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Analysis and Optimal Control of a Frictional Contact Problem with Normal Damped Response and a Semiconductor Piezoelectric Material

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ABSTRACT: This article examines a quasistatic contact problem with friction involving a piezoelectric material and a semi-conductive foundation. The material's behavior is captured by an electroviscoelastic constitutive model. Contact interactions are modeled using a classical normal damped response condition along with a frictional law. We examine the variational framework associated with our model and demonstrate the uniqueness of the solution in the weak sense. Furthermore, we establish the existence of optimal solutions for two categories of optimal control and inverse problems, specifically within the context of the piezo-viscoelastic contact issue being analyzed.

Key Words: Piezoelectric material, viscoelastic material, frictional contact, normal damped response, optimal control.

Contents

1. Introduction

Piezoelectric materials generate electricity when subjected to mechanical stress and, conversely, produce mechanical strain when exposed to an electric field. This dual capability makes them essential for advanced applications, including actuators, sensors, control systems in engineering, and smart materials and structures. The coupling effects between mechanical and electric fields in piezoelectrics are key to their importance in advanced technological development. The foundational mathematical frameworks for representing linear piezoelectric materials were established in [12]. The theory on piezoelectric materials has been expanded upon in reference [13].

Different studies have examined quasistatic contact problems with viscoelastic bodies, with some including friction (see [14,15,16]) and others excluding it (see [17,18]). Over the past decade, numerous contact problems involving the piezoelectric effect have been investigated from both variational and numerical perspectives (see, for example, [2,5,6,7], and more recently [8,9,10,11]). In [8], the authors investigate both variational and numerical methods for modeling quasistatic contact interactions between a viscoelectroelastic material and an electrically conductive substrate, addressing issues such as damped normal interaction and localized friction mechanisms.

References [30,31,32,34] explore optimal control methods for different contact issues involving elastic materials. The application of optimal control in mixed variational, hemivariational, and quasi-hemivariational inequalities has been widely demonstrated in modeling problems in contact mechanics, physics, and engineering, as highlighted in [20,21,23], and more recently in [23,24,25,33]. Notable advancements in the application of optimal control and inverse problems within the framework of variational inequalities are documented in [26,27,28]. Recently, Bouallala in [29] focused on an optimal control

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problem concerning frictional contact involving a piezo-viscoelastic material and a conductive foundation.

This article has two primary objectives. First, it presents a novel contact model for piezoelectric materials. The innovations in this model pertain both to the constitutive law used to characterize the material's behavior and to the electrical conductivity condition applied to the contact surface. The material is considered to be viscoelastic and piezoelectric, with dynamic contact modeled by a standard damped normal response condition, where the foundation's reaction is speed-dependent [19]. This scenario introduces complexities, such as nonlinearity due to contact and friction conditions, along with the foundation's conductivity.
Materials with piezo-viscoelastic properties and partial electrical conductivity exhibit a combination of piezoelectric and viscoelastic characteristics, with the electrical conductivity affecting their response to both electric and mechanical fields.
These characteristics give rise to a novel mathematical model. The model mathematically translates into a system that includes a nonlinear parabolic variational equation governing the displacement and a quasi-variational elliptic inequality for the electrical potential.

Our second objective is to demonstrate the application of optimal control and inverse problems, which allows us to understand how to adjust control parameters to optimize the response of materials under mechanical and electrical stress, and to infer material properties from experimental data.

The structure of the paper is organized as follows: **Problem Statement** defines the notation, outlines the mechanical issue, presents the assumptions, and derives the weak formulation. In **Existence and Uniqueness Result**, we prove the existence and uniqueness of a weak solution for the model using techniques from quasi-variational inequalities and nonlinear variational equalities. Finally, **Optimal Control for Mechanical Models** introduces an optimal control strategy for the contact model described by the variational framework and establishes its solvability.

2. Problem Statement

We consider a piezoelectric material situated in a domain $\Omega \subset \mathbb{R}^2$. The material is bounded by a smooth surface $\partial\Omega = \Gamma$, which is associated with a unit outward normal vector denoted as ν. We analyze the material over a time interval $[0, T]$, where T is a positive constant. Within the domain $\Omega \times (0, T)$, the material is subjected to volume forces described by the density vector ϕ_m and has volume electric charge densities given by ϕ_e.

Additionally, mechanical constraints are imposed on it through its boundary. To achieve this objective, we posit that the boundary Γ can be partitioned into separate, non-overlapping segments denoted as Γ_D, Γ_N, and Γ_C, with the stipulation that Γ_D possesses a non-zero measure. The object is immobilized across Γ_D. A partition of $\Gamma_D \cup \Gamma_N$ into two open parts Γ_a and Γ_b. We assume that the boundaries Γ_D and Γ_a have strictly positive measures.

We define the following regions: $\Xi = \Omega \times (0, T)$, $\Xi_D = \Gamma_D \times (0, T)$, $\Xi_N = \Gamma_N \times (0, T)$, $\Xi_C = \Gamma_C \times (0, T)$, $\Xi_a = \Gamma_a \times (0, T)$, and $\Xi_b = \Gamma_b \times (0, T)$.
Moreover, surface tractions characterized by the density vector ϕ_N are applied to Ξ_N. We also assume that the electrical potential is zero on Ξ_a and a surface electrical charge with density ϕ_b is specified on Ξ_b. Within the region Ξ_C, the object may experience frictional contact with a deformable foundation. Note that the indices i, j, k, and l are assumed to range from 1 to d, unless otherwise stated.

We represent the space of second-order symmetric tensors on $\mathbb{R}^2$ by $\mathbb{S}^2$. In this context, the symbols "$\cdot$" and "$\|\cdot\|$" signify the inner product and Euclidean norm on $\mathbb{R}^2$ and $\mathbb{S}^2$, respectively.

We denote $\vartheta := \Xi \to \mathbb{R}^2$ as the displacement field, $\Theta := \Xi \to \mathbb{S}^2$ the stress tensor, $\kappa := \Xi \to \mathbb{R}$ the electric potential and $\Psi := \Xi \to \mathbb{R}^2$ the displacement electric field.

Furthermore, we represent the linearized strain tensor by $\varepsilon(\vartheta) : \Xi \to \mathbb{S}^2$, where it is defined as:

$$\varepsilon(\vartheta) = (\varepsilon_{ij}(\vartheta) = \frac{1}{2}(\vartheta_{i,j} + \vartheta_{j,i}),\ i, j = 1, ..., d \text{ where } \vartheta_{i,j} = \partial\vartheta_i/\partial x_j.$$

We define the notations ϑ_ν and ϑ_τ to denote the normal and tangential displacements, where $\vartheta_\nu = \vartheta \cdot \nu$ and $\vartheta_\tau = \vartheta - \vartheta_\nu \nu$.
Moreover, we employ Θ_τ and Θ_ν to represent the tangential and the normal stress tensors. These are defined as $\Theta_\nu = (\Theta \cdot \nu) \cdot \nu = \Theta_{ij}\nu_i\nu_j$ and $\Theta_\tau = \Theta \cdot \nu - \Theta_\nu \nu$, respectively.
We express "Div" and "div" as the divergence operators for tensor and vector functions, respectively, i.e., $Div(\Theta) = (\Theta_{ij,j})$ and $div(\Psi) = \Psi_{i,i}$.

