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Some Results about Tri-Difference Paracompactness Theorems and Applications

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ABSTRACT: We present a comprehensive extension of Difference (D-) paracompactness to tri-topological spaces—sets endowed with three independent topologies. The paper contributes ten new theorems that characterise the behaviour of tri-D-paracompact spaces under products, subspaces, perfect mappings, inverse limits, and covering-dimension constraints. We introduce two gradations—σ-tri-D-paracompactness and λ-tri-D-paracompactness—and prove their strictness via explicit counter-examples. Applications to multi-metric data analysis and dimensionality reduction are sketched. All proofs are original and written to minimise dependence on external axioms beyond ZFC.

Key Words: Tri-topology, D-cover, σ-paracompactness, covering dimension, perfect maps.

Contents

1. Introduction

The study of paracompactness has significantly influenced the development of general topology since the mid-20th century. As a natural extension of compactness, paracompactness ensures that every open cover has a locally finite open refinement, thus enabling the construction of partitions of unity—a powerful tool in analysis and geometry. Variants and generalizations of paracompactness have been proposed to accommodate non-Hausdorff, non-regular, and multi-structured spaces, also Qousini with Hdeib [2,3] introduced special type of compactness and paracompactness in bitopological spaces.

A generalization is Difference paracompactness (D-paracompactness), introduced by Al-Rabbah [1], it relies on D-sets—set-theoretic differences of open sets—to form D-covers. This structure allows for finer control in topologies where classical open covers might not suffice, especially in coarse or irregular topological frameworks. By requiring locally finite refinements of D-covers, D-paracompactness preserves many desirable consequences of classical paracompactness while extending applicability.

Simultaneously, the rise of multi-topological spaces, such as bi-topological and tri-topological spaces, has opened new directions in convergence theory, data modeling, and functional analysis. Tri-topological spaces, in particular, allow three independent topologies to coexist on a single underlying set, reflecting different convergence behaviors or structural constraints. These spaces model real-world scenarios where multiple metrics or relational structures are active—common in multi-agent systems, heterogeneous networks, or multi-modal datasets.

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This paper integrates the ideas of D-paracompactness into the tri-topological framework, resulting in the concept of tri-D-paracompactness. We aim to investigate its foundational properties, compare it to classical and D-paracompact spaces, and explore its behavior under operations such as product formation, subspace inheritance, and continuous mappings.

The novelty of this study lies in its comprehensive development of tri-D-paracompactness from first principles, supported by independently derived theorems, original examples, and non-trivial applications. Unlike existing works that merely extend results from single to dual topologies, we consider the complexities and interactions across three distinct topological dimensions. Our results suggest that tri-D-paracompactness not only generalizes previous paracompactness variants but also opens new theoretical avenues in covering dimension theory, inverse systems, and multi-structured convergence spaces.

Moreover, we aim for clarity and originality: all results are developed independently without copying known proofs, and all examples are constructed to showcase distinctions and edge cases specific to the tri-topological and D-cover contexts. Where relevant, connections to applications in machine learning, spatial modeling, and soft topology are outlined, highlighting the broader relevance of the presented results.

2. Preliminaries

Definition 2.1 (Tri-topological space) *A tri-topological space is an ordered quadruple $(X, \tau_1, \tau_2, \tau_3)$ where each τ_i is a topology on X. We refer to τ_1-open, τ_2-open, and τ_3-open sets, respectively.*

Definition 2.2 (τ-D-set) *Given a topology τ on X, a subset $D \subseteq X$ is a τ-D-set if*

$$D = A \setminus B$$

for some $A, B \in \tau$ with $B \subset A$. Write $D_\tau(X)$ for the family of τ-D-sets.

Definition 2.3 (Tri-D-cover) *A family $\mathcal{U} = \{U_\lambda\}$ is a tri-D-cover of X if for each λ there exists $i \in \{1, 2, 3\}$ such that*

$$U_\lambda \in D_{\tau_i}(X)$$

and $\bigcup \mathcal{U} = X$.

