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Stabile and Critical Pitchfork Domination in Graphs

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ABSTRACT: Let G be a finite, simple, undirected graph and without isolated vertices. A subset D of V is a pitchfork dominating set if $j \leq |N(v) \cap (V - D)| \leq k$ for every $v \in D$ and any j and k integers. In this paper, The effects of adding or removing an edge and removing vertex from the graph are studied on the pitchfork domination number $\gamma_{pf}(G)$ for $j = 1$ and $k = 2$. Some graphs didn't effects on this changing, while $\gamma_{pf}(G)$ of other graphs were increasing or decreasing. The study of these effects has an important advantages to learn the ways of treatments to any added or damaged of any nods (vertices) or links (edges) of the system or networks to avoid losing some properties of the system and to give the best services with minimum costs.

Key Words: pitchfork dominating set, pitchfork domination number, changing and unchanging.

Contents

1. Introduction

Let $G = (V, E)$ be a graph without isolated vertices has a vertex set V of order n and an edge set E of size m. For any vertex $v \in V$, the degree of v is defined as the number of edges incident on v and denoted by $deg(v)$. The subgraph of G induced by the vertices in A is denoted by $G[A]$. The complement $\overline{G}$ of a simple graph G with vertex set $V(G)$ is the graph in which two vertices are adjacent if and only if they are not adjacent in G. If $e \in E$, then $G - e$ means the graph obtained from G after removing e. Also, if $v \in V$, then $G - v$ means the graph obtained from G after removing v and all the edges incident on it. If $e \notin E$ is an edge of $\overline{G}$, then $G + e$ means the graph getting from G after adding e. For $D \subseteq V$ and $u \in D$, the private neighbour set of u with respect to D is defined as $pn[u, D] = \{v | N[v] \cap D = \{u\}\}$. For graph theoretic terminology we refer to [27]. The study of domination and related subset problems is one of the fastest growing areas in graph theory. For a detailed survey of domination one can see [19]. There are very more sorts of domination such as [1,2,3,8,9,10,11,13,14,15,16,17,18,20,21,22,23,24,25,26,28,29,30]. A new model of domination in graphs called the pitchfork domination is introduced by Al-Harere and Abdlhusein [4,5,6,7,12]. A subset D of V is a pitchfork dominating set if every vertex $v \in D$ dominates at least j and at most k vertices of $V - D$, for any j and k integers. The domination number of G, denoted by $\gamma_{pf}(G)$ is a minimum cardinality over all pitchfork dominating sets in G. The study of the effects of removing or adding an edge or removing a vertex from the graph has large advantages applications in any system or network. Such as how to deal with any damaged device (vertex), or any cutting in link (edge) of network. Also, for adding other link (edge) to the network to avoid lose some properties of the system and to give the best services with minimum costs.
In this paper, the effects of adding or remove an edge and remove vertex from the graph are studied on $\gamma_{pf}(G)$. Some graphs don't effects on this changing, while $\gamma_{pf}(G)$ of other graphs will increase or decrease.

2. Changing and Un-Changing $\gamma_{pf}(G)$

In this section we discuss the effects on $\gamma_{pf}(G)$ when G is modified by deleting a vertex or deleting or adding an edge. If $G - v$ has a pitchfork dominating set, then we partition the vertices of G into three sets:

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$V^0 = \{v \in V : \gamma_{pf}(G - v) = \gamma_{pf}(G)\}$.
$V^+ = \{v \in V : \gamma_{pf}(G - v) > \gamma_{pf}(G)\}$.
$V^- = \{v \in V : \gamma_{pf}(G - v) < \gamma_{pf}(G)\}$.
Similarly, edges set can be partitioned into:
$E^0_* = \{e \in E : \gamma_{pf}(G * e) = \gamma_{pf}(G)\}$.
$E^+_* = \{e \in E : \gamma_{pf}(G * e) > \gamma_{pf}(G)\}$.
$E^-_* = \{e \in E : \gamma_{pf}(G * e) < \gamma_{pf}(G)\}$, where $* = \begin{cases} - & ,if\ e \in G \\ + & ,if\ e \in \overline{G} \end{cases}$