The conventional expression of the contact problem can be presented as follows:
Problem (P): Determine a displacement field $\vartheta : \Xi \longrightarrow \mathbb{R}^2$ and an electric potential $\kappa : \Xi \longrightarrow \mathbb{R}$ such that for a.e. $t \in (0, T)$

$$\Theta(t) = \mathcal{E}\varepsilon(\vartheta(t)) + \mathcal{V}\varepsilon\left(\dot{\vartheta}(t)\right) - \mathcal{A}^\top E(\kappa(t)) \quad \text{in } \Xi, \tag{2.1}$$

$$\Psi(t) = \mathcal{A}\varepsilon(\vartheta(t)) + \mathcal{B}E(\kappa(t)) \quad \text{in } \Xi, \tag{2.2}$$

$$Div(\Theta(t)) = -\phi_m(t) \quad \text{in } \Xi, \tag{2.3}$$

$$div(\Psi(t)) = \phi_e(t) \quad \text{in } \Xi, \tag{2.4}$$

$$\vartheta(t) = 0 \quad \text{on } \Xi_D, \tag{2.5}$$

$$\Theta(t)\nu = \phi_N(t) \quad \text{on } \Xi_N, \tag{2.6}$$

$$\kappa(t) = 0 \quad \text{on } \Xi_a, \tag{2.7}$$

$$\Psi(t) \cdot \nu = \phi_b \quad \text{on } \Xi_b, \tag{2.8}$$

$$\Theta_\nu(t) = -\phi_\nu\left(\dot{\vartheta}_\nu(t)\right) \quad \text{on } \Xi_C, \tag{2.9}$$

$$\Theta_\tau(t) = -\phi_\tau\left(\dot{\vartheta}_\tau(t)\right) \quad \text{on } \Xi_C, \tag{2.10}$$

$$|\Psi(t) \cdot \nu| \leq k,\ \ \Psi(t) \cdot \nu = k\frac{\kappa(t)}{|\kappa(t)|} \ \text{ if } \ \kappa(t) \neq 0 \quad \text{on } \Xi_C, \tag{2.11}$$

$$\vartheta(0) = \vartheta_0 \quad \text{in } \Xi. \tag{2.12}$$

Equations (2.1) and (2.2) describe the electro-viscoelastic constitutive law of the material. Here, $\mathcal{E} = (e_{ijkl})$ represents the elasticity tensor, $\mathcal{V} = (v_{ijkl})$ denotes the viscosity operator, $\mathcal{A} = (a_{ijk})$ is the third-order piezoelectric tensor, $\mathcal{A}^\top$ is its transpose, $\mathcal{B} = (b_{ij})$ signifies the electric permittivity tensor, and $E(\kappa) = -\nabla\kappa = -(\kappa_{,i})$ is the electric field. It is important to note that $\mathcal{A}$ and $\mathcal{A}^\top$ satisfy the following relationship:

$$\mathcal{A}\alpha \cdot \beta = \alpha : \mathcal{A}^\top\beta, \ \text{ for all } \ \alpha \in \mathbb{S}^2,\ \beta \in \mathbb{R}^2.$$

Relations (2.3) and (2.4) correspond to the equations of motion and the balance equation for the electric displacement field. The displacement and traction boundary conditions are given by equations (2.5) and (2.6), whereas equations (2.7) and (2.8) define the electric boundary conditions.
Equation (2.9) represents a normal damped response contact condition (see [1,2]), indicating that the foundation exhibits reactive behavior. The normal damped response function ϕ_ν is defined for $r \leq 0$, where no contact occurs. As an example, consider the function (see [2]),

$$\phi_\nu(r) = c_c r^+, \ \ \forall r \in \mathbb{R}^2, \tag{2.13}$$

where $c_c > 0$ is a positive constant representing the deformability coefficient and $r^+ = \max\{r, 0\}$. Additionally, the law specified in (2.10) involves ϕ_τ, a constitutive function characterized by properties such that

$$\phi_\tau(r) = c_f r, \quad \forall r \in \mathbb{R}^2. \tag{2.14}$$

Here, c_f is the friction coefficient, and the tangential shear force is proportional to the tangential velocity.

Furthermore, condition (2.11) specifies the foundation's conductivity, as detailed in [3]. Equation (2.12) presents the initial condition, with ϑ_0 denoting the initial displacement.

To derive the variational formulation for **Problem (P)**, we utilize the following function spaces:

$$\begin{aligned} X &= \left\{ \vartheta \in H^1(\Omega)^2 \text{ such that } \vartheta = 0 \text{ on } \Gamma_D \right\}, \\ Y &= \left\{ \kappa \in H^1(\Omega) \text{ such that } \kappa = 0 \text{ on } \Gamma_a \right\}, \\ \mathcal{Q} &= \left\{ \Theta \in \mathbb{S}^2 \text{ with } \Theta_{ij} = \Theta_{ji} \in L^2(\Omega) \right\}. \end{aligned} \tag{2.15}$$

These are actual Hilbert spaces, each equipped with an inner product defined as follows:

$$\begin{aligned} (\vartheta, \alpha)_X &= \int_\Omega \varepsilon(\vartheta) \cdot \varepsilon(\alpha)\, dx, \text{ for all } \vartheta, \alpha \in X, \\ (\kappa, \beta)_Y &= \int_\Omega \nabla\kappa \cdot \nabla\beta\, dx, \text{ for all } \kappa, \beta \in Y, \\ (\Theta, \tau)_{\mathcal{Q}} &= \int_\Omega \Theta_{ij}\tau_{ij}\, dx, \text{ for all } \Theta, \tau \in \mathcal{Q}, \end{aligned} \tag{2.16}$$

along with the associated norms $\|\cdot\|_X$, $\|\cdot\|_Y$, and $\|\cdot\|_{\mathcal{Q}}$.
For the spaces X and Y, we define the inner products and corresponding norms as follows:

$$\begin{aligned} &(\vartheta, \alpha)_X = (\varepsilon(\vartheta), \varepsilon(\alpha))_{\mathcal{Q}}, \quad \|\vartheta\|_X = \|\varepsilon(\vartheta)\|_{\mathcal{Q}}, \text{ for all } \vartheta, \alpha \in X, \\ &(\kappa, \beta)_Y = (\nabla\kappa, \nabla\beta)_{L^2(\Omega)}, \quad \|\kappa\|_Y = \|\nabla\kappa\|_{L^2(\Omega)}, \text{ for all } \kappa, \beta \in Y. \end{aligned} \tag{2.17}$$

Let X and Y be two Banach spaces, with X^* representing the dual space of X. We have $X \subset L^2(\Omega)^2 = \left(L^2(\Omega)^2\right)^* \subset X^*$.
Moreover, given that Y is dense in $L^2(\Omega)$, this inclusion can be written as $Y \subset L^2(\Omega) \subset Y^*$.

