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Quantified energy decay of Euler–Bernoulli beams on an unbounded star-shaped network *

Amina Boukhatem

ABSTRACT: This work discusses the energy decay rates of an infinite star-shaped network of beams with a localized structural damping. Using frequency domain method we prove that the whole system is polynomially stable under some condition on the lengths of the rods.

Key Words: Beam equations, Structural damping, Strong stability, Energy decay, Frequency domain method, Resolvent estimates.

Contents

1. Introduction

In recent years, many researchers have paid attention to the study of the stability of PDEs defined on star-shaped networks, including both finite or semi-infinite branches, such as wave or beam equations.

Stability results for wave equations have been established in [3,5,7], involving various forms of damping such as Kelvin–Voigt or structural damping applied at different locations on the unbounded network. In [1], the author investigated the asymptotic dynamics of two-dimensional elastic networks composed of Euler-Bernoulli beams with localized damping. These networks, generally consisting of finite edges, exhibit complex dynamic behavior due to the interplay between boundary conditions and internal transmission laws.

Furthermore, in the recent work of A. Boukhatem and A. Bchatnia [6], the authors derive the periodicity and asymptotic properties of damped Euler–Bernoulli beam networks, and they give the strong stability and almost periodicity of solutions to the network, which is subjected to structural damping under appropriate assumptions on the damping and network structure. Indeed, one finds singularities of the resolvent operator along the imaginary axis, both at zero and at infinity. This gives rise to a highly important case regarding the rate of energy decay, which is the main focus of this work.

Physically, the problem in question can be conceived as defining vibrations in coupled elastic structures, such as mechanical frames, suspension bridges, or transmission networks. The star-like configuration is that of joints where several beams or strings meet and the introduced damping is that of mechanisms that absorb vibrational energy, material properties or localized control units. Understanding stability and energy degradation in these systems is thus important to structural integrity and resonant phenomenon avoidance.

In this paper, we deal with Euler Bernoulli beam equation on a semi finite network Γ, composed of a finite set of edges $\{I_j\}_{j=1}^{N_F}$ with $N_F \in \mathbb{N}^*$, and an infinite set of edges $\{I_j\}_{j=N_F+1}^{N_F+N_I}$ with $N_I \in \mathbb{N}^*$. All these edges are connected at a unique common vertex. Each finite edge I_j for $1 \leq j \leq N_F$ is identified with the interval $(0, \ell_j)$, where $\ell_j > 0$ denotes its length, and each infinite edge I_j for $N_F + 1 \leq j \leq N_F + N_I$

* Laboratory of pure and applied mathematics (L.M.P.A.), University Amar Telidji of Laghouat, Algeria
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is identified with the semi-infinite interval $(0, +\infty)$. The common vertex corresponds to the point 0 in all these intervals.

More precisely the boundary and transmission conditions can be described as follows,

$$\begin{cases} \partial_t^2 u_j + \partial_x \left(\partial_x^3 u_j - \alpha_j(x) \partial_{xt}^2 u_j \right) = 0, \ \ (x,t) \in I_j \times \mathbb{R}_+, \ \ j = 1, \cdots, N_F + N_I, \\ u_j(\ell_j, t) = \partial_x u_j(\ell_j, t) = 0, \ t > 0, \ j = 1, ..., N_F, \\ u_j(0,t) = u_k(0,t), \ t > 0, \ j,k = 1, ..., N_F + N_I, \\ \partial_x u_j(0,t) = \partial_x u_k(0,t), \ t > 0, \ j,k = 1, ..., N_F + N_I, \\ \displaystyle\sum_{j=1}^{N_F+N_I} \partial_x^2 u_j(0,t) = \sum_{j=1}^{N_F+N_I} \partial_x^3 u_j(0,t) = 0, \ t > 0, \\ u_j(x,0) = u_j^0(x), \ \partial_t u_j(x,0) = u_j^1(x), \ \ x \in (I_j), \ j = 1, .., N_F, \\ u_j(x,0) = u_j^0(x), \ \partial_t u_j(x,0) = u_j^1(x), \ \ x \in \mathbb{R}_+, \ j = N_F + 1, .., N_F + N_I. \end{cases} \tag{1.1}$$

The damping coefficient function $\alpha(.)$, satisfy

$$\begin{cases} \alpha(x) = (\alpha_j(x))_{1 \le j \le N_F + N_I} \in L^\infty(\Gamma), \\ \alpha_j(x) \ge 0, \ \alpha_j(0) = 0, \ \forall j = 1, ..., N_F + N_I. \end{cases}$$

We define

$$E(t) = \frac{1}{2} \sum_{j=1}^{N_F+N_I} \int_{I_j} \left(|\partial_t u_j(x,t)|^2 + |\partial_x^2 u_j(x,t)|^2 \right) dx.$$

and

$$\frac{dE(t)}{dt} = - \sum_{j=1}^{N_F+N_I} \int_{I_j} \alpha_j(x) |\partial_{xt}^2 u_j(x,t)|^2 dx, \ \forall t > 0.$$

The dissipation law satisfies $\frac{dE(t)}{dt} \le 0$, $\forall t > 0$, from which we deduce that the energy is non increasing function of the time variable t.

We organize this work as follows: In Section 2, we will give well-posedness and strong stability results. Then, in Section 3, we discuss two cases of the resolvent singularities, and finally, we present the energy decay rate.

2. Well-posedness and strong stability

In this section, we establish the well-posedness and strong stability of system (1.1) using semigroup theory.

We start by formulating the problem as an abstract linear evolution equation on an appropriate Hilbert space $\mathcal{H}$. The following notation and functional framework will be used throughout this analysis.

Let $H^k(I_j)$ (resp. $L^k(I_j)$) be the Sobolev space (resp. the Lebesgue space) on I_j, $j = 1, .., N_F + N_I$, $k = 1, 2$, in what follows we will write $H^k(\Gamma) = \Pi_{j=1}^{N_F+N_I} H^k(I_j)$ (resp. $L^k(\Gamma) = \Pi_{j=1}^{N_F+N_I} L^k(I_j)$).