Remark 2.1 For any graph G having a pitchfork domination, we have:
1. If $\gamma_{pf}(G) = 1$. For any $v \in V$, $e \in E$ and $\acute{e} \notin E$, if $G - v$, $G - e$, or $G + \acute{e}$ has a pitchfork dominating set, then $v \in V^0$, $e \in E^0_-$ and $\acute{e} \in E^0_+$.
2. If G has a support vertex u, then $G - u$ has no pitchfork domination.
3. If G has an end vertex v, then $G - e$ has no pitchfork domination where e lies on v.

Theorem 2.1 *Let G be a graph having pitchfork domination, then $v \in V^0$ if and only if one of the following statements hold:*
1. If $v \in V - D$ and for each vertex in D that dominates v, dominates other vertex.
2. If $v \in D$ and it dominates only one vertex say $w \in V - D$ or two vertices $w, u \in V - D$ such that $w \in pn[v, D]$ and $1 \le |N(w) \cap (V - D)| \le 2$.
3. If $v \in D$ and it dominates one or two non-adjacent vertices say w_1 and w_2 such that $w_1, w_2 \in pn[v, D]$ and they are adjacent to another vertex in $V - D$, say w_3 such that $N(w_3) \cap V - D = \{w_1, w_2\}$ and all vertices dominate w_3 in D dominates another vertices in $V - D$.

Proof: Assume that D is a γ_{pf}-set in G.
1. It is obvious.
2. Since the vertex w is dominated by only v. So, in $D - \{v\}$, the set $D - \{v\}$ does not dominate the vertex w. Thus, vertex w must be insert to set $D - \{v\}$. It is clear that set $D - \{v\} \cup \{w\}$ is a pitchfork dominating set since $1 \le |w \cap (V - D)| \le 2$ and it is a minimum. See Fig. 1 and Fig. 2.
3. As the same manner in case 2, the two vertices w_1 and w_2 are not dominated by set $D - \{v\}$. Also, each vertex in D that dominates w_3, dominates another vertex in $V - D$. Thus, vertex w_3 dominates w_1 and w_2 in a set $D - \{v\} \cup \{w_3\}$ which is a minimum pitchfork dominating set. See Fig. 3 and Fig. 4.

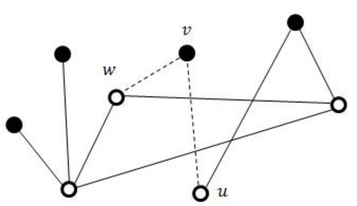

Figure 1: G

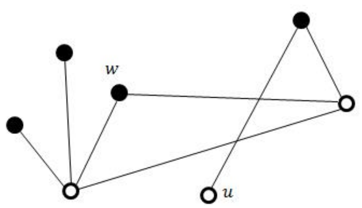

Figure 2: $G - v$

□

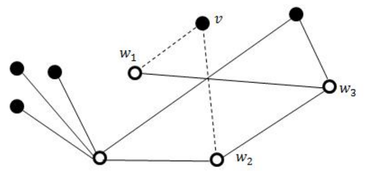

Figure 3: G

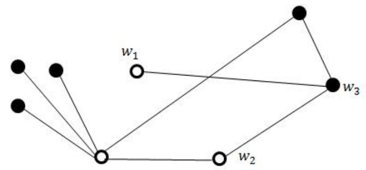

Figure 4: $G - v$

Theorem 2.2 *Let G be a graph having pitchfork domination, then $v \in V^+$ if $v \in D$ and $w_1,\ w_2 \in pn[v, D]$ such that w_1 and w_2 are non-adjacent, where $1 \leq |N(w_i) \cap (V - D)| \leq 2$; $i = 1,\ 2$ and $|N(w_1) \cap (V - D)| \neq |N(w_2) \cap (V - D)|$.*