Since Γ_D has positive measure, Korn's inequality is applicable:

$$m_K\|\vartheta\|_{H^1(\Omega)^2} \leq \|\varepsilon(\vartheta)\|_{\mathcal{Q}}, \text{ for all } \vartheta \in X, \tag{2.18}$$

where $m_K > 0$ is a constant that depends only on Γ and Γ_D.
Additionally, the following Friedrichs-Poincaré inequality holds:

$$\|\nabla\kappa\|_{L^2(\Omega)} \geq m_F\|\kappa\|_Y, \text{ for all } \kappa \in Y. \tag{2.19}$$

Moreover, the Sobolev trace theorem ensures the existence of constants m_D and m_E, which depend on the domains Ω, Γ_a, and Γ_C, such that

$$\|\vartheta\|_{L^2(\Gamma_C)^2} \leq m_D\|\vartheta\|_X, \text{ for all } \vartheta \in X, \tag{2.20}$$

$$\|\kappa\|_{L^2(\Gamma_C)} \leq m_E\|\kappa\|_Y, \text{ for all } \kappa \in Y. \tag{2.21}$$

Assuming that the vector field Θ and Ψ consist of sufficiently well-behaved functions, the following Green's formulae are valid:

$$\int_\Omega \Theta \cdot \varepsilon(\vartheta)\,dx + \int_\Omega \operatorname{div}(\Theta) \cdot \vartheta\,dx = \int_\Gamma \Theta \cdot \nu\,\vartheta\,da, \tag{2.22}$$

$$\int_\Omega \Psi \cdot \nabla\kappa\,dx + \int_\Omega \operatorname{div}(\Psi) \cdot \kappa\,dx = \int_\Gamma \Psi \cdot \nu\,\kappa\,da. \tag{2.23}$$

Now, we introduce the following operators:

$$\begin{aligned}
&e : X \times X \to \mathbb{R},\ \ e(\vartheta,\alpha) := \left(\mathcal{E}\varepsilon(\vartheta), \varepsilon(\alpha)\right)_{\mathcal{Q}},\\
&v : X \times X \to \mathbb{R},\ \ v(\vartheta,\alpha) := \left(\mathcal{V}\varepsilon(\vartheta), \varepsilon(\alpha)\right)_{\mathcal{Q}},\\
&b : Y \times Y \to \mathbb{R},\ \ b(\kappa,\beta) := \left(\mathcal{B}\nabla\kappa, \nabla\beta\right)_{L^2(\Omega)},\\
&a : X \times Y \to \mathbb{R},\ \ a(\vartheta,\beta) := \left(\mathcal{A}\varepsilon(\vartheta), \nabla\beta\right)_{L^2(\Omega)} = \left(\mathcal{A}^\top\nabla\beta, \varepsilon(\vartheta)\right)_{L^2(\Omega)^2}.
\end{aligned} \tag{2.24}$$

According to the Riesz representation theorem, there exist functions $\phi : [0,T] \to X^*$ and $q : [0,T] \to Y^*$ such that

$$(\phi(t),\alpha)_{X^*\times X} := \int_\Omega \phi_m(t) \cdot \alpha\ dx + \int_{\Gamma_N} \phi_N(t) \cdot \alpha\ da, \text{ for all } \alpha \in X, \tag{2.25}$$

$$(q(t),\beta)_{Y^*\times Y} := \int_\Omega \phi_e(t)\beta\ dx + \int_{\Gamma_b} \phi_b(t)\beta\ da, \text{ for all } \beta \in Y. \tag{2.26}$$

Also, we consider the mappings $j_m : X \times X \to \mathbb{R}$ and $j_e : Y \to \mathbb{R}$ such that

$$j_m(\vartheta(t),\alpha) := \int_{\Gamma_C} \phi_\nu(\vartheta_\nu(t))\alpha_\nu da + \int_{\Gamma_C} \phi_\tau(\vartheta_\tau(t)) \cdot \alpha_\tau\ da, \text{ for all } \alpha \in X, \tag{2.27}$$

$$j_e(\kappa(t)) := \int_{\Gamma_C} k\,|\kappa(t)|\ da, \text{ for all } \kappa \in Y. \tag{2.28}$$

Let's now enumerate the assumptions concerning the data for the given problem.

H(1): The following standard symmetry condition hold

$$\begin{aligned}
&e_{ijkl} = e_{jikl} = e_{lkij} \in L^\infty(\Omega), \quad v_{ijkl} = v_{jikl} = v_{lkij} \in L^\infty(\Omega),\\
&a_{ijk} = a_{kij} \in L^\infty(\Omega), \quad b_{ij} = b_{ji} \in L^\infty(\Omega).
\end{aligned}$$

This imply that there exist a positive constants M_e, M_v, M_b and M_a such that

$$\begin{aligned}
&|e(\vartheta,\alpha)| \le M_e\|\vartheta\|_X\|\alpha\|_X,\ \ |v(\vartheta,\alpha)| \le M_v\|\vartheta\|_X\|\alpha\|_X,\\
&|b(\kappa,\beta)| \le M_b\|\kappa\|_Y\|\beta\|_Y,\ \ |a(\vartheta,\beta)| \le M_a\|\vartheta\|_X\|\beta\|_Y.
\end{aligned}$$

H(2): The operator $\mathcal{E}$, $\mathcal{V}$ and $\mathcal{A}$ are strictly positive, i.e. there exist m_e, m_v and m_a such that

$$\begin{aligned}
&e(\vartheta,\vartheta) \ge m_e\|\vartheta\|_X^2,\\
&v(\vartheta,\vartheta) \ge m_v\|\vartheta\|_X^2,\\
&b(\kappa,\kappa) \ge m_b\|\kappa\|_Y^2.
\end{aligned}$$

H(3): The regularity condition are imposed on the forces, traction, volume and surface charge densities as follows:

$$\phi_m \in C(0,T,L^2(\Omega)^2), \quad \phi_N \in C(0,T,L^2(\Gamma_N)^2),$$

$$\phi_e \in W^{1,2}(0,T,L^2(\Omega)), \quad \phi_b \in W^{1,2}(0,T,L^2(\Gamma_b)).$$

H(4): The electric conductivity coefficient and the initial condition are such that

$$k : \Gamma_C \to \mathbb{R}^+,\ k(x) \geq 0,\ k \in L^\infty(\Gamma_C) \text{ a.e. } x \in \Gamma_C,\ \vartheta_0 \in X.$$

