Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–8.
ISSN-0037-8712
doi:10.5269/bspm.77330

Generalization of Common Fixed Points Theorems for C-T-Contraction Mappings with Application to Partial Differential Equations and Modified Meir-Keeler's Theorems

M. Iadh Ayari* and M. Boussoffra

ABSTRACT: In this paper, we prove two common fixed point theorems for pairs of self-mappings satisfying L-weak commuting condition. Then we prove some fixed point theorems for more general self-mappings which do not depend on L-weakly commuting condition called T-contractions, which include a class that satisfies a generalized Meir-Keeler type contractive condition using C-Functions. We also present examples that support and strengthen our results. Finally, we consider an application in partial differential equations, ensuring the existence of a common fixed point that provides an exact solution of a nonlinear equation.

Key Words: L-Weakly commuting mappings, C-class functions, C-Meir-Keeler-type contraction.

Contents

1. Introduction

Common fixed points of self-mappings satisfying certain contractive types of conditions have been the focus of many researchers. Some of these works dealt with commuting or L-weak commuting mappings, which were first introduced by Sessa [1]. In 1986, Jungck [2] proposed the definition of compatible mappings. Also in the same year, Tivari and Singh [3] introduced asymptotic commutativity.

In the present paper, using C-functions introduced by Ansari [4], we suggest the notion of C-T-contraction mappings. We prove theorems of existence and uniqueness of common fixed points under the assumption of C-T-contraction and L-weak commuting mappings. Additionally, we establish common fixed point theorems for L-weak commuting pairs satisfying a modification of Meir and Keeler's condition using C-functions.

We then suggest some common fixed point theorems for self-mappings called T-contractions satisfying a modified Meir-Keeler type contraction. Several examples are proposed to strengthen our theorems. Finally, we consider an application in partial differential equations, ensuring the existence of a common fixed point that provides an exact solution of a nonlinear equation.

2. Backgrounds and Notations

Definition 2.1 Let (X, d) be a metric space and let S and T be two self-mappings of X. The mappings S and T are called L-weakly commuting if there exists a positive number L such that

$$d(ST(x), TS(x)) \leq L\, d(S(x), T(x)), \quad \text{for all} \quad x \in X.$$

Ansari introduced the following definition:

* Corresponding author.
2010 *Mathematics Subject Classification:* 47H10, 54H25.
Submitted June 15, 2025. Published September 24, 2025

Definition 2.2 [4] A continuous function $J : [0,\infty)^2 \to \mathbb{R}$ is considered a C-*class* function if it satisfies:

1. $J(a,b) \le a$;
2. $J(a,b) = a$ implies that either $a = 0$ or $b = 0$, for all $a, b \in [0,\infty)$.

C-class functions are denoted by $\mathcal{C}$.

Example 2.1 *[4] The following functions $J : [0,\infty)^2 \to \mathbb{R}$ belong to $\mathcal{C}$ for all $a, b \in [0,\infty)$:*

1. $J(a,b) = a - b$;
2. $J(a,b) = ma$ *with* $0 < m < 1$;
3. $J(a,b) = \frac{a}{(1+b)^r}$ *with* $r \in (0,\infty)$;
4. $J(a,b) = \frac{\log(b+s^a)}{1+b}$ *with* $s > 1$;
5. $J(a,b) = \frac{\ln(1+s^a)}{2}$ *with* $s > e$;
6. $J(a,b) = (a+l)^{(1/(1+b)^s)} - t$ *with* $t > 1$ *and* $s \in (0,\infty)$;
7. $J(a,b) = a \log_{b+r} r$ *with* $r > 1$;
8. $J(a,b) = a - \left(\frac{1+a}{2+a}\right)\left(\frac{b}{1+b}\right)$;
9. $J(a,b) = a\beta(a)$ *where* $\beta : [0,\infty) \to [0,1)$ *is continuous;*
10. $J(a,b) = s - \frac{b}{k+b}$;
11. $J(a,b) = a - \Lambda(b)$ *where* $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous and* $\Lambda(b) = 0 \Leftrightarrow b = 0$;
12. $J(a,b) = a\zeta(a,b)$ *where* $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous and* $\zeta(b,a) < 1$ *for all* $a, b > 0$;
13. $J(a,b) = a - \left(\frac{2+b}{1+b}\right) b$;
14. $J(a,b) = \sqrt[n]{\ln(1+a^n)}$;
15. $J(a,b) = \phi(a)$ *where* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *is upper semi-continuous,* $\phi(0) = 0$, *and* $\phi(b) < b$ *for all* $b > 0$;
16. $J(a,b) = \frac{a}{(1+a)^r}$ *with* $r \in (0,\infty)$.