Proof: Assume that D is a γ_{pf}-set in G and $v \in D$. Since $w_1,\ w_2 \in pn[v, D]$, then w_1 and w_2 are not dominated by $D - v$. So we add w_1 and w_2 to the set $D - v$ since every one of them dominate one or two different vertices of $V - D$. Therefore, $(D - v) \cup \{w_1, w_2\}$ is a minimum pitchfork dominating set and $\gamma_{pf}(G - v) > \gamma_{pf}(G)$. See Fig. 5 and Fig. 6 □

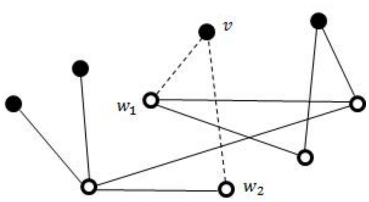

Figure 5: G

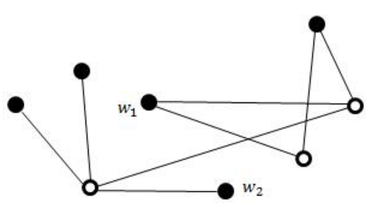

Figure 6: $G - v$

Theorem 2.3 *Let G be a graph having pitchfork domination, then $v \in V^-$ if one of the following statements hold:*
1. If $v \in D$ and every vertex in $V - D$ that is dominated by v is also dominated by other vertex in D.
2. If there is a component of cycle C_n such that $n \equiv 1\ (mod\,3\,)$.
3. If there is a component of path of order n such that $n \equiv 1\ (mod\,3\,)$, then $\gamma_{pf}(G - v) < \gamma_{pf}(G)$ if $G - v \cong P_{n-1}$ or $G - v \cong P_w \cup P_z$ such that $w = z \equiv 0\ (mod\,3\,)$.

4. If there exist $v \in V - D$ such that each vertex in D that dominates v dominates another vertex in $V - D$ and one of these vertices (say w) is adjacent to one of them and $pn[w, V - D] = \phi$.

Proof: Assume that D is a $\gamma_{pf}-$set in G.
1. Let v dominates two vertices in $V - D$ say w_1, w_2 where every one of them has other neighbour in D. Then, $D - v$ is a $\gamma_{pf}-$set of $G - v$ and $\gamma_{pf}(G - v) < \gamma_{pf}(G)$. For example see K_5.
2. Let v be any vertex in C_n, since $\gamma_{pf}(C_n) = \lceil \frac{n}{3} \rceil$ by Observation 1.7 in [12], then $\gamma_{pf}(C_{n-1}) = \lceil \frac{n}{3} \rceil - 1$. Thus, $\gamma_{pf}(G - v) < \gamma_{pf}(G)$.
3. In $G - v$, two cases are obtained as follows:
Case 1: If v is a pendent vertex, then P_n convert to P_{n-1} and its clear that $n - 1 \equiv 0\,(mod\,3)$, then $\frac{n-1}{3} = t < \lceil \frac{n}{3} \rceil$. Thus, $\gamma_{pf}(G - v) < \gamma_{pf}(G)$ and $v \in V^-$.
Case 2: If P_n convert to two paths P_{n-1} such that $P_n - v \cong P_w \cup P_z$ and $w = z \equiv 0\,(mod\,3)$, then $\frac{w}{3} + \frac{z}{3} < \frac{w+z+1}{3}$. Thus, $\gamma_{pf}(G - v) < \gamma_{pf}(G)$ and $v \in V^-$.
4. The vertex v_t can be putted instead of v in $(V - D) - v$. Therefore, $\gamma_{pf}(G - v) < \gamma_{pf}(G)$. See Fig. 7 and Fig. 8.

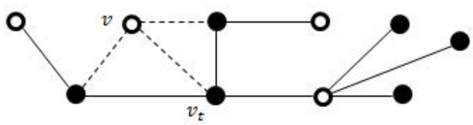

Figure 7: G

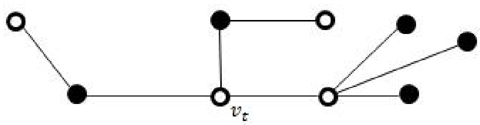

Figure 8: $G - v$

□

Notice that, when we add an edge $e \in \overline{G}$ to a graph G, then D may be effected until if e incident on two vertices both of them belong to D or to $V - D$ as the following two results.