H(5): The function $\phi_s : \Gamma_C \times \mathbb{R} \to \mathbb{R}^+$, for $s = \nu$ or $s = \tau$, satisfies the following conditions:

i) The function $x \mapsto \phi_s(x,l)$ is measurable for every $l \in \mathbb{R}^2$, on Γ_C,

ii) $\phi_s(x,l) = 0$, for all $l \leq 0$ and almost everywhere on Γ_C,

iii) $|\phi_s(x,l)| \leq c_s$, where $c_s > 0$,

iv) There exists a constant $L_s > 0$ such that for all $l, w \in \mathbb{R}^+$, the inequality $|\phi_s(\cdot,l) - \phi_s(\cdot,w)| \leq L_s|l-w|$ holds,

v) For any $l_1, l_2 \in \mathbb{R}^2$, almost everywhere on Γ_C, it holds that

$$(p_s(x,l_1) - p_s(x,l_2)) \cdot (l_1 - l_2) \geq 0.$$

H(6): The operator j_e is continuous on Y

We now focus on the variational formulation of **Problem (P)**. For this purpose, we assume that ϑ, Θ, κ, and Ψ are regular functions that satisfy the conditions given in (2.1)–(2.12).
Let $\alpha \in X$ and $\beta \in Y$ be arbitrary. Using (2.22) and (2.23), we obtain

$$\begin{aligned}&\int_\Omega \mathcal{E}\varepsilon(\vartheta(t)) \cdot \varepsilon(\alpha)\, dx + \int_\Omega \mathcal{V}\varepsilon(\dot{\vartheta}(t)) \cdot \varepsilon(\alpha)\, dx + \int_\Omega \mathcal{A}\varepsilon(\alpha) \cdot \nabla\kappa(t)\, dx \\ &= \int_\Omega \phi_m(t) \cdot \alpha\, dx + \int_{\Gamma_N} \phi_N(t) \cdot \alpha\, da + \int_{\Gamma_C} \Theta_\tau(t) \cdot \alpha_\tau\, da + \int_{\Gamma_C} \Theta_\nu(t)\alpha_\nu\, da,\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}&\int_\Omega \mathcal{B}\nabla\kappa(t) \cdot \nabla\beta\, dx - \int_\Omega \mathcal{A}\varepsilon(\vartheta(t)) \cdot \nabla\beta dx = \int_\Omega \phi_e(t)\beta\, dx \\ &- \int_{\Gamma_b} \phi_b(t)\beta\, da - \int_{\Gamma_C} D(t) \cdot \nu\beta\, da.\end{aligned} \tag{2.30}$$

Taking account (2.24)-(2.25) and (2.27), we deduce that

$$e(\vartheta(t),\alpha) + v(\dot{\vartheta}(t),\alpha) + a(\kappa(t),\alpha) + j_m(\dot{\vartheta}(t),\alpha) = (\phi(t),\alpha)_{X^*\times X}. \tag{2.31}$$

Now, by the conductivity relation (2.11) combined by (2.24), (2.26) and (2.28), we find that

$$b(\kappa(t),\beta-\kappa(t)) - a(\vartheta(t),\beta-\kappa(t)) + j_e(\beta) - j_e(\kappa(t)) \geq (q(t),\beta-\kappa(t))_{Y^*\times Y}. \tag{2.32}$$

We then deduce the ensuing variational formulation of (2.1)-(2.10).
Problem (PV): Find a displacement field $\vartheta : [0,T] \to X$, an electric potential $\kappa : [0,T] \to Y$ for all $\alpha \in X$, $\beta \in Y$ and for a.e. $t \in [0,T]$

$$e(\vartheta(t),\alpha) + v(\dot{\vartheta}(t),\alpha) + a(\kappa(t),\alpha) + j_m(\dot{\vartheta}(t),\alpha) = (\phi(t),\alpha)_{X^*\times X}, \tag{2.33}$$

$$b(\kappa(t),\beta-\kappa(t)) - a(\vartheta(t),\beta-\kappa(t)) + j_e(\beta) - j_e(\kappa(t)) \geq (q(t),\beta-\kappa(t))_{Y^*\times Y}, \tag{2.34}$$

$$\vartheta(0) = \vartheta_0. \tag{2.35}$$

3. Existence and Uniqueness Result

In this section, we demonstrate the existence of solutions for the variational problem defined by equations (2.33) through (2.35).

Theorem 3.1 *Assuming the conditions (H1) through (H7) are satisfied, there exists at least one solution to the system described by (2.33) to (2.35) with the following regularity:*

$$\vartheta \in C^1(0,T;X), \ \kappa \in W^{1,2}(0,T;Y). \tag{3.1}$$

Theorem 3.1 will be proven in several phases, using compactness arguments, lower semi-continuity principles, monotone operator theory, and fixed-point theorems.
Throughout this text, the symbol c denotes a positive generic constant, with its specific value potentially varying from one occurrence to another.
First, let $z \in C(0,T;X)$ and $y \in W^{1,2}(0,T;Y)$ be given. We then consider the following auxiliary problem.
Problem (M): Find a displacement field $\vartheta_z : [0,T] \to X$ such that

$$v(w_z(t),\alpha) + j_m(w_z(t),\alpha) + (z(t),\alpha)_{X^*\times X} = (\phi(t),\alpha)_{X^*\times X}, \tag{3.2}$$

$$w_z(0) = w_0, \tag{3.3}$$

for all $\alpha \in X$ and $t \in [0,T]$.
We obtain the following result:

Lemma 3.1 *1) **Problem (M)** has a unique solution w_z, which satisfies the regularity condition $w_z \in C(0,T;X)$.*

*2) For two solutions ϑ_{z_1} and ϑ_{z_2} of **Problem (M)**, associated with z_1 and z_2 in $C(0,T;X)$, there exists a constant $c > 0$ such that*

$$\|\vartheta_{z_1}(t) - \vartheta_{z_2}(t)\|_X \le c\int_0^t \|z_1(s) - z_2(s)\|_X\, ds. \tag{3.4}$$

Proof:

1) By applying Riesz's representation theorem, we find the operator $E : X \to X$ and the element $\phi_z(t) \in Y$ defined as follows:

$$E(w_z(t),\alpha) := v(w_z(t),\alpha) + j_m(w_z(t),\alpha), \tag{3.5}$$

$$(\phi_z,\alpha)_{X^*\times X} := (\phi(t),\alpha)_{X^*\times X} - (z(t),\alpha)_{X^*\times X}. \tag{3.6}$$

Thus, equation (3.2) can be expressed as:

$$E(w_z(t),\alpha) = (\phi_z(t),\alpha)_{X^*\times X}. \tag{3.7}$$

For w_{z_1} and w_{z_2} in X, using (3.5), we get:

$$\begin{aligned} E(w_{z_1}(t) - w_{z_2}(t), w_{z_1}(t) - w_{z_2}(t)) &= v(w_{z_1}(t) - w_{z_2}(t), w_{z_1}(t) - w_{z_2}(t)) \\ &+ j_m(w_{z_1}(t), w_{z_1}(t) - w_{z_2}(t)) - j_m(w_{z_2}(t), w_{z_1}(t) - w_{z_2}(t)). \end{aligned} \tag{3.8}$$

By applying **H(2)** and **H(5)**, it follows that:

$$E(w_{z_1}(t) - w_{z_2}(t), w_{z_1}(t) - w_{z_2}(t)) \ge m_v \|w_{z_1}(t) - w_{z_2}(t)\|_X^2. \tag{3.9}$$

Thus, the operator E is strongly monotone on X. Furthermore, we have:

$$E(w_{z_1}(t) - w_{z_2}(t),\alpha) = v(w_{z_1}(t) - w_{z_2}(t),\alpha) + j_m(w_{z_1}(t) - w_{z_2}(t),\alpha), \tag{3.10}$$

and using **H(2)**, **H(5)**, and (2.20), we obtain:

$$|E(w_{z_1}(t) - w_{z_2}(t), \alpha)| \leq (M_v + m_D(L_\nu + L_\tau)) \|w_{z_1}(t) - w_{z_2}(t)\|_X \|\alpha\|_X. \tag{3.11}$$

Consequently, the operator E is both continuous and strongly monotone on X, ensuring the existence of a unique element $w_z \in C(0,T;X)$ that satisfies (3.7).
Thus, **Problem (M)** has a unique solution $\vartheta_z \in C^1(0,T;X)$.