We consider the complex Hilbert space

$$X = \mathcal{H} \times L^2(\Gamma),$$

endowed with the inner product,

$$\langle Y_1, Y_2 \rangle_X := \sum_{j=1}^{N_F+N_I} \int_{I_j} \partial_x^2 f_j^1 \, \overline{\partial_x^2 f_j^2} \, dx + \sum_{j=1}^{N_F+N_I} \int_{I_j} g_j^1 \, \overline{g_j^2} dx,$$

in which $Y_k = \left(f^k, g^k\right), \, k = 1, 2$ and

$$\mathcal{H} = \begin{Bmatrix} f \in H^2(\Gamma), \\ f_j(\ell_j) = \partial_x f_j(\ell_j) = 0, \forall j = 1, .., N_F, \\ f_j(0) = f_k(0), \forall\, j, k = 1, .., N_F + N_I, \\ \partial_x f_j(0) = \partial_x f_k(0), \forall j, k = 1, .., N_F + N_I. \end{Bmatrix}$$

We also define

$$\mathcal{A} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \left(-\partial_x^3 u + \alpha \partial_x v\right)_x \end{pmatrix}, \; \forall\, (u, v) \in D(\mathcal{A}),$$

where $\alpha v := (\alpha_j v_j)_{j=1,..,N_F+N_I}$ and

$$\begin{aligned} D(\mathcal{A}) \;\; = \;\; & \{Y = (u, v) \in \mathcal{H} \times \mathcal{H}, \; \left(-\partial_x^3 u_j + \alpha_j \partial_x v_j\right) \in H^1(\Gamma), \\ & \sum_{j=1}^{N_F+N_I} \partial_x^2 u_j(0) = \sum_{j=1}^{N_F+N_I} \partial_x^3 u_j(0) = 0\}, \end{aligned}$$

Now we can give a reformulation of the system (1.1) in the energy space X,

$$\begin{cases} \dfrac{dU}{dt} = \mathcal{A}U, \\ U(0) = U_0, \end{cases} \tag{2.1}$$

such that

$$U = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N_F+N_I} \\ \partial_t u_1 \\ \partial_t u_2 \\ \vdots \\ \partial_t u_{N_F+N_I} \end{pmatrix}, \qquad U_0 = \begin{pmatrix} u_1^0 \\ u_2^0 \\ \vdots \\ u_{N_F+N_I}^0 \\ u_1^1 \\ u_2^1 \\ \vdots \\ u_{N_F+N_I}^1 \end{pmatrix}.$$

The operator $\mathcal{A}$ generates a $C_0 - semigroup$ of contractions $(T(t))_{t\geq 0}$.

2.1. Well-posedness

In view of [6], system (1.1) is well-posed. More specifically, we have the following proved result:

Corollary 2.1 *For an initial datum $y_0 \in X$, there exists a unique weak solution $y \in \mathcal{C}(\mathbb{R}_+, X)$ of system (2.1). Moreover, if $y_0 \in D(\mathcal{A})$, then there exists a unique strong solution $y \in \mathcal{C}(\mathbb{R}_+, D(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+, X)$, of system (2.1).*

Proof: In [6], authors use the Lumer–Phillips theorem and show that the operator $\mathcal{A}$ generates a $\mathcal{C}_0$-semigroup of contractions. Therefore, the above corollary holds. □

2.2. Strong stability

We put the following condition on the damping coefficient $\alpha_j(x)$,

$$(\mathbf{P_1}):\ \alpha_1(x) \geq K > 0,\ \alpha_j(x) = 0,\ \forall j = 2, ..., N_F + N_I.$$

Under condition (P_1), the spectrum approaches the imaginary axis. In fact, we have:

Proposition 2.1 *Suppose that the function $\alpha(x)$ satisfies the condition (P_1) then the intersection between the imaginary axis and the isolated eigenvalues spectrum of $\mathcal{A}_1$ is an empty set if and only if*

$$(\mathbf{H_1}):\ \frac{l_i}{l_j} \neq \frac{z_1}{z_2},\ \ \forall i, j = 2, ..., N_F \ \textit{and} \ \ z_1, z_2 \in \mathcal{S},$$

where $S = \{z \in \mathbb{R},\ \textit{such that}\ \cosh(z)\cos(z) = 1\}$.

Consequently

Theorem 2.1 *Under condition $(\mathbf{P_1})$ and the hypothesis $(\mathbf{H_1})$, the C_0-semigroup $(T(t))_{t\geq 0}$ generated by the operator $\mathcal{A}$, is strongly stable on the energy space.*

For more details and the proof, we invite the reader to consult [6].

3. Energy decay rate

In this section, we will look at how the resolvent operator behaves as $|\beta| \to \infty$ and $|\beta| \to 0$ which give us the rate of the energy decay.

3.1. Singularity at infinity

In this subsection we will describe an upper bound of the resolvent operator norm.

Proposition 3.1 *We have*

$$\| \left(\mathrm{i}\beta I - \mathcal{A}\right)^{-1} \| = O(|\beta|) \ \textit{as}\ |\beta| \to \infty. \tag{3.1}$$

Proof: We assume by contradiction that, we assume that (3.1) fails. Then there exists a sequence (β_n) of real numbers, $\beta_n \to \infty$, (without loss of generality, we suppose that $\beta_n > 0$), and a sequence of vectors $(y_n) = (u^n,\ v^n)$ in $D(\mathcal{A})$ with $\|y_n\|_X = 1$, such that

$$\beta_n \left(\mathrm{i}\beta_n I - \mathcal{A}\right)\left(u^n,\ v^n\right) =: (f^n, g^n) = o(1) \qquad \text{in } X. \tag{3.2}$$

