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A note on the energy of graphs with self-loops

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ABSTRACT: In this paper, we compute the spectra of some extended (bipartite) double graphs with self-loops and compare their energies. We also give a method to construct cospectral graphs with self loops.

Key Words: Self-loops, adjacency matrix, energy of a self-loop graph.

Contents

1. Introduction

Throughout the paper, G denotes a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let S be a set consisting of σ number of vertices of G. The self-loop graph G_S is obtained from G by adding a self-loop at all those vertices which are in the set S. The adjacency matrix of G_S, denoted by $A(G_S)$, is a $(0, 1)$-sqaure matrix of order n and is defined as $A(G_S) = A(G) + D_S(G)$, where $A(G)$ is the adjacency matrix of G and $D_S(G)$ is the diagonal matrix with its i-th diagonal entry equal to 1 if $v_i \in S$ or 0 otherwise. Let $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ be the eigenvalues of $A(G)$ and let the eigenvalues of $A(G_S)$ be $\lambda_1^S(G) \geq \lambda_2^S(G) \geq \cdots \geq \lambda_n^S(G)$. Two non-isomorphic graphs G_S and $H_{S'}$ are co-spectral if the spectrum of $A(G_S)$ and $A(H_{S'})$ are same. The energy of G, denoted by $\mathcal{E}(G)$, is the sum of absolute values of λ_i, where $i = 1, 2, \ldots, n$. Recently, the concept of energy of a graph with self-loops was introduced by Gutman et al. in [3]. It is denoted by $\mathcal{E}(G_S)$ and is defined as $\mathcal{E}(G_S) = \sum_{i=1}^{n} \left| \lambda_i^S(G) - \frac{\sigma}{n} \right|$. The authors in [3] showed that the relation $\mathcal{E}(G_S) = \mathcal{E}(G_{V(G)\setminus S})$ holds if G is a bipartite graph. Also, in [3], a McClelland–type upper bound for the energy of graphs with self-loops is obtained and it was conjectured that $\mathcal{E}(G_S) > \mathcal{E}(G)$ for $1 < \sigma < n - 1$. This conjecture was disproved in [5] by means of counterexamples. In [1], Akbari et al. showed that energy of a bipartite graph G_S is always greater than or equal to its ordinary energy. Later this result was improved for an unbalanced bipartite graph in [11] by Rakshith et al. Relations between energy of a graph and energy of a graph with self-loops were also obtained in [11]. In [8], Popat et al. obtained a family of graphs which satisfies the relation $\mathcal{E}(G_S) = \mathcal{E}(G)$ and $0 < \sigma < n$. Two non-isomorphic self-loop graphs are called equienergetic if their energies are same. In [9], pairs of equienergetic graphs with self loops are presented. Some bounds on energy of self-loop graph and spectral related properties of self-loop graphs are presented in [3,1,10]

Motivated by the concept of energy of graphs with self-loops, in this note, we compute the spectra of some extended (bipartite) double graphs with self-loops and compare their energies. We also give a method to construct cospectral graphs with self loops.

2. Main Results

The extended double graph and extended bipartite double graph of G are defined as follows.

Definition 2.1 *The extended double graph of G is obtained by taking two copies of G, and then adding an edge from a vertex v_i of a copy of G to a vertex v_j of another copy of G if and only if $i = j$ or $v_j v_j$ is an edge in G. It is denoted by $ED(G)$.*

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Definition 2.2 *The extended bipartite double graph of G is a bipartite graph with vertex partition sets $\{v_{11}, v_{12}, \ldots, v_{1n}\} \cup \{v_{21}, v_{22}, \ldots, v_{2n}\}$ and two vertices v_{1i} and v_{2j} are adjacent if and only if $i = j$ or $v_i v_j$ is an edge in G. It is denoted by $EBD(G)$.*

Let $ED(G)_\sigma$ be the graph obtained from $ED(G)$ by attaching self-loops at each of the vertices belonging to one of the copies of G in $ED(G)$. In the following theorem, we give the adjacency spectrum of $ED(G)_\sigma$.