2) Using (3.2), we obtain:

$$\begin{aligned} &v(w_{z_1}(t) - w_{z_2}(t), w_{z_1}(t) - w_{z_2}(t)) + (z_1 - z_2, w_{z_1}(t) - w_{z_2}(t))_{X^* \times X} \\ &+ j_m(w_{z_1}, w_{z_1}(t) - w_{z_2}(t)) - j_m(w_{z_2}, w_{z_1}(t) - w_{z_2}(t)) = 0. \end{aligned} \tag{3.12}$$

Based on **H(5)** and (2.27), we have:

$$j_m(w_{z_1}, w_{z_1}(t) - w_{z_2}(t)) - j_m(w_{z_2}, w_{z_1}(t) - w_{z_2}(t)) \geq 0, \tag{3.13}$$

and considering **H(2)**, it follows that:

$$v(w_{z_1}(t) - w_{z_2}(t), w_{z_1}(t) - w_{z_2}(t)) \geq m_v \|w_{z_1}(t) - w_{z_2}(t)\|_X^2. \tag{3.14}$$

Combining these inequalities, we deduce:

$$\|w_{z_1}(t) - w_{z_2}(t)\|_X \leq c \|z_1(t) - z_2(t)\|_X. \tag{3.15}$$

According to (3.1), we obtain:

$$\|\vartheta_{z_1}(t) - \vartheta_{z_2}(t)\|_X \leq c \int_0^t \|w_{z_1}(s) - w_{z_2}(s)\|_X \, ds, \tag{3.16}$$

and applying Gronwall's inequality gives:

$$\|\vartheta_{z_1}(t) - \vartheta_{z_2}(t)\|_X \leq c \int_0^t \|z_1(s) - z_2(s)\|_X \, ds. \tag{3.17}$$

□

In this stage, we utilize the solution ϑ_z obtained from lemma 3.1 and examine the following auxiliary problem related to the potential:
Problem (E): Determine the electrical potential $\kappa_y : [0,T] \to Y$ such that

$$b(\kappa_y(t), \beta - \kappa_y(t)) - (y(t), \beta - \kappa_y(t))_{Y^* \times Y} + j_e(\beta) - j_e(\kappa_y(t)) \geq (q(t), \beta - \kappa_y(t))_{Y^* \times Y}, \tag{3.18}$$

$$\kappa_y(0) = \kappa_0, \tag{3.19}$$

for all $t \in [0,T]$ and $\beta \in Y$. The following result holds for **Problem (E)**:

Lemma 3.2 *1) Assuming **H(1)-H(6)**, there exists a unique solution $\kappa_y \in W^{1,2}(0,T;Y)$ to **Problem (E)**.*

*2) Let κ_{y_1} and κ_{y_2} be two solutions of **Problem (E)**, corresponding to y_1 and y_2. Then there exists a constant $c > 0$ such that*

$$\|\kappa_{y_1}(t) - \kappa_{y_2}(t)\|_Y \leq c \|y_1(t) - y_2(t)\|_Y. \tag{3.20}$$

Proof:

1) Similarly to (3.6), we define the functional $q_y : [0,T] \to Y$ as

$$(q_y(t), \beta)_{Y^* \times Y} := (q(t), \beta) + (y(t), \beta)_{Y^* \times Y}. \tag{3.21}$$

Given the regularity of ϕ_e, ϕ_b, and y, it follows that $q_y \in W^{1,2}(0,T; L^2(\Omega))$.
Hence, the variational inequality (3.18) can be rewritten as

$$b(\kappa_y(t), \beta - \kappa_y(t)) + j_e(\beta) - j_e(\kappa_y(t)) \geq (q_y(t), \beta - \kappa_y(t)). \tag{3.22}$$

According to **H(1)** and **H(2)**, the operator b is bilinear, continuous, and coercive.
Furthermore, the functional j_e is a continuous semi-norm on Y. Taking into account **H(6)** and applying the result from [4, Theorem 3.12, p. 57], we assume the existence of a solution to Problem (E) such that $\kappa_y \in W^{1,2}(0,T;Y)$.

2) Let κ_{y_1} and κ_{y_2} be two solutions of **Problem (E)** corresponding to y_1 and y_2, respectively. Applying inequality (3.18), we obtain

$$b(\kappa_{y_1}(t) - \kappa_{y_2}(t), \kappa_{y_1}(t) - \kappa_{y_2}(t)) \leq (y_1(t) - y_2(t), \kappa_{y_1}(t) - \kappa_{y_2}(t))_{Y^* \times Y}. \tag{3.23}$$

Using the coercivity of b along with Young's inequality, we deduce that

$$\|\kappa_{y_1}(t) - \kappa_{y_2}(t)\|_Y^2 \leq c \, \|y_1(t) - y_2(t)\|_Y^2 . \tag{3.24}$$

□

In the final step, for all $(\alpha, \beta) \in X \times Y$ and $t \in [0,T]$, we define the following operator

$$\varpi(z,y)(t) := (\varpi_1(z,y)(t), \varpi_2(z,y)(t)), \tag{3.25}$$

where

$$\varpi_1(z,y)(t) = e(\vartheta_z(t), \alpha) + a(\kappa_y(t), \alpha), \text{ for all } \alpha \in X, \tag{3.26}$$

$$\varpi_2(z,y)(t) = a(\vartheta_z(t), \beta), \text{ for all } \beta \in Y. \tag{3.27}$$

The following result holds:

Lemma 3.3 *The operator ϖ is continuous and has a unique fixed point (z^*, y^*) in $C(0,T;X) \times W^{1,2}(0,T;Y)$.*

Proof: Let $(z,y) \in C(0,T;X) \times W^{1,2}(0,T;Y)$ and t_1, t_2 be in $[0,T]$. Using **H(1)**, we have

$$|\varpi_1(z,y)(t_1) - \varpi_1(z,y)(t_2)| \leq \sup(M_e, M_a) \left(\|\vartheta_z(t_1) - \vartheta_z(t_2)\|_X + \|\kappa_y(t_1) - \kappa_y(t_2)\|_Y \right). \tag{3.28}$$

Additionally,

$$|\varpi_2(z,y)(t_1) - \varpi_2(z,y)(t_2)| \leq M_a \, \|\vartheta_z(t_1) - \vartheta_z(t_2)\|_X . \tag{3.29}$$

Given the regularity of ϑ_z and κ_y, the operator ϖ is continuous.