3. Main Results

We begin by introducing the following concept:

Definition 3.1 Let T and S be two self-mappings on a metric space (X,d). S is said to be a C-T-contraction if there exists $J \in \mathcal{C}$ such that

$$d(Sx, Sy) < J(d(Tx,Ty), d(Tx,Ty)) \quad \text{for all} \quad x, y \in X.$$

Theorem 3.1 *Let (X,d) be a complete metric space and let S and T be L-weakly commuting self-mappings of X satisfying:*

(i) There exists $J \in \mathcal{C}$ such that S is a C-T-contraction;

(ii) $S(X) \subset T(X)$;

(iii) Either S or T is continuous.

Then S and T have a unique common fixed point.

Proof: Let ξ_0 be an arbitrary point in X. Since $S(X) \subset T(X)$, there exists $\xi_1 \in X$ such that $S\xi_0 = T\xi_1$. Inductively, we construct a sequence $\{\xi_n\}$ in X such that $S\xi_n = T\xi_{n+1}$ for $n \geq 0$. Then

$$\begin{aligned} d(S\xi_n, S\xi_{n+1}) &< J(d(T\xi_n, T\xi_{n+1}), d(T\xi_n, T\xi_{n+1})) \\ &= J(d(S\xi_{n-1}, S\xi_n), d(S\xi_{n-1}, S\xi_n)) \\ &\leq d(S\xi_{n-1}, S\xi_n). \end{aligned}$$

Thus, $\{d(S\xi_n, S\xi_{n+1})\}_{n=0}^{\infty}$ is a decreasing sequence of positive real numbers and converges to a limit $l \geq 0$. Suppose $l > 0$. Then,

$$d(S\xi_n, S\xi_{n+1}) < J(d(S\xi_{n-1}, S\xi_n), d(S\xi_{n-1}, S\xi_n)) \leq d(S\xi_{n-1}, S\xi_n).$$

Taking $n \to \infty$ and using the continuity of J, we obtain $l \leq J(l, l) \leq l$, a contradiction. Hence, $l = 0$.

Now, we show that $\{S\xi_n\}$ is a Cauchy sequence. Suppose not. Then there exists $\beta > 0$ and subsequences $\{S\xi_{m_p}\}$ and $\{S\xi_{n_p}\}$ such that for all $p \in \mathbb{N}$ with $m_p > n_p > p$, we have $d(S\xi_{m_p}, S\xi_{n_p}) \geq \beta$ and $d(S\xi_{m_p}, S\xi_{n_p-1}) < \beta$. By the triangle inequality,

$$\beta \leq d(S\xi_{m_p}, S\xi_{n_p}) \leq d(S\xi_{m_p}, S\xi_{m_p-1}) + d(S\xi_{m_p-1}, S\xi_{n_p}) < \beta + d(S\xi_{m_p}, S\xi_{m_p-1}).$$

As $p \to \infty$, $d(S\xi_{m_p}, S\xi_{n_p}) \to \beta$. Also,

$$d(S\xi_{m_p}, S\xi_{n_p}) \leq J(d(S\xi_{m_p-1}, S\xi_{n_p-1}), d(S\xi_{m_p-1}, S\xi_{n_p-1})) \leq d(S\xi_{m_p-1}, S\xi_{n_p-1}).$$

Letting $p \to \infty$, we get $\beta \leq J(\beta, \beta) \leq \beta$, implying $\beta = 0$, a contradiction. Thus, $\{S\xi_n\}$ is Cauchy and converges to some $\xi \in X$. Similarly, $\{T\xi_n\}$ converges to ξ.