Theorem 2.1 *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $A(G)$. Then the spectrum of $A(ED(G)_\sigma)$ consists of $\lambda_i + \frac{1}{2} \pm \frac{1}{2}\sqrt{4\,{\lambda_i}^2 + 8\,\lambda_i + 5}$ for $i = 1, 2, \ldots, n$.*

Proof: Let $\Gamma_\sigma \cong ED(G)_\sigma$. By proper labeling of the vertices of Γ_σ, we get the adjacency matrix $A(\Gamma_\sigma)$ of Γ_σ as follows:

$$\begin{pmatrix} A(G)+I_n & A(G)+I_n \\ A(G)+I_n & A(G) \end{pmatrix}.$$

Since $A(G)$ is real symmetric, it has n orthogonal eigenvectors. Let $X_1, X_2, \ldots, X_n$ be a set of n orthogonal eigenvectors of $A(G)$ corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively. Let $Z_i = \begin{bmatrix} X_i \\ \delta_i X_i \end{bmatrix}$ for $i = 1, 2, \ldots, n$. Then Z_i's are orthogonal and

$$A(\Gamma_\sigma)Z_i = \begin{bmatrix} (\lambda_i+1)(\delta_i+1)X_i \\ \\ (\lambda_i(\delta_i+1)+1)X_i \end{bmatrix}.$$

Thus, Z_i is an eigenvector of $A(\Gamma_\sigma)$ if and only if $\frac{(\lambda_i(\delta_i+1)+1)}{(\lambda_i+1)(\delta_i+1)} = \delta_i$ and the corresponding eigenvalue is $(\lambda_i+1)(\delta_i+1)$. Hence for $\lambda_i \neq -1$ and $\delta_i = \frac{-1 \pm \sqrt{4\lambda_i^2 + 8\lambda_i + 5}}{2(\lambda_i+1)}$, $\lambda_i + \frac{1}{2} \pm \frac{1}{2}\sqrt{4\,{\lambda_i}^2 + 8\,\lambda_i + 5}$ is an eigenvalue of $A(\Gamma_\sigma)$ corresponding to the eigenvector Z_i.
Now, if $\lambda_i = -1$, then it is easy to see that $\begin{bmatrix} \mathbf{0} \\ X_i \end{bmatrix}$ and $\begin{bmatrix} X_i \\ \mathbf{0} \end{bmatrix}$ are orthogonal eigenvectors of $A(\Gamma_\sigma)$ associated with the eigenvalues -1 and 0, respectively. Thus we have listed $2n$ orthogonal eigenvectors of $A(\Gamma_\sigma)$ along with their corresponding eigenvalues. □

The following corollary is immediate from the above theorem.

Corollary 2.1 *Let G_1 and G_2 be two non-isomorphic cospectral graphs on n vertices. Then $ED(G_1)_\sigma$ and $ED(G_2)_\sigma$ are cospectral graphs with self-loops.*

Let $EBD(G)_\sigma$ be the graph obtained from $EBD(G)$ by attaching self-loops at each of the vertices belonging to one of the partition sets of $EBD(G)$. In the following theorem, we give the adjacency spectrum of $EBD(G)_\sigma$.

Theorem 2.2 *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $A(G)$. Then the spectrum of $A(EBD(G)_\sigma)$ consists of $\frac{1}{2} \pm \frac{1}{2}\sqrt{4\,{\lambda_i}^2 + 8\,\lambda_i + 5}$ for $i = 1, 2, \ldots, n$.*

Proof: Let $\Gamma'_\sigma \cong EBD(G)_\sigma$. By proper labeling of the vertices of Γ'_σ, we get the adjacency matrix $A(\Gamma'_\sigma)$ of Γ'_σ as follows:

$$\begin{pmatrix} I_n & A(G)+I_n \\ A(G)+I_n & \mathbf{0} \end{pmatrix}.$$

Since $A(G)$ is real symmetric, it has n orthogonal eigenvectors. Let $X_1, X_2, \ldots, X_n$ be a set of n orthogonal eigenvectors of $A(G)$ corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively. Let $Z_i = \begin{bmatrix} X_i \\ \delta_i X_i \end{bmatrix}$ for

$i = 1, 2, \ldots, n$. Then Z_i's are orthogonal and

$$A(\Gamma'_\sigma)Z_i = \begin{bmatrix} (\delta_i(\lambda_i + 1) + 1)X_i \\ \\ (\lambda_i + 1)X_i \end{bmatrix}.$$