Now, let (z_1, y_1) and (z_2, y_2) be in $C(0,T;X) \times W^{1,2}(0,T;Y)$. By a similar argument to (3.28)-(3.29), we obtain

$$\begin{aligned} |\varpi(z_1,y_1)(t) - \varpi(z_2,y_2)(t)| &\leq c \left(\|\vartheta_{z_1}(t) - \vartheta_{z_2}(t)\|_X + \|\kappa_{y_1}(t) - \kappa_{y_2}(t)\|_Y \right) \\ &\leq c \, \|(z_1,y_1) - (z_2,y_2)\|_{C(0,T;X \times Y)} . \end{aligned} \tag{3.30}$$

Iterating this inequality for n iterations, we get

$$|\varpi^n(z_1,y_1)(t) - \varpi^n(z_2,y_2)(t)| \leq \frac{c^n}{n!} \, \|(z_1,y_1) - (z_2,y_2)\|_{C(0,T;X \times Y)} . \tag{3.31}$$

For sufficiently large n, the operator ϖ^n becomes a contraction in $C(0,T;X\times Y)$.
Thus, ϖ has a unique fixed point denoted by (z^*,y^*). □

We are now ready to demonstrate Theorem 3.1.

Proof: [Proof of Theorem 3.1] *Existence:* Consider $z^*\in C(0,T;X)$ and $y^*\in W^{1,2}(0,T;Y)$, which are fixed points of the operator ϖ. Let ϑ_{z^*} and κ_{y^*} represent the solutions associated with these fixed points. By substituting $z=z^*$ and $y=y^*$ into the definition of ϖ, we establish that $(\vartheta_{z^*},\kappa_{y^*})$ satisfies **Problem (PV)**.
Uniqueness: The uniqueness of the solution follows from the fact that the fixed point of the operator ϖ, as described in (3.25), is unique. □

4. Optimal Control for Mechanical Models

In this section, we introduce an optimal control strategy for a mechanical model described by the weak formulation in equations (2.33) through (2.35). The control variables for **Problem (PV)** are defined as follows:

$$\lambda=(\phi_m,\phi_N,\phi_e,\phi_b)\in U_c, \tag{4.1}$$

where the control space U_c is defined by

$$U_c=C(0,T;L^2(\Omega)^2)\times C(0,T;L^2(\Gamma_N)^2)\times W^{1,2}(0,T;L^2(\Omega))\times W^{1,2}(0,T;L^2(\Gamma_b)). \tag{4.2}$$

For every control $\lambda\in U_c$, **Problem (PV)** has a distinct solution, denoted by $(\vartheta_\lambda,\kappa_\lambda)$, which varies with the selection of λ. The control problem is defined as follows:

Let U_c^{ad} denote a nonempty subset of U_c, which represents the set of admissible controls. We then consider the objective functional given by:

$$\begin{aligned}&\mathcal{F}:U_c\times K\to\mathbb{R},\\ &(\lambda,\vartheta_\lambda,\kappa_\lambda)\mapsto\mathcal{F}\left(\lambda,\vartheta_\lambda,\kappa_\lambda\right),\end{aligned} \tag{4.3}$$

where $K=C^1(0,T;X)\times W^{1,2}(0,T;Y)$ and $\mathcal{F}$ is defined by

$$\begin{aligned}\mathcal{F}\left(\lambda,\vartheta_\lambda,\kappa_\lambda\right)=&\frac{1}{2}\int_{\Gamma_C}\left(\|\vartheta_\lambda-\vartheta_d\|^2+\|\kappa_\lambda-\kappa_d\|^2\right)\,\mathrm{d}\Gamma\\ &+\frac{\varsigma}{2}\left(\|\phi_m\|^2_{L^2(\Omega)^2}+\|\phi_N\|^2_{L^2(\Gamma_N)^2}+\|\phi_e\|^2_{L^2(\Omega)}+\|\phi_b\|^2_{L^2(\Gamma_b)}\right),\end{aligned}$$

where $\vartheta_d\in L^2(\Gamma_C)^2$ and $\kappa_d\in L^2(\Gamma_C)$ are given, and ς is a given Chekhov regularization parameter. Our goal is to find $(\phi_m,\phi_N,\phi_e,\phi_b)$ which leads to the desired state (ϑ_d,κ_d) on Γ_C.

This section focuses on solving the following problem:
Problem (K): Determine $(\lambda^*,\vartheta^*,\kappa^*)\in U_c^{\mathrm{ad}}\times K$ such that

$$\mathcal{F}\left(\lambda^*,\vartheta^*,\kappa^*\right)=\inf_{\lambda\in U_c^{\mathrm{ad}}}\mathcal{F}\left(\lambda,\vartheta_\lambda,\kappa_\lambda\right), \tag{4.4}$$

where (ϑ^*,κ^*) represents the unique solution to **Problem (PV)** corresponding to the specific control λ^*. For every $\mu\in\mathbb{N}$, let $\lambda_\mu=\left(\phi_{m_\mu},\phi_{N_\mu},\phi_{e_\mu},\phi_{b_\mu}\right)\subset U_c$ be a sequence converging weakly to $\lambda=(\phi_m,\phi_N,\phi_e,\phi_b)$ in the space U_c. We then consider the following problem related to λ_μ:
Problem (PVMU): Determine a displacement field $\vartheta_\mu:[0,T]\to X$ and an electric potential $\kappa_\mu:[0,T]\to Y$ such that for all $\alpha\in X$, $\beta\in Y$, and almost every $t\in[0,T]$, the following conditions are satisfied:

$$e(\vartheta_\mu(t),\alpha)+v(\dot{\vartheta}_\mu(t),\alpha)+a(\kappa_\mu(t),\alpha)+j_m(\dot{\vartheta}_\mu(t),\alpha)=(\phi_\mu(t),\alpha)_{X^*\times X}, \tag{4.5}$$

$$b(\kappa_\mu(t),\beta-\kappa_\mu(t))-a(\vartheta_\mu(t),\beta-\kappa_\mu(t))+j_e(\beta)-j_e(\kappa_\mu(t))\geq(q_\mu(t),\beta-\kappa_\mu(t))_{Y^*\times Y}, \tag{4.6}$$

$$\vartheta_\mu(0)=\vartheta_0. \tag{4.7}$$

Where

$$\left(\phi_{m_\mu}(t), \alpha\right)_{X^*\times X} = \int_\Omega \phi_\mu(t)\cdot\alpha\, dx + \int_{\Gamma_N} \phi_{N_\mu}(t)\cdot\alpha\, d\Gamma,\ \text{for all } \alpha\in X, \tag{4.8}$$

$$(\phi_\mu(t), \beta)_Y = \int_\Omega \phi_{e_\mu}(t)\beta\, dx + \int_{\Gamma_b} \phi_{b_\mu}(t)\xi\, d\Gamma,\ \text{for all } \beta\in Y, \tag{4.9}$$

$$j_m(\vartheta_\mu(t), \alpha) = \int_{\Gamma_C} \phi_\nu(\vartheta_{\mu_\nu}(t))\alpha_\nu\, da + \int_{\Gamma_C} \phi_\tau(\vartheta_{\mu_\tau}(t))\cdot\alpha_\tau\, da,\ \text{for all } \alpha\in X, \tag{4.10}$$

$$j_e(\kappa_\mu(t)) = \int_{\Gamma_C} k|\kappa_\mu(t)|\, da,\ \text{for all } \kappa\in Y. \tag{4.11}$$

