1,2,..,n$ then, we have*

$$D_t{}^n\left(D_t{}^\alpha f(t)\right)=D_t{}^\alpha\left(D_t{}^n f(t)\right) \tag{2.7}$$

Proof: We begin considering $n=1$, then from definition (2.2) of $D_t^{(\alpha+1)}f(t)$, we obtain

$$D_t^\alpha\left(D_t^1 f(t)\right)=\frac{M(\alpha)}{(1-\alpha)}\int_a^t f'(\tau)E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau \tag{2.8}$$

Hence, after integration by parts and assuming $f'(a)=0$, we have

$$\begin{aligned}
D_t{}^\alpha\left(D_t{}^1 f(t)\right)&=\frac{M(\alpha)}{(1-\alpha)}\int_a^t\left(\frac{d}{d\tau}f(\tau)\right)E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau\\
&=\frac{M(\alpha)}{(1-\alpha)}\left[\int_a^t\frac{d}{d\tau}f(\tau)E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau-\frac{\alpha}{1-\alpha}\int_a^t f(\tau)E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau\right]\\
&=\frac{M(\alpha)}{(1-\alpha)}\left[f(t)-\frac{\alpha}{1-\alpha}\int_a^t f(\tau)E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau\right].
\end{aligned}$$

Otherwise

$$\begin{aligned}
D_t{}^1\left(D_t{}^\alpha f(t)\right)&=\frac{d}{dt}\left(\frac{M(\alpha)}{(1-\alpha)}\int_a^t f(\tau)E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau\right)\\
&=\frac{M(\alpha)}{(1-\alpha)}\left[f(t)-\frac{\alpha}{1-\alpha}\int_a^t f(\tau)E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau\right].
\end{aligned}$$

It is easy to generalize the proof for any $n>1$.
In the following, we suppose the function $M(\alpha)=1$. □

3. The Laplace Transform of the NFD_t

To study the properties of the NFD_t, defined in equation (2.2) with a $=0$, has priority the computation of its Laplace transform (LT) given with p variable

$$\text{LT}\left[D_t^{(\alpha)}f(t)\right]=\frac{1}{(1-\alpha)}\int_0^\infty E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}-pt\int_0^t f(\tau)E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}\left(-\frac{\alpha(t-\tau)}{(1-\alpha)}\right)d\tau dt. \tag{3.1}$$

Hence, from the property of Laplace transform of convolution, we have

$$\begin{aligned}
\text{LT}\left[D_t^{(\alpha)}f(t)\right]&=\frac{1}{(1-\alpha)}\text{LT}(f(t))LT\left(E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}\left(-\frac{\alpha t}{(1-\alpha)}\right)\right)\\
&=\frac{(pLT(f(t)-f(0))}{p+\alpha(1-p)}.
\end{aligned} \tag{3.2}$$

Similarly

$$\begin{aligned}
\text{LT}[D_t^{(\alpha+1)}f(t)]&=\frac{1}{(1-\alpha)}\text{LT}\left(\overline{f}(t)\right)LT\left(E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}-\frac{\alpha t}{(1-\alpha)}\right)\\
&=\frac{\left(p^2LT[f(t)]-pf(0)-f'(0)\right)}{p+\alpha(1-p)}.
\end{aligned} \tag{3.3}$$

Finally,

$$\begin{aligned}
\text{LT}\left[D_t^{(\alpha+n)}f(t)\right]&=\frac{1}{1-\alpha}LT\left[f^{(n+1)}(t)\right]LT\left[E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}-\frac{\alpha t}{(1-\alpha)}\right]\\
&=\frac{p^{n+1}LT[f(t)]-p^nf(0)-p^{n-1}f'(0)\dots f^{(n)}(0)}{p+\alpha(1-p)}.
\end{aligned}$$

4. Fractional Gradient Operators

In this section, we introduce a new notion of fractional gradient able to describe non local dependence in constitutive equations (see [39] and [40]). Let us consider a set $\Omega \epsilon R^3$ and a scalar function $u(\cdot) : \Omega \to R$, we define the fractional gradient of order $\alpha \in [0,1]$ as follows

$$\nabla^{(\alpha)} u(x) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_\Omega \nabla u(y) E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma} \left[-\frac{\alpha^2(x-y)^2}{(1-\alpha)^2} \right] dy \tag{4.1}$$

with $x, y \in \Omega$

It is simple to prove from definition (9) and by the property

$$\lim_{\alpha\to 1} \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma} \left[-\frac{\alpha^2(x-y)^2}{(1-\alpha)^2} \right] = \mu(x-y) \tag{4.2}$$

that

$$\nabla^{(1)} u(x) = \nabla u(x) \text{ and } \nabla^{(0)} u(x) = \int_\Omega \nabla u(y) dy. \tag{4.3}$$