Assume S is continuous. Then $S(S\xi_n) \to S\xi$ and $S(T\xi_n) \to S\xi$. Since S and T are L-weakly commuting,

$$d(S(T\xi_n), T(S\xi_n)) \leq L\, d(S\xi_n, T\xi_n).$$

Thus, $T(S\xi_n) \to S\xi$. Now, suppose $\xi \neq S\xi$. Then

$$d(S\xi_n, S(S\xi_n)) < J(d(T\xi_n, T(S\xi_n)), d(T\xi_n, T(S\xi_n))) \leq d(T\xi_n, T(S\xi_n)).$$

Letting $n \to \infty$, we get $d(\xi, S\xi) \leq J(d(\xi, S\xi), d(\xi, S\xi)) \leq d(\xi, S\xi)$, a contradiction. Hence, $\xi = S\xi$.

Since $S(X) \subset T(X)$, there exists $\psi \in X$ such that $\xi = S\xi = T\psi$. Now,

$$d(S(S\xi_n), S\psi) < J(d(T(S\xi_n), T\psi), d(T(S\xi_n), T\psi)) \leq d(T(S\xi_n), T\psi).$$

Letting $n \to \infty$, we get $S\xi = T\psi$, so $\xi = S\xi = T\psi$. Thus,

$$d(S\xi, T\xi) = d(S(T\psi), T(S\psi)) \leq L\, d(S\psi, T\psi) = 0,$$

implying $\xi = S\xi = T\xi$. Therefore, ξ is a common fixed point.

For uniqueness, suppose ξ' is another common fixed point. Then

$$d(S\xi, S\xi') < J(d(T\xi, T\xi'), d(T\xi, T\xi')) \leq d(T\xi, T\xi') = d(S\xi, S\xi'),$$

a contradiction. Hence, the common fixed point is unique. □

Corollary 3.1 *[5] Let (X, d) be a complete metric space and let S and T be L-weakly commuting self-mappings of X satisfying:*

$$d(Sx, Sy) \leq r(d(Tx, Ty)),$$

where $r : \mathbb{R}_+ \to \mathbb{R}_+$ is upper semi-continuous, $r(0) = 0$, and $r(t) < t$ for all $t > 0$. If $S(X) \subset T(X)$ and either S or T is continuous, then S and T have a unique common fixed point.

Proof: Set $J(a, b) = r(a)$, which belongs to $\mathcal{C}$ by Example **??**(15). The result follows from Theorem 3.1. □

Theorem 3.2 *Let (X, d) be a complete metric space and let S and T be L-weakly commuting self-mappings of X satisfying:*

(i) For every $\varepsilon > 0$, there exist $h(\varepsilon) > 0$ and $J(\varepsilon) \in \mathcal{C}$ such that

$$\varepsilon \leq J(d(Tx,Ty), d(Tx,Ty)) < \varepsilon + h \implies d(Sx,Sy) < \varepsilon;$$

(ii) $Tx = Ty \implies Sx = Sy$;

(iii) $S(X) \subset T(X)$;

(iv) Either S or T is continuous.

Then S and T have a unique common fixed point.

Proof: Construct a sequence $\{\xi_n\}$ such that $S\xi_n = T\xi_{n+1}$. From (i), for $Tx \neq Ty$,

$$d(Sx,Sy) < J(d(Tx,Ty), d(Tx,Ty)).$$

Thus,

$$\begin{aligned} d(S\xi_n, S\xi_{n+1}) &< J(d(T\xi_n, T\xi_{n+1}), d(T\xi_n, T\xi_{n+1})) \\ &= J(d(S\xi_{n-1}, S\xi_n), d(S\xi_{n-1}, S\xi_n)) \\ &\leq d(S\xi_{n-1}, S\xi_n). \end{aligned}$$

So $\{d(S\xi_n, S\xi_{n+1})\}$ decreases to some $l \geq 0$. Suppose $l > 0$. Then for $h > 0$, there exists $N \in \mathbb{N}$ such that for $m \geq N$,

$$l \leq d(S\xi_m, S\xi_{m+1}) < J(d(T\xi_m, T\xi_{m+1}), d(T\xi_m, T\xi_{m+1})) < l + h.$$

But $l \leq J(d(S\xi_{m-1}, S\xi_m), d(S\xi_{m-1}, S\xi_m)) \leq d(S\xi_{m-1}, S\xi_m) < l$, a contradiction. Hence, $l = 0$.