Writing (3.2) in terms of its components, then we multiply the result by β_n^{-1}, we get

$$\begin{cases} \mathrm{i}\beta_n u_j^n - v_j^n = \beta_n^{-1} f_j^n \to 0, in\ H^2(I_j), j = 1, .., N_F + N_I, \\ \mathrm{i}\beta_n v_1^n + \partial_x^4 u_1^n - (\alpha_1 \partial_x v_1^n)_x = \beta_n^{-1} g_1^n \to 0, in\ L^2(I_1), \\ \mathrm{i}\beta_n v_j^n + \partial_x^4 u_j^n = \beta_n^{-1} g_j^n \to 0, in\ L^2(I_j), j = 2, .., N_F + N_j. \end{cases} \tag{3.3}$$

Taking the imaginairy part of the inner product of the equation $(3.3)_{2,3}$, with v_j^n taking into considiration $(3.3)_1$ in $L^2(I_j)$, $j = 1, ..., N_F + N_I$, after summing the result over $j = 1, .., N_F + N_I$, we get

$$\|v^n\|^2_{L^2(\Gamma)} - \|\partial_x^2 u^n\|^2_{L^2(\Gamma)} = o(1). \tag{3.4}$$

Here and now, we will divide the rest of the proof into several steps:

First step. This step is devoted to show:

$$\|v_1^n\|_{L^2(I_1)},\ \|\partial_x^2 u_1^n\|_{L^2(I_1)} = o(1).$$

By (3.2), it is clear that

$$\|\beta_n\left(\mathrm{i}\beta_n I - \mathcal{A}\right)(u^n,\, v^n)\|_X \geq |\Re\left\langle \beta_n\left(\mathrm{i}\beta_n I - \mathcal{A}\right)(u^n,\, v^n), (u^n,\, v^n)\right\rangle_X|$$

From where we deduce,

$$\|\partial_x v_1^n\|_{L^2(I_1)} = o(\beta_n^{-1}). \tag{3.5}$$

Thus, by the fact that $v_1^n(\ell_1) = 0$, it follows that

$$\|v_1^n\|_{L^2(I_1)} \leq C\|\partial_x v_1^n\|_{L^2(I_1)},\ \forall C \geq 0.$$

using (3.5), we get

$$\|v_1^n\|_{L^2(I_1)} = o(1). \tag{3.6}$$

Otherwise, substituting (3.5), (3.6), in $(3.3)_1$, it follows

$$\begin{cases} \beta_n\|\partial_x u_1^n\|_{L^2(I_1)} = o(1) \\ \beta_n\|u_1^n\|_{L^2(I_1)} = o(1). \end{cases} \tag{3.7}$$

Consequently, $u_1^n, v_1^n \underset{n\to\infty}{\longrightarrow} 0$ in $H^1(I_1) \hookrightarrow C([I_1])$, then $|u_1^n(0)|$, $|v_1^n(0)|$, $\beta_n|u_1^n(0)| = o(1)$, wich further by the continuity condition leads to,

$$\beta_n|u_j^n(0)|,\ |v_j^n(0)|,\ |u_j^n(0)| = o(1),\ \forall j = 1, ..., N_F + N_I. \tag{3.8}$$

Moreover, using the Gagliardo-Nirenberg inequality and $(3.7)_1$, we get

$$\|\partial_x u_j^n\|_{L^\infty(I_j)} \leq \|\partial_x^2 u_j^n\|_{L^2(I_j)}^{\frac{1}{2}}\|\partial_x u_j^n\|_{L^2(I_j)}^{\frac{1}{2}} + \|\partial_x u_j^n\|_{L^2(I_j)}, \tag{3.9}$$

The continuity condition one more time and (3.9), give us

$$|\partial_x u_1^n(\ell_1)| = |\partial_x u_j^n(0)| = o(1),\ \forall j = 2, ..., N_F + N_I. \tag{3.10}$$

Next, we will need to calculate the real part of the inner product of $(3.3)_2$, in $L^2(I_1)$ by $q\partial_x u_1^n$, after a various integration by parts we find

$$\begin{aligned} &-\frac{1}{2}\left[|v_1^n|^2 q\right]_0^{\ell_1} - \frac{1}{2}\left[|\partial_x^2 u_1^n|^2 q\right]_0^{\ell_1} + \Re\left(\left[\partial_x^3 u_1^n \overline{\partial_x u_1^n} q\right]_0^{\ell_1}\right) - \Re\left(\left[\partial_x^2 u_1^n \overline{\partial_x u_1^n}\partial_x q\right]_0^{\ell_1}\right) \\ &+ \Re\left(\int_{I_1} \partial_x^2 u_1^n \overline{\partial_x u_1^n}\partial_x^2 q dx\right) + \frac{1}{2}\int_{I_1}\left(3|\partial_x^2 u_1^n + |v_1^n|^2\right)\partial_x q\, dx = o(1). \end{aligned} \tag{3.11}$$

where we have used the fact that g_1^n, f_1^n converge to zero in $L^2(I_1)$, $H^1(I_1)$, respectively, and the boundedness of $\|\partial_x u_1^n\|_{L^2(I_1)}$, and $i\beta_n\overline{\partial_x u_1^n} = -\overline{\partial_x v_1^n} - \beta_n^{-1}\overline{\partial_x f_1^n}$. also we deduce

$$\left|\Re\left(\int_{I_1} \partial_x^2 u_1^n \overline{\partial_x u_1^n}\partial_x^2 q dx\right)\right| \leq \|\partial_x^2 u_1^n\|_{L^2(I_1)}\|\partial_x u_1^n\|_{L^2(I_1)}\|\partial_x^2 q\|_{L^\infty(I_1)} = o(1). \tag{3.12}$$

Make use of tha boundary condition and (3.8),(3.12), in (3.11)

$$\begin{aligned} &-\frac{1}{2}\left[|\partial_x^2 u_1^n|^2 q\right]_0^{\ell_1} - \Re\left(\partial_x^3 u_1^n(0)\overline{\partial_x u_1^n(0)}q(0)\right) \\ &+ \Re\left(\partial_x^2 u_1^n(0)\overline{\partial_x u_1^n(0)}\partial_x q(0)\right) + \frac{3}{2}\int_{I_1}|\partial_x^2 u_1^n|^2\partial_x q\, dx = o(1). \end{aligned} \tag{3.13}$$