So, when $\alpha = 1, \nabla^{(1)} u(x)$ loses the non-locality, otherwise $\nabla^{(0)} u(x)$ is related to the mean value of $\nabla u(y)$ on Ω. In the case of a vector $u(x)$, we define the fractional tensor by

$$\nabla^{(\alpha)} u(x) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_\Omega \nabla \cdot u(y) E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma} \left[-\frac{\alpha^2(x-y)^2}{(1-\alpha)^2} \right] dy. \tag{4.4}$$

Thus, a material with the non-local property may be described by fractional constitutive equations. As an example, we consider an elastic non-local material, defined by the following constitutive equation between the stress tensor T and $\nabla^{(\alpha)} u(x) T(x,t) = A \nabla^{(\alpha)} u(x,t), \alpha \in (0,1]$ where A is a fourth order symmetric tensor, or in the integral form

$$T(x,t) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_\Omega \nabla u(y) E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma} \left[-\frac{\alpha^2(x-y)^2}{(1-\alpha)^2} \right] dy. \tag{4.5}$$

Likewise, we introduce the fractional divergence, defined for a smooth $u(x) : \Omega \to R^3$ by

$$\nabla^{(\alpha)} u(x) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_\Omega \nabla u(y) E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma} \left[-\frac{\alpha^2(x-y)^2}{(1-\alpha)^2} \right] dy. \tag{4.6}$$

Theorem 4.1 *From definitions (4.1) and (4.2), we have for any $u(x) : \Omega \to R$, such that*

$$\nabla u(x) \cdot n|_{\vartheta\Omega} = 0 \tag{4.7}$$

the following identity

$$\nabla \cdot \nabla^{(\alpha)} u(x) = \nabla^{(\alpha)} \cdot \nabla u(x). \tag{4.8}$$

Proof: Employing (4.6), we obtain

$$\nabla \cdot \nabla^{(\alpha)} u(x) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_\Omega \nabla u(y) \cdot \nabla_x E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma} \left[-\frac{\alpha^2(x-y)^2}{(1-\alpha)^2} \right] dy \tag{4.9}$$

$$\nabla \cdot \nabla^{(\alpha)} u(x) = -\frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_\Omega \nabla u(y) \cdot \nabla E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma} \left[-\frac{\alpha^2(x-y)^2}{(1-\alpha)^2} \right] dy \tag{4.10}$$

$$= \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_\Omega \nabla \cdot \nabla u(y) E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma} \left[-\frac{\alpha^2(x-y)^2}{(1-\alpha)^2} \right] dy -$$

$$\frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_{\partial\Omega} \nabla u(y).n E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma} \left[-\frac{\alpha^2(x-y)^2}{(1-\alpha)^2} \right] dy$$

hence, from the boundary condition (4.7), the identity (4.8) is proved, because (4.8) coincides with

$$\nabla^{(\alpha)} \cdot \nabla u(x) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_\Omega \nabla \cdot \nabla u(y) E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma} \left[-\frac{\alpha^2(x-y)^2}{(1-\alpha)^2} \right] dy.$$

□

5. Fourier Transform of Fractional Gradient and Divergence

For a smooth function $u(\mathbf{x}) : R^3 \to R$ the Fourier transform (FT) of the fractional gradient is defined by

$$\mathrm{FT}\left(\nabla^{(\alpha)} u(\mathbf{x})\right)(\xi) = \int_{R^3}^{\cdot\cdot} \nabla^{(\alpha)} u(\mathbf{x}) E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}[-2\pi i \xi \cdot \mathbf{x}] \mathbf{d}x. \tag{5.1}$$

Thus, if we consider the gradient of (4.1), the Fourier transform is given by

$$\begin{aligned} FT\left(\nabla^{(\alpha)} u\right)(\xi) &= \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} FT\left(\int_{R^3}^{\cdot\cdot} \nabla u(\mathbf{y}) E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}\left[-\frac{\alpha^2 (x-y)^2}{(1-\alpha)^2}\right] dy\right)(\xi) \\ &= \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} FT(\nabla u)(\xi) FT\left(E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}\left[-\frac{\alpha^2 (x-y)^2}{(1-\alpha)^2}\right]\right)(\xi) \end{aligned}$$

where

$$\mathrm{FT}\left(E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}\left[-\frac{\alpha^2 (x-y)^2}{(1-\alpha)^2}\right]\right)(\xi) = \frac{(1-\alpha)\sqrt{\pi}}{\alpha} E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}\left[\frac{\pi^2 (1-\alpha)^2 \xi^2}{\alpha^2}\right].$$