We now have the following conditions:

$$\phi_{m_\mu} \rightharpoonup \phi_m \quad \text{in} \quad L^2(0,T;L^2(\Omega)^2) \quad \text{as} \quad \mu\to\infty, \tag{4.12}$$

$$\phi_{N_\mu} \rightharpoonup \phi_N \quad \text{in} \quad L^2(0,T;L^2(\Gamma_N)^2) \quad \text{as} \quad \mu\to\infty, \tag{4.13}$$

$$\phi_{e_\mu} \rightharpoonup \phi_e \quad \text{in} \quad L^2(0,T;L^2(\Omega)) \quad \text{as} \quad \mu\to\infty, \tag{4.14}$$

$$\phi_{b_\mu} \rightharpoonup \phi_b \quad \text{in} \quad L^2(0,T;L^2(\Gamma_b)) \quad \text{as} \quad \mu\to\infty. \tag{4.15}$$

Additionally, there are positive constants ρ_1 and ρ_2 such that

$$\left\|\phi_{m_\mu}\right\|_{L^2(\Omega)^2} + \left\|\phi_{N_\mu}\right\|_{L^2(\Gamma_N)^2} \le c_1, \tag{4.16}$$

$$\left\|\phi_{e_\mu}\right\|_{L^2(\Omega)} + \left\|\phi_{b_\mu}\right\|_{L^2(\Gamma_b)} \le c_2. \tag{4.17}$$

The following theorem is established:

Theorem 4.1 *Assuming that the conditions and hypotheses outlined in Theorem 3.1 are met, it follows that the mapping $\lambda\mapsto(\vartheta_\lambda,\kappa_\lambda)$ is upper semicontinuous.*

The lemma below is the initial component in the proof of Theorem 4.1.

Lemma 4.1 *Let $\left\{\lambda_\mu = \left(\phi_{m_\mu}, \phi_{N_\mu}, \phi_{e_\mu}, \phi_{b_\mu}\right)\right\} \subset U_c$ be a sequence that weakly converges to $\lambda = (\phi_m, \phi_N, \phi_e, \phi_b)$ in U_c. Given the condition*

$$m_v > m_D^2(L_\tau + L_\nu), \tag{4.18}$$

there exists a positive constant c such that

$$\left\|\dot\vartheta_\mu\right\|_{L^2(0,T;X)} + \|\vartheta_\mu(t)\|_X + \|\kappa_\mu\|_{L^2(0,T;Y)} \le c,\ \forall t\in[0,T], \tag{4.19}$$

*where $(\vartheta_\mu, \kappa_\mu)$ denotes the solution of **Problem (PV)** corresponding to λ_μ.*

Proof: Consider the sequence $\left\{\lambda_\mu = \left(\phi_{m_\mu}, \phi_{N_\mu}, \phi_{e_\mu}, \phi_{b_\mu}\right)\right\} \subset U_c$, which weakly converges to $\lambda = (\phi_m, \phi_N, \phi_e, \phi_b)$ in U_c. For each integer μ, the pair $(\vartheta_\mu, \kappa_\mu)$ solves **Problem (PVMU)**.
By setting $\alpha = \dot\vartheta_\mu(t)$ in (4.5), we obtain:

$$\begin{aligned} &e\left(\vartheta_\mu(t), \dot\vartheta_\mu(t)\right) + v\left(\dot\vartheta_\mu(t), \dot\vartheta_\mu(t)\right) + a\left(\kappa_\mu(t), \dot\vartheta_\mu(t)\right) \\ &\quad + j_m\left(\dot\vartheta_\mu(t), \dot\vartheta_\mu(t)\right) = \left(\phi_\mu(t), \dot\vartheta_\mu(t)\right)_{X^*\times X}. \end{aligned} \tag{4.20}$$

Using (4.10) and **H(5)**, we derive:

$$\left| j_m \left(\dot{\vartheta}_\mu(t), \dot{\vartheta}_\mu(t) \right) \right| \leq m_D^2 (L_\tau + L_\nu) \left\| \dot{\vartheta}_\mu(t) \right\|_X^2 . \tag{4.21}$$

From **H(1)** and **H(2)**, we get:

$$\begin{aligned} &\left(m_v - m_D^2(L_\tau + L_\nu) \right) \left\| \dot{\vartheta}_\mu(t) \right\|_X^2 + \frac{1}{2} \frac{d}{dt} e\left(\vartheta_\mu(t), \vartheta_\mu(t) \right) \\ &\leq M_a \left\| \dot{\vartheta}_\mu(t) \right\|_X \left\| \kappa_\mu(t) \right\|_Y + \left\| \dot{\vartheta}_\mu(t) \right\|_X \left\| \phi_\mu(t) \right\|_X . \end{aligned} \tag{4.22}$$

Setting $\beta = 0$ in (4.6) yields:

$$b\left(\kappa_\mu(t), \kappa_\mu(t)\right) - a\left(\vartheta_\mu(t), \kappa_\mu(t)\right) + j_e\left(\kappa_\mu(t)\right) \leq \left(q_\mu(t), \kappa_\mu(t)\right)_{Y^* \times Y} . \tag{4.23}$$

By utilizing (4.11) and condition **H(2)**, we can determine a positive constant c such that:

$$\left| j_e\left(\kappa_\mu(t)\right) \right| \leq c \left\| \kappa_\mu(t) \right\|_Y . \tag{4.24}$$

By **H(1)** and **H(2)**, we derive:

$$m_b \left\| \kappa_\mu(t) \right\|_Y^2 \leq M_a \left\| \vartheta_\mu(t) \right\|_X \left\| \kappa_\mu(t) \right\|_Y + c \left\| \kappa_\mu(t) \right\|_Y + \left\| \vartheta_\mu(t) \right\|_X \left\| \kappa_\mu(t) \right\|_Y . \tag{4.25}$$

By considering equations (4.22) through (4.25), integrating from 0 to t, and subsequently applying Young's inequality, we can show that there exists a positive constant c such that:

$$\begin{aligned} &\left\| \dot{\vartheta}_\mu \right\|_{L^2(0,T;X)}^2 + \left\| \kappa_\mu \right\|_{L^2(0,T;Y)}^2 + \left\| \vartheta_\mu(t) \right\|_X^2 \\ &\leq c \left\{ \left\| \phi_\mu \right\|_{L^2(0,T;X)}^2 + \left\| q_\mu \right\|_{L^2(0,T;Y)}^2 + \left\| \vartheta_0 \right\|_X^2 + \int_0^t \left\| \kappa_\mu(s) \right\|_Y \, ds \right\}. \end{aligned} \tag{4.26}$$

The convergence of the sequence $\{\lambda_\mu\}$ guarantees that it is bounded within U_c. As a result, by utilizing Gronwall's inequality, we can identify a positive constant c such that:

$$\left\| \vartheta_\mu \right\|_{L^2(0,T;X)}^2 + \left\| \vartheta_\mu(t) \right\|_X^2 + \left\| \kappa_\mu \right\|_{L^2(0,T;Y)}^2 \leq c. \tag{4.27}$$

□

We are now equipped with all the essential elements needed to establish Theorem 4.1.