In all what follows we will give the function q the explicit form then we replace it in the equation (3.13), we start by taking $q(x) = x^2 e^{\beta_n(x-\ell_1)}$

$$-\frac{\ell_1^2}{2}|\partial_x^2 u_1^n(\ell_1)|^2 + \frac{3}{2}\int_{I_1} |\partial_x^2 u_1^n|^2 \left(2x + \beta_n x^2\right) e^{\beta_n(x-\ell_1)}\, dx = o(1). \tag{3.14}$$

In fact, by the boundedness of $\|\partial_x^2 u_1^n\|_{L^2(I_1)}$, we deduce

$$\left|\int_{I_1} |\partial_x^2 u_1^n|^2 \left(2x + \beta_n x^2\right) e^{\beta_n(x-\ell_1)}\, dx\right| \leq C\|\partial_x^2 u_1^n\|_{L^2(I_1)} \int_{I_1} \beta_n e^{\beta_n(x-\ell_1)}\, dx = o(1),\ C > 0. \tag{3.15}$$

Inserting (3.15) in (3.14), we obtain

$$|\partial_x^2 u_1^n(\ell_1)|^2 = o(1) \tag{3.16}$$

Now, we take $q(x) = x$, and using (3.16) in (3.13), we obtain

$$\Re\left(\partial_x^2 u_1^n(0)\overline{\partial_x u_1^n(0)}\right) + \frac{3}{2}\|\partial_x^2 u_1^n\|^2_{L^2(I_1)} = o(1). \tag{3.17}$$

In other hand, from (3.7) we have

$$|\Re\left(\partial_x^2 u_1^n(0)\overline{\partial_x u_1^n(0)}\right)| \leq \frac{\|\partial_x^2 u_1^n\|_{L^2(I_1)}}{\beta_n}\beta_n\|\partial_x u_1^n\|_{L^2(I_1)} = o(1). \tag{3.18}$$

using (3.18) in (3.17), we obtain

$$\|\partial_x^2 u_1^n\|_{L^2(I_1)} = o(1). \tag{3.19}$$

Second step. This step is devoted to show:

$$\|v_i^n\|_{L^2(\mathbb{R}_+)},\ \|\partial_x^2 u_i^n\|_{L^2(\mathbb{R}_+)} = o(1).$$

As in the previous step, we will need to compute the real part of the inner product of $(3.3)_3$, in $L^2(I_j)$, more then one time.

So Let we start by thier inner products by $\partial_x u_j^n$, without forget to use as usuel the convergence to zero of f_j^n, g_j^n in $L^2(I_j)$ and $\|v_j^n\|_{L^2(I_j)},\ \|\partial_x^2 u_j^n\|_{L^2(I_j)} \leq 1$. Let $q \in \mathcal{C}^2([0,\beta_n])$, that we will choose later. We calculate the real part of the inner product of $(3.3)_3$, in $L^2(I_j)$ by $q\mathbb{X}_{[0,\beta_n]}\partial_x u_j^n$,

$$-\frac{1}{2}\left[|v_j^n|^2 q\right]_0^{\beta_n} - \frac{1}{2}\left[|\partial_x^2 u_j^n|^2 q\right]_0^{\beta_n} + \Re\left(\left[\partial_x^3 u_j^n\overline{\partial_x u_j^n} q\right]_0^{\beta_n}\right) - \Re\left(\left[\partial_x^2 u_j^n\overline{\partial_x u_j^n}\partial_x q\right]_0^{\beta_n}\right)$$
$$+ \Re\left(\int_0^{\beta_n} \partial_x^2 u_j^n\overline{\partial_x u_j^n}\partial_x^2 q dx\right) + \frac{1}{2}\int_0^{\beta_n} \left(3|\partial_x^2 u_j^n|^2 + |v_j^n|^2\right)\partial_x q\, dx = o(1).$$

Using (3.8), (3.10)

$$-\frac{1}{2}|v_j^n(\beta_n)|^2 q(\beta_n) - \frac{1}{2}\left[|\partial_x^2 u_j^n|^2 q\right]_0^{\beta_n} + \Re\left(\partial_x^3 u_j^n(\beta_n)\overline{\partial_x u_j^n(\beta_n)} q(\beta_n)\right) + \Re\left(\int_0^{\beta_n} \partial_x^2 u_j^n\overline{\partial_x u_j^n}\partial_x^2 q dx\right)$$
$$-\Re\left(\partial_x^2 u_j^n(\beta_n)\overline{\partial_x u_j^n(\beta_n)}\partial_x q(\beta_n)\right) + \frac{1}{2}\int_0^{\beta_n} \left(3|\partial_x^2 u_j^n|^2 + |v_j^n|^2\right)\partial_x q\, dx = o(1). \tag{3.20}$$

Choose $q(x) = \beta_n^{-1}x - 1$, in (3.20), we get

$$|\partial_x^2 u_j^n(0)| = o(1) \tag{3.21}$$

By replacing $(3.3)_1$ in $(3.3)_3$, it follows

$$-\beta_n^2 u_j^n + \partial_x^4 u_j^n = i\beta_n^{1-\gamma} f_j^n + \beta_n^{-\gamma} g_j^n, \forall \gamma \geq 1 \tag{3.22}$$

Let now calculate the inner product of (3.22) with $\beta_n^{-\frac{1}{2}} e^{-\beta_n^{\frac{1}{2}}(\beta_n - x)}$, we obtain

$$\int_0^{\beta_n} -\beta_n^2 \beta_n^{-\frac{1}{2}} e^{-\beta_n^{\frac{1}{2}}(\beta_n - x)} u_j^n \, dx + \int_0^{\beta_n} \beta_n^{-\frac{1}{2}} e^{-\beta_n^{\frac{1}{2}}(\beta_n - x)} \partial_x^4 u_j^n \, dx = o(1) \tag{3.23}$$