Proposition 2.1 *Let G be a graph having pitchfork domination and $e \in \overline{G}$ such that $G+e$ has a pitchfork domination. If $e = u\,v$ such that $u, v \in V - D$, then $E_+^- \neq \phi$.*

Proof: Let G be a graph has a support vertex $v \in D$ adjacent with two leaves and with a vertex $t \in D$ where every vertex in D that adjacent with t except v dominates only one vertex. If t dominates an isolated vertex in $V - D$ say w that is dominated by other vertex in D which has a property (every vertex that is dominated by it has another neighbours in D). Now, let $e = z\,u$, then $D = \{z, w\}$ is a minimum pitchfork dominating set of $T + e$. Therefore, $e \in E_+^-$ and $E_+^- \neq \phi$. See Fig. 9 and Fig. 10.

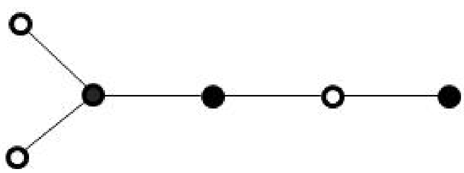

Figure 9: G

□

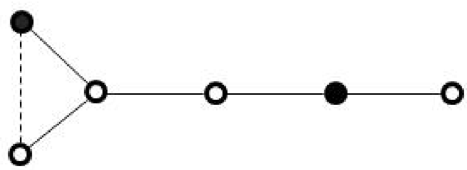

Figure 10: $G + e$

Theorem 2.4 *Let G be a graph having pitchfork domination with a support vertex adjacent with n end vertices for $n \geq 3$ and $e \in \overline{G}$ such that $G + e$ has a pitchfork domination. If $e = u\,v$ and u, $v \in D$, then $e \in E^{-}_{+}$.*

Proof: Since the minimum pitchfork dominating set D of G contains all these n end vertices. Let e incident on any two end vertices of D, then we get a cycle C_3 which is dominated by only one vertex. Thus, the order of D in $G + e$ is less by one than it in G. Therefore, $e \in E^{-}_{+}$. See Fig. 11 and Fig. 12. □

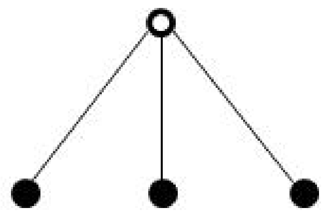

Figure 11: G

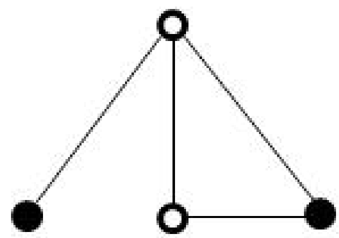

Figure 12: $G + e$

Proposition 2.2 *For any graph G having pitchfork domination and $e \in \overline{G}$ such that $e = v\,w$ where $v \in D$ and $w \in V - D$, then v dominates only one vertex if and only if $e \in E^{0}_{+}$.*

Proof: Since every vertex in D can be dominates at most two vertices, and v dominates only one vertex in G. Then, in $G + e$ the vertex v dominates exactly two vertices. Therefore, D is a minimum pitchfork dominating set in $G + e$. Thus, $\gamma_{pf}(G + e) = \gamma_{pf}(G)$ and $e \in E^{0}_{+}$.
To prove the converse, suppose that v dominates two vertices in G, then it will dominate three vertices in $G + e$, which is contradict pitchfork domination definition. □

Theorem 2.5 *For any graph G having pitchfork domination and $e \in \overline{G}$ such that $e = v\,w$ where $v \in D$ and $w \in V - D$. Let v dominates two vertices u, z. If $G + e$ has pitchfork domination, then $e \in E^{0}_{+}$ if one of the following conditions hold:*

1. If w is adjacent to vertex y in D such that all vertices which are dominated by y are dominated by $D-\{y\}\cup\{w\}$.
2. If $t = u$, z or w where t has degree at most one in $V - D$ and $t \notin pn[r, D]$ for any $r \in D$. Also, v is an isolated vertex in D or is adjacent to a vertex that dominates only one vertex.