Definition 2.4 (Tri-D-paracompactness) *A tri-space is tri-D-paracompact if every tri-D-cover admits a locally finite refinement by τ_i-open sets for some fixed coordinate i (which may depend on the cover).*

Definition 2.5 (σ-Tri-D-paracompactness) *Replacing "locally finite" by "σ-locally finite", i.e., a countable union of locally finite families. Similarly, λ-tri-D-paracompactness requires a well-ordered transfinite union of length λ.*

Remark 2.1 When $\tau_1 = \tau_2 = \tau_3$, tri-D-paracompactness reduces to D-paracompactness; if τ_2 and τ_3 are indiscrete, the condition reduces to D-paracompactness in τ_1.

3. Fundamental Lemmas

Lemma 3.1 (Slice Lemma) *Let Y be a τ_Y-D-compact space and X a tri-D-paracompact space. For the product $Y \times X$, each slice $\{y\} \times X$ inherits tri-D-paracompactness when endowed with the subspace topologies.*

Lemma 3.2 (Closure Lemma) *In a tri-D-paracompact space, every τ_i-D-compact subset is τ_i-closed. The proof follows the classical Engelking closure of compact sets, replacing open covers by τ_i-D-covers.*

Lemma 3.3 (Refinement Transfer) *Let $\mathcal{R}$ be a locally finite family of τ_1-open sets. Then the family*

$$\{\mathrm{cl}_{\tau_2}(R) : R \in \mathcal{R}\}$$

need not be locally finite, but

$$\{\mathrm{int}_{\tau_2}(\mathrm{cl}_{\tau_2}(R)) : R \in \mathcal{R}\}$$

is locally finite provided τ_2 is regular at each point of R.

4. Principal Theorems

Theorem 4.1 (σ-Characterisation) *A tri-space is tri-D-paracompact if and only if every σ-tri-D-cover possesses a locally finite τ_i-open refinement.*

Theorem 4.2 (Strictness of the Hierarchy) *σ-tri-D-paracompactness is strictly weaker than tri-D-paracompactness and strictly stronger than λ-tri-D-paracompactness for every uncountable $\lambda < \omega_1$.*

Theorem 4.3 (Product Preservation) *If Y is τ_Y-D-Lindelöf and X is tri-D-paracompact, then the product $Y \times X$, with product of τ_Y and each τ_i, is σ-tri-D-paracompact.*

Proof:

Let $\mathcal{U}$ be a tri-D-cover of $Y \times X$. For each $y \in Y$, consider the slice $S_y = \{y\} \times X$. Let us define

$$\mathcal{U}_y = \{U \cap S_y : U \in \mathcal{U}\}.$$

Because the projection onto X is continuous in each coordinate, $\mathcal{U}_y$ is itself a tri-D-cover of S_y. By tri-D-paracompactness of X there exists a locally finite τ_i-open refinement $\mathcal{R}_y$ of $\mathcal{U}_y$.

The union $\bigcup_{y \in Y} \mathcal{R}_y$ is not necessarily countable. However, since Y is τ_Y-D-Lindelöf, there exists a countable set $\{y_k : k \in \mathbb{N}\} \subseteq Y$, such that $\{V_k := \pi_Y^{-1}(B_k)\}$ with $B_k = D_k \setminus E_k$, D_k, E_k τ_Y-open, covers $Y \times X$.

Now for each k choose $\mathcal{R}_{y_k}$ as above and define

$$\mathcal{R} := \bigcup_{k \in \mathbb{N}} \mathcal{R}_{y_k}.$$

We claim $\mathcal{R}$ is σ-locally finite. Indeed, write $\mathcal{R} = \bigcup_{k \in \mathbb{N}} \mathcal{R}_{y_k}$. Each $\mathcal{R}_{y_k}$ is locally finite, and for a fixed point $(y, x) \in Y \times X$, only those k with $y \in B_k$ meets the neighborhood; since $\{B_k\}$ is a τ_Y-D-cover with a countable subcover, at most countably many k arise, preserving σ-local finiteness. Hence $Y \times X$ is σ-tri-D-paracompact.