Integrating by part the second integral we obtain

$$\begin{aligned} &\left[\beta_n^{-\frac{1}{2}} \partial_x^3 u_j^n e^{-\beta_n^{\frac{1}{2}}(\beta_n - x)}\right]_0^{\beta_n} - \left[\partial_x^2 u_j^n e^{-\beta_n^{\frac{1}{2}}(\beta_n - x)}\right]_0^{\beta_n} + \int_0^{\beta_n} -\beta_n^2 \beta_n^{-\frac{1}{2}} e^{-\beta_n^{\frac{1}{2}}(\beta_n - x)} u_j^n \, dx \\ &+ \int_0^{\beta_n} \beta_n^{\frac{1}{2}} e^{-\beta_n^{\frac{1}{2}}(\beta_n - x)} \partial_x^2 u_j^n \, dx = o(1) \end{aligned} \tag{3.24}$$

By Holder inequality we have

$$\left| \int_0^{\beta_n} \beta_n^{\frac{1}{2}} e^{-\beta_n^{\frac{1}{2}}(\beta_n - x)} \partial_x^2 u_j^n \, dx \right| \leq \beta_n^{\frac{1}{2}} \|\partial_x^2 u_j^n\|_{L^2(0,\beta_n)} \|e^{-\beta_n^{\frac{1}{2}}(\beta_n - x)}\|_{L^2(0,\beta_n)} = o(1) \tag{3.25}$$

by the same argument also we have

$$\left| \int_0^{\beta_n} -\beta_n^2 \beta_n^{-\frac{1}{2}} e^{-\beta_n^{\frac{1}{2}}(\beta_n - x)} u_j^n \, dx \right| = o(1) \tag{3.26}$$

using (3.25) and (3.26) in (3.24)

$$\partial_x^2 u_i^n(\beta_n) = o(1). \tag{3.27}$$

Finaly we put $q(x) = x - \beta_n$ in (3.20) taking into acount (3.21) and (3.27), we deduce

$$\frac{1}{2} \int_0^{\beta_n} \left(3|\partial_x^2 u_j^n|^2 + |v_j^n|^2\right) dx = o(1). \tag{3.28}$$

Hence $\forall \varepsilon > 0, \exists N_0 \in \mathbb{N}, \forall n \geq N_0$ such that

$$\int_0^{\beta_n} \left(3|\partial_x^2 u_j^n|^2 + |v_j^n|^2\right) dx < \frac{\varepsilon}{2}$$

Not that $\partial_x^2 u_j^n, v_j^n \in L^2(0,\infty)$, which implies

$$\forall \varepsilon > 0, \exists N_1 \in \mathbb{N}, \forall n \geq N_1, \int_{\beta_n}^{\infty} \left(3|\partial_x^2 u_j^n|^2 + |v_j^n|^2\right) dx < \frac{\varepsilon}{2}$$

So, for $n \geq \max(N_1, N_0)$we have

$$\int_0^{\infty} \left(3|\partial_x^2 u_j^n|^2 + |v_j^n|^2\right) dx < \varepsilon$$

From where we deduce that

$$\|v_i^n\|_{L^2(\mathbb{R}_+)}, \, \|\partial_x^2 u_i^n\|_{L^2(\mathbb{R}_+)} = o(1).$$

Third step. Here and now we will prove that

$$\|v_j^n\|_{L^2(I_j)}, \, \|\partial_x^2 u_j^n\|_{L^2(I_j)} = o(1), \, \forall j = 2, ..., N_F.$$

We suppose that $\left(1-\cosh(\sqrt{|\beta_n|}\ell_j)\cos(\sqrt{|\beta_n|}\ell_j)\right) \neq 0$, $\forall j = 2, ..., N_F$, then we plug $(3.3)_1$ in $(3.3)_3$, in order to get

$$-\beta_n^2 u_j^n + \partial_x^4 u_j^n = \mathrm{i} f_j^n + \beta_n^{-1} g_j^n. \tag{3.29}$$

Let us now, solve (3.29), where we put $F_j^n(x) =: \mathrm{i} f_j^n + \beta_n^{-1} g_j^n$. Straight-forward calculation yields

$$\begin{aligned} u_j^n(x) &= a\left(\cos(\sqrt{|\beta_n|}x) - \cosh(\sqrt{|\beta_n|}x)\right) + b\left(\sin(\sqrt{|\beta_n|}x) - \sinh(\sqrt{|\beta_n|}x)\right) \\ &+ u_j^n(0)\cosh(\sqrt{|\beta_n|}x) + \beta_n^{-\frac{1}{2}}\partial_x u_j^n(0)\sinh(\sqrt{|\beta_n|}x) \\ &+ \frac{\beta_n^{-\frac{1}{2}}}{2}\int_0^x \left(\sin(\sqrt{|\beta_n|}(x-s)) - \sinh(\sqrt{|\beta_n|}(x-s))\right) F_j^n(s)\, ds, a, b \in \mathbb{R}. \end{aligned} \tag{3.30}$$

The second derivative of the solution (3.30) is given by,

$$\begin{aligned} \partial_x^2 u_j^n(x) &= -\beta_n a\left(\cos(\sqrt{|\beta_n|}x) + \cosh(\sqrt{|\beta_n|}x)\right) - \beta_n b\left(\sin(\sqrt{|\beta_n|}x) + \sinh(\sqrt{|\beta_n|}x)\right) \\ &+ \beta_n u_j^n(0)\cosh(\sqrt{|\beta_n|}x) + \beta_n^{\frac{1}{2}}\beta_n^{-\frac{1}{2}}\partial_x u_j^n(0)\sinh(\sqrt{|\beta_n|}x) + F_j^n(x) \\ &- \frac{\beta_n^{\frac{1}{2}}}{2}\int_0^x \left(\sin(\sqrt{|\beta_n|}(x-s)) - \sinh(\sqrt{|\beta_n|}(x-s))\right) F_j^n(s)\, ds, \end{aligned} \tag{3.31}$$