Proof: 1. By replacing w instead of its dominating vertex in D. For example see Fig. 13 and Fig. 14.
2. Without loss of the generality, all vertices of the graph are dominated by $D-\{v\}\cup\{t\}$ which is a $\gamma_{pf}-$set of $G+e$. For example see Fig. 15 and Fig. 16. □

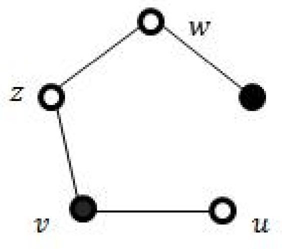

Figure 13: G

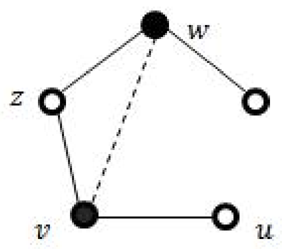

Figure 14: G+e

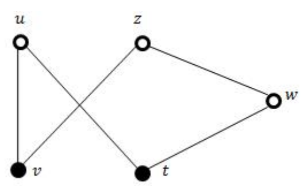

Figure 15: G

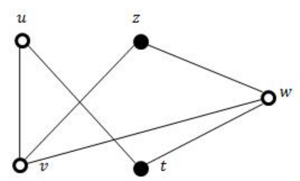

Figure 16: G+e

Theorem 2.6 *For any graph G having pitchfork domination and $e \in \overline{G}$ such that $e = v\,w$ where $w \in V - D$ and $v \in D$ that dominates two non-adjacent vertices u, z in $V - D$. If $G + e$ has pitchfork domination, then $e \in E_+^+$, if u, $z \in pn[v, D]$.*

Proof: Suppose that v is an isolated vertex in $G[D]$ or adjacent with a vertex in D that dominates only one vertex. If $deg(u) \leq 1$ and $deg(z) \leq 1$ in $G[V - D]$, then v must be get out from D. Thus, $D = D - \{v\} \cup \{u,\, z\}$ is the minimum pitchfork dominating set in $G + e$. For example see Fig. 17 and Fig. 18. But, if $deg(u) = 2$ in $V - D$, then $D = D \cup \{u\}$ where v stay in D to avoid u dominates three vertices. For example see Fig. 19 and Fig. 20. Hence, $\gamma_{pf}(G + e) > \gamma_{pf}(G)$. □

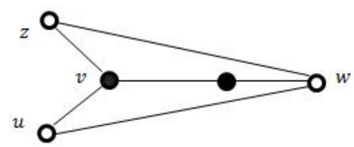

Figure 17: G

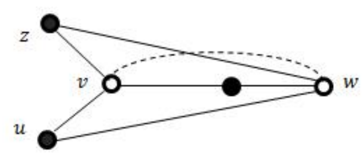

Figure 18: $G + e$

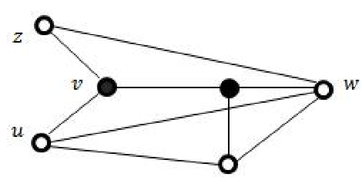

Figure 19: G

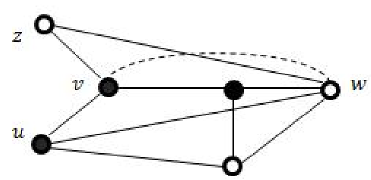

Figure 20: $G + e$

Theorem 2.7 *For any graph G having pitchfork domination and $e \in \overline{G}$ such that $e = v\,w$ where $w \in V - D$ and $v \in D$. If $G + e$ has pitchfork domination, then $e \in E_+^-$, if v dominates two vertices u, z such that u, $z \notin pn[v, D]$. And w is a pendent vertex dominated by a support vertex $t \in D$. Where $deg(t) \leq 1$ and $deg(v) = 0$ in $G[D]$.*

Proof: Since every vertex in $V - D$ that is dominated by v is dominated by other vertex in D, then we can delete v from D to be dominated by w which is putting in D instead of its dominating vertex t.