The sequence $\{S\xi_n\}$ is Cauchy (proof similar to Theorem 3.1) and converges to $\xi \in X$. Similarly, $\{T\xi_n\} \to \xi$. Assume S is continuous. Then $S(S\xi_n) \to S\xi$ and $S(T\xi_n) \to S\xi$. By L-weak commutativity,

$$d(S(T\xi_n), T(S\xi_n)) \leq L\, d(S\xi_n, T\xi_n),$$

so $T(S\xi_n) \to S\xi$. Suppose $\xi \neq S\xi$. Then no subsequence of $\{S(S\xi_n)\}$ or $\{T(S\xi_n)\}$ converges to ξ. Thus, there exists $a > 0$ and integers s, t such that for $n \geq s$, $m \geq t$, $\inf d(S\xi_n, S(S\xi_m)) = a$. But from (i), $\inf d(S\xi_n, S(S\xi_m)) < a$, a contradiction. Hence, $\xi = S\xi$. The rest follows as in Theorem 3.1. □

Definition 3.2 Let (X, d) be a metric space. A mapping $T : X \to X$ is sequentially convergent if for every sequence $\{y_n\}$, convergence of $\{Ty_n\}$ implies convergence of $\{y_n\}$.

Theorem 3.3 *Let (X, d) be a complete metric space and $S, T : X \to X$ be continuous mappings satisfying:*

(i) There exists $J \in \mathcal{C}$ such that S is a C-T-contraction;

(ii) T is injective and sequentially convergent.

Then S has a unique fixed point in X.

Proof: Let $\xi_0 \in X$. Define $\xi_{n+1} = S\xi_n = S^{n+1}\xi_0$ and $\psi_n = T\xi_n$. If $\psi_{n_0+1} = \psi_{n_0}$ for some n_0, then $T\xi_{n_0+1} = T\xi_{n_0}$, so $\xi_{n_0+1} = \xi_{n_0}$ by injectivity, and $S\xi_{n_0} = \xi_{n_0}$. Assume $d(\psi_n, \psi_{n+1}) > 0$ for all n. Then

$$\begin{aligned} d(TS\xi_n, TS\xi_{n+1}) &< J(d(T\xi_n, T\xi_{n+1}), d(T\xi_n, T\xi_{n+1})) \\ &= J(d(\psi_n, \psi_{n+1}), d(\psi_n, \psi_{n+1})) \\ &\leq d(\psi_n, \psi_{n+1}). \end{aligned}$$

Thus $d(\psi_{n+1}, \psi_{n+2}) \leq d(\psi_n, \psi_{n+1})$, so $\{d(\psi_n, \psi_{n+1})\}$ decreases to $\varepsilon \geq 0$. If $\varepsilon > 0$, then

$$d(\psi_n, \psi_{n+1}) < J(d(\psi_{n-1}, \psi_n), d(\psi_{n-1}, \psi_n)) \leq d(\psi_{n-1}, \psi_n).$$

Letting $n \to \infty$, $\varepsilon \leq J(\varepsilon, \varepsilon) \leq \varepsilon$, contradiction. Hence $\varepsilon = 0$.

Now $\{\psi_n\}$ is Cauchy (proof similar to Theorem 3.1) and converges to $\psi \in X$. Since T is sequentially convergent, $\{\xi_n\}$ converges to $\xi \in X$. By continuity of T, $T\xi = \psi$. By continuity of TS,

$$\psi = \lim_{n\to\infty} TS\xi_n = TS\xi.$$

Thus $TS\xi = T\xi$, so $S\xi = \xi$ by injectivity. Uniqueness follows as before. □

Theorem 3.4 *Let (X, d) be a complete metric space and $S, T : X \to X$ be continuous mappings satisfying:*

(i) For every $\varepsilon > 0$, there exist $k(\varepsilon) > 0$ and $J(\varepsilon) \in \mathcal{C}$ such that

$$\varepsilon \leq J(d(Tx, Ty), d(Tx, Ty)) < \varepsilon + k \implies d(TSx, TSy) < \varepsilon;$$

(ii) T is injective and sequentially convergent.

Then S has a unique fixed point in X.