Proof: [Proof of Theorem 4.1]
Consider a sequence $\{\lambda_\mu\} \subset U_c$ that weakly converges to $\{\lambda\}$ within U_c. Let $(\vartheta_\mu, \kappa_\mu)$ and (ϑ, κ) denote the unique solutions to **Problem (PV)** corresponding to λ_μ and λ, respectively. Applying Theorem 3.1 and, if necessary, selecting an appropriate subsequence, we can assume that

$$\begin{aligned} &\vartheta_\mu \rightharpoonup \widehat{\vartheta}, \quad \dot{\vartheta}_\mu \rightharpoonup \dot{\widehat{\vartheta}} \text{ in } L^2(0,T;X), \\ &\kappa_\mu \rightharpoonup \widehat{\kappa} \text{ in } L^2(0,T;Y). \end{aligned} \tag{4.28}$$

Utilizing the compactness of the embeddings $X \times Y \hookrightarrow L^2(\Omega)^2 \times L^2(\Omega)$, we obtain

$$\begin{aligned} &\vartheta_\mu \rightharpoonup \widehat{\vartheta}, \quad \dot{\vartheta}_\mu \rightharpoonup \dot{\widehat{\vartheta}} \text{ in } L^2(0,T;L^2(\Omega)^2), \\ &\kappa_\mu \rightharpoonup \widehat{\kappa} \text{ in } L^2(0,T;L^2(\Omega)). \end{aligned} \tag{4.29}$$

Considering the continuity of the trace operator $\iota : X \times Y \to L^2(\Gamma_C)^2 \times L^2(\Gamma_C)$, we can derive the following convergence:

$$\begin{aligned} &\vartheta_\mu \rightharpoonup \widehat{\vartheta}, \quad \dot{\vartheta}_\mu \rightharpoonup \dot{\widehat{\vartheta}}, \text{ in } L^2(0,T;L^2(\Gamma_C)^2), \\ &\kappa_\mu \rightharpoonup \widehat{\kappa} \text{ in } L^2(0,T;L^2(\Gamma_C)). \end{aligned} \tag{4.30}$$

Given the bilinear nature of the functions e, v, a, and b, it follows that for any $\alpha \in X$ and $\beta \in Y$:

$$\begin{aligned} &e\left(\vartheta_\mu(t), \alpha\right) \to e\left(\widehat{\vartheta}(t), \alpha\right), \text{ in } \mathbb{R}, \\ &v\left(\dot{\vartheta}_\mu(t), \alpha\right) \to v\left(\dot{\widehat{\vartheta}}(t), \alpha\right), \text{ in } \mathbb{R}, \\ &a\left(\kappa_\mu(t), \beta - \kappa_\mu(t)\right) \to a\left(\widehat{\kappa}(t), \beta - \kappa_\mu(t)\right), \text{ in } \mathbb{R}, \\ &b\left(\kappa_\mu(t), \beta - \kappa_\mu(t)\right) \to b\left(\widehat{\kappa}(t), \beta - \widehat{\kappa}(t)\right), \text{ in } \mathbb{R}, \\ &a\left(\vartheta_\mu(t), \beta - \kappa_\mu(t)\right) \to a\left(\widehat{\vartheta}(t), \beta - \widehat{\kappa}(t)\right), \text{ in } \mathbb{R}. \end{aligned} \tag{4.31}$$

Based on the convergence results from equations (4.12) through (4.13), we derive:

$$\begin{aligned} &(\phi_\mu(t), \alpha)_{X^* \times X} \to (\phi(t), \alpha)_{X^* \times X}, \text{ in } \mathbb{R}, \\ &(q_\mu(t), \beta - \kappa_\mu(t))_{Y^* \times Y} \to (q(t), \beta - \widehat{\kappa}(t))_{Y^* \times Y}, \text{ in } \mathbb{R}. \end{aligned} \tag{4.32}$$

By applying (4.30), we find that

$$\begin{aligned} &j_m(\dot{\vartheta}_\mu(t), \alpha) \to j_m(\dot{\widehat{\vartheta}}(t), \alpha), \text{ in } \mathbb{R}, \\ &j_e(\kappa_\mu(t)) \to j_e(\widehat{\kappa}(t)), \text{ in } \mathbb{R}. \end{aligned} \tag{4.33}$$

Thus, using (4.31) through (4.33), we conclude that as $\mu \to \infty$,

$$e\left(\widehat{\vartheta}(t), \alpha\right) + v\left(\dot{\widehat{\vartheta}}(t), \alpha\right) + a\left(\widehat{\kappa}(t), \alpha\right) + j_m\left(\dot{\widehat{\vartheta}}(t), \alpha\right) = \left(\phi(t), \alpha\right)_{X^* \times X}, \tag{4.34}$$

and

$$b\left(\widehat{\kappa}(t), \beta - \widehat{\kappa}(t)\right) - a\left(\widehat{\vartheta}(t), \beta - \widehat{\kappa}(t)\right) + j_e\left(\beta\right) - j_e\left(\widehat{\kappa}(t)\right) \geq \left(q(t), \beta - \widehat{\kappa}(t)\right)_{Y^* \times Y}. \tag{4.35}$$

Thus, we deduce that $\left(\widehat{\vartheta}, \widehat{\kappa}\right) = (\vartheta, \kappa)$. With the uniqueness of the solution to **Problem (PV)** established, we have successfully completed the proof of Theorem 4.1. □

Next, we address the existence of an optimal solution for the problem described in (4.4).

Theorem 4.2 *Under the assumptions given in Theorem 3.1, where U_c^{ad} is a weakly compact subset of U_c and $\mathcal{F}$ is a lower semi-continuous function, there exists an optimal solution $(\lambda^*, \vartheta^*, \kappa^*)$ for the optimization problem (4.4).*

Proof: Consider a minimizing sequence $(\lambda_\mu, \vartheta_\mu, \kappa_\mu)$ for the problem (4.4), where $\lambda_\mu \in U_c$ and $(\vartheta_\mu, \kappa_\mu)$ solves **Problem (PV)** corresponding to λ_μ, such that

$$\lim_{\mu \to +\infty} \mathcal{F}\left(\lambda_\mu, \vartheta_\mu, \kappa_\mu\right) = \inf_{\lambda \in U_c^{ad}} \mathcal{F}\left(\lambda, \vartheta_\lambda, \kappa_\lambda\right) = f \in [-\infty, +\infty). \tag{4.36}$$

By virtue of the compactness of U_c^{ad}, there exists a subsequence of λ_μ, denoted again by λ_μ, such that

$$\lambda_\mu \rightharpoonup \lambda^* \text{ in } U_c \text{ with } \lambda^* \in U_c^{ad}. \tag{4.37}$$

Applying Theorem 4.1 yields

$$(\vartheta_\mu, \kappa_\mu) \rightharpoonup (\vartheta^*, \kappa^*) \text{ in } L^2(0,T; X \times Y), \tag{4.38}$$

where (ϑ^*, κ^*) represents the solution to **Problem (PV)** for λ^*.
Furthermore, due to the lower semi-continuity of the cost function $\mathcal{F}$, we have

$$f \leq \mathcal{F}(\lambda^*, \vartheta^*, \kappa^*) \leq \lim_{\mu \to +\infty} \mathcal{F}(\lambda_\mu, \vartheta_\mu, \kappa_\mu) = f. \tag{4.39}$$

Hence, we have demonstrated the existence of an optimal solution, thereby concluding the proof of Theorem 4.2. □

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