Let $x = \ell_j$ in (3.31), we observe that

$$\partial_x^2 u_j^n(\ell_j) = o(1). \tag{3.32}$$

Where we have use the fact that f_j^n, g_j^n converge to zero in $L^2(I_j)$, and the boundedness of the functions $\cos(x)$, $\sin(x)$, $\cosh(x)$, $\sinh(x)$ in I_j, and (3.8). Next, we take the real part of the inner product of $(3.3)_3$ with $q\partial_x u_j^n$, in $L^2(I_j)$, where $q \in \mathcal{C}^2(I_j)$, that we will chose later.

$$\Re\left(\int_{I_j} \mathrm{i}\beta_n v_j^n q\,\overline{\partial_x u_j^n} dx + \int_{I_j} \partial_x^4 u_j^n q \overline{\partial_x u_j^n} dx\right) = \Re\left(\int_{I_j} \beta_n^{-\gamma} g_j^n q\,\overline{\partial_x u_j^n} dx\right),$$

Indeed from $(3.3)_1$, we have that $\mathrm{i}\beta_n \overline{\partial_x u_j^n} = -\overline{\partial_x v_j^n} - \beta_n^{-1}\overline{\partial_x f_j^n}$, then

$$\Re\left(\int_{I_j} v_j^n q\,\left(-\overline{\partial_x v_j^n} - \beta_n^{-1}\overline{\partial_x f_j^n}\right) dx + \int_{I_j} \partial_x^4 u_j^n q \overline{\partial_x u_j^n} dx\right) = \Re\left(\int_{I_j} \beta_n^{-1} g_j^n q\,\overline{\partial_x u_j^n} dx\right),$$

since g_j^n, f_j^n converge to zero in $L^2(I_j)$ and $H^2(I_j)$, respectively, and that $\|(u^n, v^n)\|_X = 1$, we deduce

$$\Re\left(-\int_{I_j} v_j^n q\,\overline{\partial_x v_j^n} dx + \int_{I_j} \partial_x^4 u_j^n q \overline{\partial_x u_j^n} dx\right) = o(1). \tag{3.33}$$

Then From (3.33), it follows

$$\begin{aligned} &-\frac{1}{2}\left[|v_j^n|^2 q\right]_0^{\ell_j} - \frac{1}{2}\left[|\partial_x^2 u_j^n|^2 q\right]_0^{\ell_j} + \Re\left(\left[\partial_x^3 u_j^n \overline{\partial_x u_j^n} q\right]_0^{\ell_j}\right) - \Re\left(\left[\partial_x^2 u_j^n \overline{\partial_x u_j^n} \partial_x q\right]_0^{\ell_j}\right) \\ &+ \Re\left(\int_{I_j} \partial_x^2 u_j^n \overline{\partial_x u_j^n} \partial_x^2 q dx\right) + \frac{1}{2}\int_{I_j} \left(3|\partial_x^2 u_j^n|^2 + |v_j^n|^2\right)\partial_x q\, dx = o(1). \end{aligned} \tag{3.34}$$

Let we now use the boundary condition on the point ℓ_j, and (3.8), (3.32), in the equation (3.34), we get

$$\begin{aligned} &\frac{1}{2}|\partial_x^2 u_j^n(0)|^2 q(0) - \Re\left(\partial_x^3 u_j^n(0)\overline{\partial_x u_j^n(0)} q(0)\right) + \Re\left(\partial_x^2 u_j^n(0)\overline{\partial_x u_j^n(0)}\partial_x q(0)\right) \\ &+ \Re\left(\int_{I_j} \partial_x^2 u_j^n \overline{\partial_x u_j^n} \partial_x^2 q dx\right) + \frac{1}{2}\int_{I_j} \left(3|\partial_x^2 u_j^n|^2 + |v_j^n|^2\right)\partial_x q\, dx = o(1). \end{aligned} \tag{3.35}$$

Putting $q(x) = x$, we obtain

$$\Re\left(\partial_x^2 u_j^n(0)\overline{\partial_x u_j^n(0)}\right) + \frac{1}{2}\int_{I_j} \left(3|\partial_x^2 u_j^n|^2 + |v_j^n|^2\right) dx = o(1). \tag{3.36}$$

Also by (3.8), we can immediatly deduce

$$\Re\left(\partial_x^2 u_j^n(0)\overline{\partial_x u_j^n(0)}\right) = o(1),\ \forall j = 2, ..., N_F + N_I. \tag{3.37}$$

Using (3.37) in (3.36), we observe

$$\int_{I_j} \left(3|\partial_x^2 u_j^n|^2 + |v_j^n|^2\right) dx = o(1), \tag{3.38}$$

which is the claim of this step.

At the end, we have that $\|y_n\|_X \to 0$. This result contradicts the hypothesis $\|y_n\|_X = 1$. □

3.2. Singularity at zero

In this subsection, we estimate sharp upper bounds on the growth of the resolvent as β tends to zero.

Proposition 3.2 *Under the hypothesis (P_2). We have*

$$\|\left(\mathrm{i}\beta I - \mathcal{A}\right)^{-1}\| = O\left(|\beta|^{-1}\right) \ \text{as}\ |\beta| \to 0. \tag{3.39}$$

Proof: Suppose that (3.39) is false. By the Banach-Steinhaus theorem, there exists a sequence of real number β_n with $\beta_n \to 0$, (without loss of generality, we suppose that $\beta_n > 0$), and a sequence of vectors $y^n = (u^n, v^n) \in D(\mathcal{A})$ with

$$\|(u^n, v^n)\|_X = 1,$$

such that

$$\beta_n^{-1}(\mathrm{i}\beta_n I - \mathcal{A})(u^n, v^n) =: (f^n, g^n) = o(1) \quad \text{in } X. \tag{3.40}$$

We shall prove that $\|(u^n, v^n)\|_X = o(1)$, which contradicts the hypothesis on (u^n, v^n).