Proof: Similar to Theorem 3.3, define ξ_n and ψ_n. The sequence $\{d(\psi_n, \psi_{n+1})\}$ decreases to $l \geq 0$. If $l > 0$, then for $k > 0$, there exists $R \in \mathbb{N}$ such that for $m \geq R$,

$$l \leq d(TS\xi_m, TS\xi_{m+1}) < J(d(T\xi_m, T\xi_{m+1}), d(T\xi_m, T\xi_{m+1})) < l + k.$$

Then $l \leq J(d(\psi_m, \psi_{m+1}), d(\psi_m, \psi_{m+1})) \leq d(\psi_m, \psi_{m+1}) < l$, contradiction. Hence $l = 0$. The rest follows as in Theorems 3.2 and 3.3. □

Theorem 3.5 *Let (X, d) be a complete metric space, $G : X \to X$ continuous, and $T : X \to X$ injective, continuous, and sequentially convergent. Suppose for every $\varepsilon > 0$, there exist $\mu > 0$ and $J \in \mathcal{C}$ such that for all $x, y \in X$,*

$$\varepsilon \leq J(K_T(x, y), K_T(x, y)) < \varepsilon + \mu \implies d(TGx, TGy) < \varepsilon, \tag{3.1}$$

where

$$K_T(x, y) = \max\left\{d(Tx, Ty), d(Tx, TGx), d(Ty, TGy), \frac{1}{2}\left[d(Tx, TGy) + d(Ty, TGx)\right]\right\}.$$

Then G has a unique fixed point in X.

Proof: Let $\xi_0 \in X$. Define $\xi_{n+1} = G\xi_n$ and $\psi_n = T\xi_n$. If $\mu_n = d(\psi_n, \psi_{n+1}) = 0$ for some n, then ξ_n is fixed point. Assume $\mu_n > 0$ for all n. Suppose $\mu_{n-1} < \mu_n$ for some n. Then

$$\begin{aligned} K_T(\xi_{n-1}, \xi_n) &= \max\left\{d(\psi_{n-1}, \psi_n), d(\psi_{n-1}, \psi_n), d(\psi_n, \psi_{n+1}), \frac{1}{2}\left[d(\psi_{n-1}, \psi_{n+1}) + d(\psi_n, \psi_n)\right]\right\} \\ &\leq \max\left\{\mu_{n-1}, \mu_n, \frac{1}{2}d(\psi_{n-1}, \psi_{n+1})\right\} \\ &< \mu_n + \mu_{n-1}. \end{aligned}$$

But $K_T(\xi_{n-1}, \xi_n) \geq d(\psi_{n-1}, \psi_n) = \mu_{n-1}$. Thus $K_T(\xi_{n-1}, \xi_n) = \mu_n$, and by (3.1),

$$\mu_n = d(TG\xi_{n-1}, TG\xi_n) < J(K_T(\xi_{n-1}, \xi_n), K_T(\xi_{n-1}, \xi_n)) \leq K_T(\xi_{n-1}, \xi_n) < \mu_n + \mu_{n-1},$$

contradiction. Hence $\mu_n \leq \mu_{n-1}$ for all n, so $\{\mu_n\}$ decreases to $l \geq 0$. If $l > 0$, then for $\delta > 0$, there exists $R \in \mathbb{N}$ such that for $n \geq R$, $l < \mu_n < l + \delta$. Then for $n \geq R+1$,

$$l \leq K_T(\xi_{n-1}, \xi_n) < l + \delta,$$

so $d(TG\xi_{n-1}, TG\xi_n) < l$, contradiction. Thus $l = 0$.

Now $\{\psi_n\}$ is Cauchy (proof similar to Theorem 3.2) and converges to $\psi \in X$. By sequential convergence of T, $\{\xi_n\}$ converges to $\xi \in X$. By continuity of T, $T\xi = \psi$. By continuity of G, $G\xi = \xi$. Uniqueness follows as before. □

4. Illustrating Examples

Example 4.1 *Let $X = \mathbb{R}$ with $d(x,y) = 2|x-y|$. Define $Sx = 1$, $Tx = 2x-1$. Then $S(X) = \{1\} \subset T(X) = \mathbb{R}$. For $x \neq y$,*

$$d(Sx, Sy) = 0 < J(d(Tx,Ty), d(Tx,Ty)) = |x-y| \quad \text{with} \quad J(a,b) = \frac{1}{2}a.$$

Also, $d(STx, TSx) = 0 \leq Ld(Sx,Tx) = L|1-(2x-1)| = 2L|1-x|$ for any $L \geq 0$. Conditions of Theorem 3.1 hold, and 1 is the unique common fixed point.