Thus, we obtain:

$$\mathrm{FT}\left(\nabla^{(\alpha)} u\right)(\xi) = \sqrt{\pi^{1-\alpha}}\, \mathrm{FT}(\nabla u)(\xi) E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}\left[-\frac{\pi^2 \left(1-\alpha^2\right) \xi^2}{\alpha^2}\right]. \tag{5.2}$$

The Fourier transform of fractional divergence is defined by

$$FT\left(\nabla^{(\alpha)} \cdot u\right)(\xi) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} FT\left(\int_{\Omega}^{\cdot\cdot} \nabla \cdot u(\mathbf{y}) E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}\left[-\frac{\alpha^2 (\mathbf{x}\text{-}\mathbf{y})^2}{(1-\alpha)^2}\right] d\mathbf{y}\right)(\xi) \tag{5.3}$$

from which we have

$$\mathrm{FT}\left(\nabla^{(\alpha)} \cdot \mathbf{u}\right)(\xi) = \sqrt{\pi^{1-\alpha}}\, \mathrm{FT}(\nabla \cdot u)(\xi) E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}\left[-\frac{\pi^2 (1-\alpha)^2 \xi^2}{\alpha^2}\right]. \tag{5.4}$$

6. Fractional Laplacian

In the study of partial differential equations, there is a great interest in fractional Laplacian. Using the definitions of fractional gradient and divergence, we can suggest the representation of fractional Laplacian for a smooth function $f(x) : \Omega \to R^3$, such that $\nabla f(x) \cdot n|_{\vartheta\Omega} = 0$, as

$$\left(\nabla^2\right)^{(\alpha)} f(x) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_{\Omega}^{\cdot\cdot} \nabla \cdot \nabla f(y) E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}\left[-\frac{\alpha^2 (x-y)^2}{(1-\alpha)^2}\right] dy. \tag{6.1}$$

By Theorem 4.1, we have $\left(\nabla^2\right)^{(\alpha)} f(x) = \nabla \cdot (\nabla)^{(\alpha)} f(x) = (\nabla)^{(\alpha)} \cdot \nabla f(x)$.

Now, we suppose that $(x) = 0$, on $\partial\Omega$ then we extend the function $f(x) = 0$ on $R^3 \backslash \Omega$. So, we consider the Fourier transform

$$\begin{aligned} \mathrm{FT}\left(\left(\nabla^2\right)^{(\alpha)} f(x)\right) &= \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} FT\left(\int_{R^3}^{\cdot\cdot} \nabla^2 f(y) E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}\left[-\frac{\alpha^2 (x-y)^2}{(1-\alpha)^2}\right] dy\right)(\xi) \\ &= \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} FT(\nabla \cdot \nabla f(x))(\xi) FT\left(E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}\left[-\frac{\alpha^2 x^2}{(1-\alpha)^2}\right]\right)(\xi) \\ &= 4\pi|\xi|^2 FT(f(x))(\xi) \sqrt{\pi^{1-\alpha}} E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}\left[-\frac{(1-\alpha)^2 \xi^2}{\alpha^2}\right]. \end{aligned} \tag{6.2}$$

Finally, if $\alpha = 1$ we obtain from (6.2)

$$\begin{aligned} \mathrm{FT}\left(\nabla^2 f(x)\right) &= -\lim_{\substack{\alpha \to 1 \\ k \to 1}} 4\pi|\xi|^2 FT(f(x))(\xi) \sqrt{\pi^{1-\alpha}} E^{\delta,\varepsilon}_{\vartheta,\beta,\gamma}\left[-\frac{(1-\alpha)^2 \xi^2}{\alpha^2}\right] \\ &= -4\pi|\xi|^2 LT(f(x))(\xi). \end{aligned} \tag{6.3}$$

This completes the analysis.

7. Conclusion

In this work, we have derived the new results by using advanced definition of fractional derivative without singular kernel by using new generalized five parameter Mittag-Leffler function. The results of the advanced definition of fractional derivative without singular kernel are same as the Michele Caputo and Mauro Fabrizio [43].

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