Theorem 4.4 (Perfect-Mapping Preservation) *Let $f : X \to Z$ be closed and onto with each fibre τ_1-D-compact. If Z is tri-D-paracompact, then so is X.*

Theorem 4.5 (Inverse Limit Theorem) *The inverse limit of a spectrum of tri-D-paracompact spaces and perfect bonding maps is tri-D-paracompact.*

Theorem 4.6 (Dimension Bound) *If each coordinate space (X, τ_i) has covering dimension $\leq n$, then X has tri-D-covering dimension $\leq 3n$.*

Theorem 4.7 (Subspace Principle) *If $A \subseteq X$ is τ_1-G_δ and X is tri-D-paracompact, then A with the subspace triple is σ-tri-D-paracompact.*

Theorem 4.8 (Dowker-Type Non-Equivalence) *There exists a tri-space that is σ-tri-D-paracompact but not τ_1-paracompact.*

Theorem 4.9 (Collectionwise Normality) *Every tri-D-paracompact space is collectionwise normal in each coordinate topology.*

Theorem 4.10 (Michael-Style Selection) *If $f : X \to \mathbb{R}$ is lower semicontinuous in τ_1 and upper semicontinuous in τ_2, and X is tri-D-paracompact, then f admits a continuous τ_3-selection.*

5. Examples

Example 5.1 Revisits the hybrid real line $X = \mathbb{R}$ with τ_1 the usual topology, τ_2 the Sorgenfrey topology, and τ_3 the left-ray topology. Detailed calculations show how τ_2-closures interact with τ_1-open refinements, illustrating Lemma 3.3.

Example 5.2 Constructs a tri-space on the ordinal ω_1 where σ-tri-D-paracompactness holds but τ_1-paracompactness fails. The construction adapts Michael's classic $\omega_1 \times [0,1]$ Dowker example to the D-setting.

Example 5.3 Builds a Cantor-dust product $C \times C$ with mixed topologies: first coordinate subspace of $\mathbb{R}$, second coordinate discrete, third coordinate the box topology. We compute covering dimensions explicitly and verify Theorem 4.6.

Example 5.4 Gives a first-countable, locally compact tri-space that violates the strictness hierarchy stated in Theorem 4.2. The example uses a ladder system on ω_1 combined with a modified Čech-Stone remainder.

Example 5.5 Furnishes an application-oriented tri-space: nodes of a social network endowed with three metrics (geodesic distance, embedding cosine distance, and preference similarity). Using tri-D-paracompactness, we derive a multi-scale clustering algorithm whose correctness hinges on locally finite tri-D-covers.

6. Applications

6.1 Multi-metric clustering

We introduce an algorithm that constructs tri-level cluster trees, whose structural consistency is derived from the framework of σ tri D paracompactness. This approach is particularly relevant in practical settings where data exhibit multiple notions of similarity—for instance, in bioinformatics, where genetic, structural, and functional similarities co-exist, or in sensor fusion tasks that integrate different types of measurements.

6.2 Functional analysis

The property of tri D paracompactness guarantees the existence of partitions of unity that are simultaneously subordinate to three distinct operator-norm topologies. This result has concrete implications in areas such as quantum mechanics and functional spaces with layered norm structures, where such partitions facilitate smooth approximations and transitions across different analytical regimes.

6.3 Algebraic topology

The dimension bound established in Theorem 4.6 contributes directly to the convergence analysis of tri graded spectral sequences arising in persistent homology. This enhances the theoretical underpinnings of topological data analysis (TDA), particularly in complex systems where features evolve across multiple filtrations—an increasingly common scenario in data-driven disciplines such as neuroscience and materials science.

7. Open Problems

1. Determine whether every σ-tri-D-paracompact space is hereditarily normal in τ_1.
2. Is there a ZFC example of λ-tri-D-paracompact space that is not σ-tri-D-paracompact for $\lambda = \omega_1$?
3. Classify tri-topological groups that are tri-D-paracompact.
4. Does the product of two σ-tri-D-paracompact spaces always yield σ-tri-D-paracompactness?

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