Example 4.2 *Let $X = [0,1]$, $d(x,y) = |x-y|$, $Sx = \frac{x}{x+3}$, $Tx = x$. Then $d(Sx,Sy) \leq \frac{1}{3}|x-y| < \frac{1}{2}|x-y| = J(d(Tx,Ty), d(Tx,Ty))$ with $J(a,b) = \frac{1}{2}a$. S and T commute, so L-weakly commuting. By Theorem 3.1, 0 is the unique common fixed point.*

Example 4.3 *Let $X = [0,1]$, $d(x,y) = |x-y|$, $Sx = \frac{1}{2}x$, $Tx = x$, $J(x,y) = \ln(1+2x)$. For $\varepsilon > \frac{3}{4}$, choose $\lambda = 2\ln 2 - \varepsilon$. Then*

$$\varepsilon \leq J(d(Tx,Ty), d(Tx,Ty)) = \ln(1+2|x-y|) < \varepsilon + \lambda \implies |x-y| < \frac{3}{2} \implies d(Sx,Sy) = \frac{1}{2}|x-y| < \frac{3}{4} < \varepsilon.$$

S and T commute. By Theorem 3.2, 0 is the unique common fixed point.

Example 4.4 *Let $X = [0,1]$, $Sx = \frac{1}{4}x^2$, $Tx = x^2$, $J(x,y) = \ln(1+x)$. For $\varepsilon > \frac{3}{4}$, choose $\rho = 2\ln 2 - \varepsilon$. Then*

$$\varepsilon \leq J(d(Tx,Ty), d(Tx,Ty)) < \varepsilon + \rho \implies |x^2 - y^2| < 3 \implies d(Sx,Sy) = \frac{1}{4}|x^2-y^2| < \frac{3}{4} < \varepsilon.$$

$d(STx, TSx) = \frac{3}{4}x^4 \leq L \cdot \frac{3}{4}x^2$ for $L \geq x^2 \leq 1$. By Theorem 3.2, 0 is the unique common fixed point.

Example 4.5 *Let $X = [0,1]$, $d(x,y) = |x-y|$, $Sx = \frac{x+1}{3}$, $Tx = x$. Then $d(Sx,Sy) = \frac{1}{3}|x-y| < \frac{1}{2}|x-y| = J(d(Tx,Ty), d(Tx,Ty))$ with $J(a,b) = \frac{1}{2}a$. T is injective and sequentially convergent. By Theorem 3.3, $\frac{1}{2}$ is the unique fixed point of S.*

Example 4.6 *Let $X = [1,20]$, $d(x,y) = |x-y|$,*

$$Sx = \begin{cases} 1 & \text{if } x \in [1,5) \\ \frac{1}{2}(x-3) & \text{if } x \in [5,20] \end{cases}, \quad Tx = x.$$

For various cases of x, y, choose $J(a,b) = \frac{1}{2}a$ and appropriate $\rho(\varepsilon)$ to satisfy condition (i) of Theorem 3.4. T is injective and sequentially convergent. By Theorem 3.4, 1 is the unique fixed point.

Example 4.7 *With X, d, T as above, and*

$$Gx = \begin{cases} 1 & \text{if } x \in [1,5) \\ \frac{1}{2}(x-3) & \text{if } x \in [5,20] \end{cases}.$$

Define $K_T(x,y)$ as in Theorem 3.5. For various cases, choose $J(a,b) = \frac{1}{2}a$ and appropriate $\mu(\varepsilon)$ to satisfy (3.1). By Theorem 3.5, 1 is the unique fixed point.