Thus, Z_i is an eigenvector of $A(\Gamma'_\sigma)$ if and only if $\dfrac{\lambda_i + 1}{\delta_i(\lambda_i + 1) + 1} = \delta_i$ and the corresponding eigenvalue is $\delta_i(\lambda_i + 1) + 1$. Hence for $\lambda_i \neq -1$ and $\delta_i = \dfrac{-1 \pm \sqrt{4\lambda_i^2 + 8\lambda_i + 5}}{2(\lambda_i + 1)}$, $\dfrac{1}{2} \pm \dfrac{1}{2}\sqrt{4\,\lambda_i{}^2 + 8\,\lambda_i + 5}$ is an eigenvalue of $A(\Gamma'_\sigma)$ corresponding to the eigenvector Z_i.
Now, if $\lambda_i = -1$, then it is easy to see that $\begin{bmatrix} \mathbf{0} \\ X_i \end{bmatrix}$ and $\begin{bmatrix} X_i \\ \mathbf{0} \end{bmatrix}$ are orthogonal eigenvectors of $A(\Gamma'_\sigma)$ associated with the eigenvalues 0 and 1, respectively. Thus we have listed $2n$ orthogonal eigenvectors of $A(\Gamma'_\sigma)$ along with their corresponding eigenvalues. □

Corollary 2.2 *Let G_1 and G_2 be two non-isomorphic cospectral graphs on n vertices. Then $EBD(G_1)_\sigma$ and $EBD(G_2)_\sigma$ are cospectral graphs with self-loops.*

At this stage, it is natural to ask "Whether $\mathcal{E}(ED(G)_\sigma) = \mathcal{E}(EBD(G)_\sigma)$ for some graph G?". In the following theorem we answer this question.

Theorem 2.3 *There exists no non-empty graph G satisfying the relation $\mathcal{E}(ED(G)_\sigma) = \mathcal{E}(EBD(G)_\sigma)$.*

Proof: From Theorems 2.1 and 2.2 , we get

$$\begin{aligned}
\mathcal{E}(ED(G)_\sigma) &= \sum_{i=1}^{n} |\lambda_i + \frac{1}{2}\sqrt{4(\lambda_i + 1)^2 + 1}| + \sum_{i=1}^{n} |\lambda_i - \frac{1}{2}\sqrt{4(\lambda_i + 1)^2 + 1}| \\
&= \sum_{\lambda_{i<-0.625}} |\lambda_i + \frac{1}{2}\sqrt{4(\lambda_i + 1)^2 + 1}| + \sum_{\lambda_{i<-0.625}} |\lambda_i - \frac{1}{2}\sqrt{4(\lambda_i + 1)^2 + 1}| \\
&+ \sum_{\lambda_{i\geq -0.625}} |\lambda_i + \frac{1}{2}\sqrt{4(\lambda_i + 1)^2 + 1}| + \sum_{\lambda_{i\geq -0.625}} |\lambda_i - \frac{1}{2}\sqrt{4(\lambda_i + 1)^2 + 1}| \\
&= 2\sum_{\lambda_{i<-0.625}} |\lambda_i| + \sum_{\lambda_{i\geq -0.625}} \sqrt{4(\lambda_i + 1)^2 + 1}.
\end{aligned}$$

and

$$\mathcal{E}(EBD(G)_\sigma) \quad = \quad \sum_{i=1}^{n} \sqrt{4(\lambda_i + 1)^2 + 1}.$$

Thus, $\mathcal{E}(ED(G)_\sigma) - \mathcal{E}(EBD(G)_\sigma) = 2\sum_{\lambda_{i<-0.625}} |\lambda_i| - \sum_{\lambda_{i<-0.625}} \sqrt{4(\lambda_i + 1)^2 + 1}$. Since $2|\lambda_i| > \sqrt{4(\lambda_i + 1)^2 + 1}$ for $\lambda_i < -0.625$ and one of the eigenvalue of G is less than or equal to -1, we get $\mathcal{E}(ED(G)_\sigma) > \mathcal{E}(EBD(G)_\sigma)$. This completes the proof. □

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