Then, $D - \{v, t\} \cup \{w\}$ is a minimum pitchfork dominating set of $G + e$. For example see Fig. 21 and Fig. 22. □

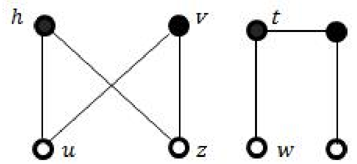

Figure 21: G

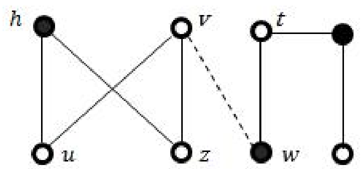

Figure 22: $G + e$

Theorem 2.8 *For any graph G having pitchfork domination and $e \in G$ such that $e = vw$ where $w \in V - D$ and $v \in D$. If $G - e$ has pitchfork domination, then $e \in E^0_-$ if $w \notin pn[v, D]$ and $deg(v) = 2$ in $G[D]$.*

Proof: Since w is dominated by v and another vertex in D and since v dominates two vertices in $V - D$. Then, deleting $e = vw$ has no changing on the minimum pitchfork dominating set D, so $\gamma_{pf}(G - e) = \gamma_{pf}(G)$. For example see Fig. 23 and Fig. 24. □

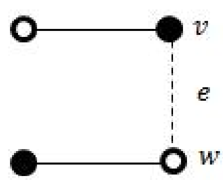

Figure 23: G

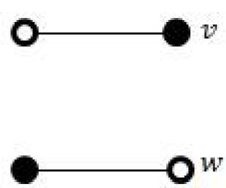

Figure 24: G-e

Theorem 2.9 *For any graph G having pitchfork domination and $e \in G$ such that $e = v\,u$ where $u \in V - D$ and $v \in D$. If $G - e$ has pitchfork domination, then $e \in E^+_-$, if v dominates two non-adjacent vertices u and w, where u, $w \in pn[v, D]$ such that $1 \leq |N(u) \cap (V - D)| \leq 2$.*

Proof: Since u is not dominated by any vertex in $G - e$ and it has one or two neighbors in $V - D$. Then, $D = D \cup \{u\}$ is the minimum pitchfork dominating set in $G - e$. Thus, $\gamma_{pf}(G - e) > \gamma_{pf}(G)$ and $e \in E^+_-$. See for example the path P_n for $n \equiv 0\,(mod\,3)$; $n \neq 3$ where $v = v_2$, $w = v_1$, $u = v_3$ and $e = v_2\,v_3$. □

Note 1 *For any graph G having pitchfork domination and $e \in G$ such that $e = v\,u$ where $u \in V - D$ and $v \in D$. Then, $\gamma_{pf}(G - e) < \gamma_{pf}(G)$ if $D - \{w\}$ is a minimum pitchfork dominating set where w is a vertex in D adjacent with v. For example see Fig. 25 and Fig. 26.*

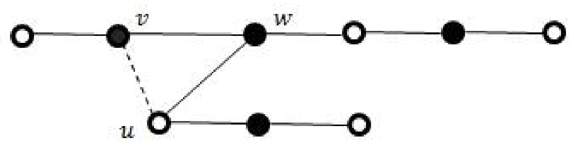

Figure 25: G

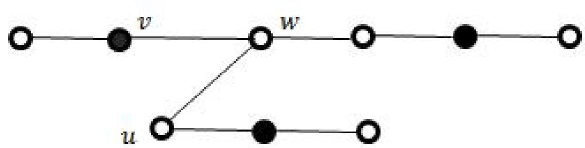

Figure 26: $G - e$

3. Conclusion

This study discussed the stability of the pitchfork domination number in three cases, when remove vertex, remove edge and add edge. Some cases increase the domination number while other cases increase it. The study of these effects has an important advantages to learn the ways of treatments to any added or damaged of any nods (vertices) or links (edges) of the system or networks to avoid losing some properties of the system and to give the best services with minimum costs.

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