5. An Application to Partial Differential Equations

We apply common fixed point theory to the nonlinear reaction-diffusion equation:

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u + f(u),$$

where $u(x,t)$ is the spatial distribution at position x and time t, α is the diffusion coefficient, ∇^2 is the Laplacian, and $f(u)$ is a nonlinear reaction term. Define operators:

- $S(u) = \alpha \nabla^2 u$ (linear diffusion),
- $T(u) = f(u)$ (nonlinear reaction).

The PDE becomes $\frac{\partial u}{\partial t} = S(u) + T(u)$. We seek u such that $S(u) = u$ and $T(u) = u$, a common fixed point in an appropriate function space X with Dirichlet boundary conditions.

Consider $f(u) = ru(1 - \frac{u}{K})$ (logistic growth). Define X as continuous functions satisfying boundary conditions. If S and T are L-weakly commuting, continuous, $S(X) \subset T(X)$, and satisfy Theorem 3.1 conditions, then a unique common fixed point exists, solving the PDE.

Iterative Solution:

1. Initialize $u^{(0)}$.
2. Iterate: $u^{(k+1)} = S(u^{(k)}) + T(u^{(k)})$.
3. Terminate when $\|u^{(k+1)} - u^{(k)}\| < \varepsilon$.

The limit is the solution. For example, with $S(u)(x) = \int_0^x u(t)dt$, $T(u)(x) = cu(x)$, the equation $u = S(u) + T(u)$ has a unique solution found iteratively.

Consider subspaces:

$$E = \{x : [-\alpha, \alpha] \to [-\lambda, \lambda] \mid x(0) = 0, \text{ continuous}\},$$
$$F = \{u : [-\alpha, \alpha] \to [-\lambda + 1, \lambda + 1] \mid u(0) = 1, \text{ continuous}\}.$$

Define $S : E \times F \to E + F$ by $S(x, u) = \alpha \frac{\partial^2 u}{\partial x^2}$, $T(x, u) = ru(1 - \frac{u}{K})$. Under appropriate conditions, S and T are L-weakly commuting, continuous, and $S(X) \subset T(X)$. By Theorem 3.1, a unique common fixed point exists, solving the PDE.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper.

Availability of Data and Materials

The data used to support the findings of this study are available from the corresponding author upon request.

Funding

This research received no external funding.

.

References

1. Sessa, S., *On a weak commutativity condition of mappings in fixed point consideration, Publ. Inst. Math. 32, 149–153, (1982).*

2. *Jungck, G.,* Compatible mappings and common fixed points, Int. J. Math. Math. Sci. 9 , 771–779, (1982).

3. Tivari, B. M. L., and Singh, S. L., *A note on recent generalizations of Jungck contraction principle, J. Uttar Pradesh Gov. Coll. Acad. Soc. 3, 13–18, (1986).*

4. *Ansari, A. H.,* Note on φ-ψ-contractive type mappings and related fixed point, The 2nd Regional Conference on Mathematics and Applications, Payame Noor University, pp. 377–380, (2014).

5. Pant, R. P., *Common fixed points of noncommuting mappings, J. Math. Anal. Appl. 188, 436–440, (1994).*

6. *Chi, K. P., Karapinar, E. and Thanh, T. D.,* A generalization of the Meir-Keeler type contraction, Arab J. Math. Sci. 18, 141–148, (2012).

7. Amini-Harandi, A., Fard, M. S., and Rhoades, B. E., *Fixed point of weakly contractive mappings and common fixed points of a family of mappings, Fixed Point Theory Appl, Article ID 18912 (2006).*

8. *Saeidi, S., Saadati, R. and Vaezi, H.,* Common fixed points of generalized weak contractive mappings in partially ordered metric spaces, Math. Sci. 7, 17–23, (2013).

9. S. Udomene and Ntouyas, S. P., *Common fixed point theorems for generalized (ϕ, ψ)-weak contractive mappings in partially ordered metric spaces, J. Inequal. Appl, Article ID 160 (2018).*

M. Iadh Ayari,
Carthage University,
National Institute of Applied Sciences and Technology, Tunisia.
E-mail address: iadh_ayari@yahoo.com

and

Community College of Qatar,
Department of Math and Science, Qatar.
E-mail address: mohammad.ayari@ccq.edu.qa

and

M. Boussoffara,
Sfax University,
Faculty of Science of Sfax, Tunisia.
E-mail address: mariem.boussoffara@yahoo.com