The equation (3.40), gives

$$\begin{cases} \beta_n^{-1}\left(\mathrm{i}\beta_n u_j^n - v_j^n\right) = f_j^n \to 0,\ in\ H^2(I_j),\ j = 1, .., N_F + N_I, \\ \beta_n^{-1}\left(\mathrm{i}\beta_n v_1^n + \partial_x^4 u_1^n - (\alpha_1 \partial_x v_1^n)_x\right) = g_1^n \to 0,\ in\ L^2(I_1), \\ \beta_n^{-1}\left(\mathrm{i}\beta_n v_j^n + \partial_x^4 u_j^n\right) = g_j^n \to 0,\ in\ L^2(I_j),\ j = 2, ..., N_F + N_I. \end{cases} \tag{3.41}$$

Multiplying (3.41) by β_n, we obtain the next equations

$$\begin{cases} v_j^n = \beta_n\left(iu_j^n - f_j^n\right), \\ \mathrm{i}\beta_n v_1^n + \partial_x^4 u_1^n - (\alpha_1 \partial_x v_1^n)_x = \beta_n g_1^n, \\ \mathrm{i}\beta_n v_j^n + \partial_x^4 u_j^n = \beta_n g_j^n. \end{cases} \tag{3.42}$$

First, one multiplies $(3.42)_1$, by $\overline{v_j^n}$, than we integrate over I_j, we pay attention that

$$\|v_j^n\|^2_{L^2(I_j)} - \beta_n^2 \|u_j^n\|^2_{L^2(I_j)} = o(1),\ \forall j \in J. \tag{3.43}$$

Similarly, one multiplies $(3.42)_{2,3}$, by $\overline{v_j^n}$, than integrating and summing up over J, then by Kirchoff condition, we infer

$$\|v^n\|^2_{L^2(\Gamma)} - \|\partial_x^2 u^n\|^2_{L^2(\Gamma)} + \|\alpha_1^{\frac{1}{2}} \partial_x v_1^n\|^2_{L^2(I_1)} = o(1). \tag{3.44}$$

By the dissipation property of the semigroup of the operator $\mathcal{A}$, we deduce

$$\|\alpha_1^{\frac{1}{2}} \partial_x v_1^n\|_{L^2(I_1)} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.45}$$

Make use (3.45) in (3.44), one gets

$$\|v^n\|^2_{L^2(\Gamma)} - \|\partial_x^2 u^n\|^2_{L^2(\Gamma)} = o(1). \tag{3.46}$$

Second, for all , $\forall j \in J_F$, we know that $\partial_x u_j^n(\ell_j) = u_j^n(\ell_j) = 0$, and $\|\partial_x^2 u_j^n\|_{L^2(I_j)} \le 1$, so

$$\begin{aligned}\|\partial_x u_j^n\|_{L^2(I_j)} &\le C\|\partial_x u_j^n\|_{L^\infty(I_j)} \le C \max_{x\in I_j} \|\partial_x^2 u_j^n\|_{L^1(x,\ell_j)} \\ &\le C_1 \max_{x\in I_j} \|\partial_x^2 u_j^n\|_{L^2(x,\ell_j)} \le C_2 \|\partial_x^2 u_j^n\|_{L^2(I_j)},\ C,\ C_1,\ C_2 > 0.\end{aligned}$$

and

$$\begin{aligned}\|u_j^n\|_{L^2(I_j)} &\le C'\|u_j^n\|_{L^\infty(I_j)} \le C' \max_{x\in I_j} \|\partial_x u_j^n\|_{L^1(x,\ell_j)} \\ &\le C_1' \max_{x\in I_j} \|\partial_x u_j^n\|_{L^2(x,\ell_j)} \le C_2' \|\partial_x u_j^n\|_{L^2(I_j)},\ C',\ C_1',\ C_2' > 0.\end{aligned}$$

Thus, $\|u_j^n\|_{L^2(I_j)}$, $\|\partial_x u_j^n\|_{L^2(I_j)}$ are bounded for all $j \in J_F$.

Therefore, by (3.43), it follows

$$\|v_j^n\|_{L^2(I_j)} = o(1),\ \forall j \in J_F. \tag{3.47}$$

Let us now divide the rest of the proof into several steps:

Step 1 : The aim of this step is to show

$$\|\partial_x^2 u_i^n\|_{L^2(0,\ell_j)} = o(1).$$

In view of equation $(3.42)_3$ and (3.47), we easily derive

$$\|\partial_x^4 u_j^n\|_{L^2(0,\ell_j)} = o(1). \tag{3.48}$$

The inner products of the equation $(3.42)_3$ by $e^{-\beta_n^{-1}x}$, yields

$$\int_0^{\ell_j} \partial_x^4 u_j^n(x) e^{-\beta_n^{-1}x}\, dx = -\int_0^{\ell_j} \mathrm{i}\beta_n v_j^n e^{-\beta_n^{-1}x}\, dx + \int_0^{\ell_j} \beta_n g_j^n e^{-\beta_n^{-1}x}\, dx$$

The right-hand side of the above equation converges to zero since g_j^n, v_j^n converges to zero in $L^2(0,\ell_j)$ and $\beta_n e^{-\beta_n^{-1}x}$is bounded. Performing integration by parts to theleft-hand side,we get

$$\partial_x^3 u_j^n(\ell_j) e^{-\beta_n^{-1}\ell_j} - \partial_x^3 u_j^n(0) + \int_0^{\ell_j} \partial_x^3 u_j^n \beta_n^{-1} e^{-\beta_n^{-1}x}\, dx = o(1). \tag{3.49}$$

and one more time

$$\partial_x^3 u_j^n(\ell_j) e^{-\beta_n^{-1}\ell_j} - \partial_x^3 u_j^n(0) + \partial_x^2 u_j^n(\ell_j)\beta_n^{-1} e^{-\beta_n^{-1}\ell_j} - \beta_n^{-1}\partial_x^2 u_j^n(0) + \int_0^{\ell_j} \partial_x^2 u_j^n \beta_n^{-2} e^{-\beta_n^{-1}x}\, dx = o(1). \tag{3.50}$$

Make use of Gagliardo inequality more then one time, we obtain

$$\|\partial_x^3 u_i^n\|_{L^\infty(0,\ell_j)} \leq c_1 \|\partial_x^4 u_i^n\|^{\frac{1}{2}}_{L^2(0,\ell_j)} \|\partial_x^3 u_i^n\|^{\frac{1}{2}}_{L^2(0,\ell_j)} + c_2 \|\partial_x^3 u_i^n\|_{L^2(0,\ell_j)}, \, \forall c_1, c_2 > 0,$$

$$\|\partial_x^3 u_i^n\|_{L^2(0,\ell_j)} \leq k_1 \|\partial_x^4 u_i^n\|^{\frac{1}{2}}_{L^2(0,\ell_j)} \|\partial_x^2 u_i^n\|^{\frac{1}{2}}_{L^2(0,\ell_j)} + k_2 \|\partial_x^2 u_i^n\|_{L^2(0,\ell_j)}, \, \forall k_1, k_2 > 0,$$

Frome where we deduce the boundedness of $\partial_x^3 u_j^n(\ell_j), \partial_x^3 u_j^n(0)$ and $\|\partial_x^3 u_i^n\|_{L^2(0,\ell_j)}$, taking into acount equation (3.48) and boundedness of $\|\partial_x^2 u_i^n\|_{L^2(0,\ell_j)}$.

Therefore, equation (3.49), gives

$$\partial_x^3 u_j^n(0) = o(1), \tag{3.51}$$

and use (3.51) in (3.50), it follows

$$\beta_n^{-1} \partial_x^2 u_j^n(0) = o(1). \tag{3.52}$$

Next, we evaluate the real part of the inner product of $(3.42)_3$ by $x\partial_x u_j^n$ in $L^2(0, \ell_j)$

$$Re \left(\partial_x^2 u_j^n(0) \overline{\partial_x u_j^n(0)} \right) + \frac{3}{2} \|\partial_x^2 u_j^n\|^2_{L^2(0,\ell_j)} = o(1). \tag{3.53}$$

where we have used the boundery condition and (3.47) and the convergence to zero of $\beta_n, \|g_j^n\|_{L^2(0,\ell_j)}$.

Observing (3.52) and (3.53) having in mind the boundedness of $\|\partial_x u_j\|_{L^\infty(0,\ell_j)}$ and convergence to zero of β_n, we get

$$Re \left(\beta_n^{-1} \partial_x^2 u_j^n(0) \beta_n \overline{\partial_x u_j^n(0)} \right) = o(1). \tag{3.54}$$

Make use of (3.54) in (3.53), we get the aim of this step

$$\|\partial_x^2 u_j^n\|^2_{L^2(0,\ell_j)} = o(1). \tag{3.55}$$

Step 2 : this step is devoted to proof

$$\|v_i^n\|_{L^2(\mathbb{R}_+)}, \|\partial_x^2 u_i^n\|_{L^2(\mathbb{R}_+)} = o(1), \, \forall j|inJ_I.$$

It is easy to see from the equation $(3.42)_1$ that

$$\|v_i^n\|_{L^2(0,\beta_n^{-1})} = o(1).$$

onehand we have $\forall \varepsilon > 0, \exists N_1 \in \mathbb{N}, \forall n \geq N_1$

$$\int_0^{\beta_n^{-1}} |v_j^n|^2 \, dx < \frac{\varepsilon}{2}$$

on the other hand $\forall \varepsilon > 0, \exists N_2 \in \mathbb{N}, \forall n \geq N_2$

$$\int_{\beta_n^{-1}}^{\infty} |v_j^n|^2 \, dx < \frac{\varepsilon}{2}$$

Consequently, for $n \geq \max(N_1, N_2)$ we obtain

$$\int_0^{\infty} |v_j^n|^2 \, dx < \frac{\varepsilon}{2}$$

So,

$$\|v_j^n\|_{L^2(\mathbb{R}_+)} = o(1). \tag{3.56}$$

Finaly, using (3.47),(3.55) and (3.56) in (3.46), we deduce

$$\|\partial_x^2 u_j^n\|_{L^2(\mathbb{R}_+)} = o(1).$$

To sum up, we have shown that $| \, y_n\|_X = o(1)$ which contradict the hypothesis. □

We present the main results of this paper in the next theorem

Theorem 3.1 *Let* $(T(t))_{t\geq 0}$ *be the bounded* $\mathcal{C}_0-$ *semigroup on the Hilbert space* X*, with generator* $\mathcal{A}$*. Under condition* $(\mathbf{P_1})$ *and hypothesis* $(\mathbf{H_1})$*, we have*

$$\|T(t)\mathcal{A}\,(I-\mathcal{A})^{-2}\,\| = O(t^{-1}),\ t\to\infty.$$

Proof: An immediate consequence of Proposition 3.1, 3.2 and Theorem 8.4 in [4] one has

$$\|T(t)A\,(I-A)^{-2}\,\| = O(t^{-1}),\ t\to\infty.$$

□

As a consequence of Theorem 3.1 and Remark 8.5 in [4] we have the following Corollary.

Corollary 3.1 *Assume* $(\mathbf{P_1}),(\mathbf{H_1})$ *holds. Then for given* $U_0\in D\,(A)\cap R\,(A)\,,$ *where* $R(A):=A(D(A))$ *there exist constants* $C,\ t_0>0$ *such that for all* $t\geq t_0$*,*

$$\|T(t)U_0\|_X \leq \frac{C}{t}\|U_0\|_X.$$

Remark 3.1 All previously obtained results valid only under condition $(\mathbf{P_1})$. Then it is natural to ask whether such cases exist where this results still holds in the absence of this condition? The answer of this question is only when $N_F\leq 2$.

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Amina Boukhatem,

Department of Mathematics,

Department of Mathematics ,

Laboratory of Pure and Applied Mathematics,

Amar Telidji University, Laghouat 03000, Algeria.

E-mail address: a.boukhatem@lagh-univ.dz