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Generalized Jordan bi-derivations on triangular algebra

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ABSTRACT: In the current investigation, our primary objective is to find the structure of generalized Jordan biderivations on triangular algebra. Infact, we establish that all generalized Jordan biderivations on a triangular algebra will be of the form of an inner derivation. Our proof contains an entirely different approach and conclusion from the existing classical theory in [13] which states that if R is a prime ring of characteristic different from 2, then any Jordan derivation of R is an ordinary derivation.

Key Words: Generalized Jordan bi-derivation, triangular algebra, module-homomorphism.

Contents

1. Introduction

The role of derivation is an iconic instance of mathematical unification as it unifies several fields of study and builds an effective method for analyzing issues of structural significance. There are many significant revolutions in the field of rings, algebra and extended algebraic structures with derivations. In the study of derivations initiated by Posner [16], he proved that the commutative structure of a prime ring must exist once a derivation occurs on it, which is centralizing and non-zero. Quite a lot of research has been done in the related field, particularly in the connections between derivations and algebraic structure.

An additive mapping $\mathcal{D} : R \to R$ is known as a derivation if $\mathcal{D}(ab) = \mathcal{D}(a)b + a\mathcal{D}(b)$ for all $a, b \in R$ and is termed as a Jordan derivation if $\mathcal{D}(b^2) = \mathcal{D}(b)b + b\mathcal{D}(b)$ is true for all $b \in R$. Every derivation is a Jordan derivation, as is evident from the definition, but the converse is often not legitimate. An essential outcome due to [13], affirms that a derivation is a Jordan derivation on a prime ring with characteristic instead of two. The extension of this result is presented for 2-torsion free semiprime ring by Cusack [10] and modified by [7]. A mapping $\mathcal{G}_{\mathcal{D}} : R \to R$ is termed as a generalized derivation if it is additive and there exists an associated derivation $\mathcal{D}$ on R such that $\mathcal{G}_{\mathcal{D}}(ab) = \mathcal{G}_{\mathcal{D}}(a)b + a\mathcal{D}(b)$ for all $a, b \in R$ and particularly if $a = b$, then $\mathcal{G}_{\mathcal{D}}$ is a generalized Jordan derivation associated with a Jordan derivation $\mathcal{D}$ on R.

A biadditive mapping $\mathfrak{J} : R \times R \longrightarrow R$ is said to be Jordan biderivation if its satisfy the condition for every $l, k \in R$

$$\mathfrak{J}(l^2, k) = \mathfrak{J}(l, k)l + l\mathfrak{J}(l, k).$$

The mapping $\mathfrak{J}$ will be called symmetric in case $\mathfrak{J}(l, k) = \mathfrak{J}(k, l)$, for each $l, k \in R$. After combining the two properties together, we can say $\mathfrak{J}$, a symmetric Jordan biderivation. In the same way, a biadditive mapping σ will be called a generalized Jordan biderivation on R if it satisfies for each $l, k \in R$

$$\sigma(l^2, k) = \sigma(l, k)l + l\mathfrak{J}(l, k),$$

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in both slots. Where $\mathfrak{J}$ will be the Jordan biderivation associated with σ.

The author of [9] presented abstract linear mappings for triangular algebra and established several noteworthy findings. He discovers Lie derivations of triangle algebras, automorphisms, commuting mappings, and the structure of derivations. To get a quick overview, see [8,9] and the relevant references. The authors of [1] achieved the findings on characterization of $(\alpha, \beta)-$ biderivations on triangular algebras and provided the generalization of [5].

our next attempt is to describe the triangular structure as: Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital algebras over a commutative ring R, and let $\mathfrak{M}$ be a unital $(\mathfrak{A}, \mathfrak{B})$-bimodule, which is faithful as a left $\mathfrak{A}$-module and also as a right $\mathfrak{B}$-module. Recall that a left $\mathfrak{A}$-module $\mathfrak{M}$ is faithful if $a\mathfrak{M} = 0$ implies that $a = 0$. The algebra

$$\mathfrak{P} = \mathrm{Tri}(\mathfrak{A}, \mathfrak{M}, \mathfrak{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \,\middle|\, a \in \mathfrak{A}, m \in \mathfrak{M}, b \in B \right\}$$

under the usual matrix operations is called a triangular algebra. Let us define two natural projections

$$\pi_{\mathfrak{A}} : \mathfrak{P} \to \mathfrak{A}, \quad \pi_{\mathfrak{A}}\left(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\right) = a, \quad \pi_{\mathfrak{B}} : \mathfrak{P} \to \mathfrak{B}, \quad \pi_{\mathfrak{B}}\left(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\right) = b.$$

By [8, Proposition 3] we know that the center the center $Z(\mathfrak{P})$ of $\mathfrak{P}$ coincides with

$$Z(\mathfrak{P}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \,\middle|\, am = mb \text{ for all } m \in \mathfrak{M} \right\}.$$

Moreover, $\pi_{\mathfrak{A}}(Z(\mathfrak{P})) \subseteq Z(\mathfrak{A})$, $\pi_{\mathfrak{B}}(Z(\mathfrak{P})) \subseteq Z(\mathfrak{B})$, and there exists a unique algebra isomorphism $\tau : \pi_{\mathfrak{A}}(Z(\mathfrak{P})) \to \pi_{\mathfrak{B}}(Z(\mathfrak{P}))$ such that $am = m\tau(a)$ for all $m \in \mathfrak{M}$.

The most natural examples of triangular algebras are upper triangular matrix algebras, block upper triangular matrix algebras and nest algebras.

In [12], authors describe the structure of a prime ring R in which g-derivations and generalized g-derivations satisfy certain algebraic identities with involution, anti-automorphism and automorphism. Another interesting idea for the development of such a hierarchy of mappings is to look at [2,3,4,5,6,11, 14,15,17,18]. Motivated by the study in the mention research, our goal is to find the characterization of generalized Jordan biderivation on triangular algebra in specific conditions.

2. Main theorems

We begin with the following lemmas, which are essential to prove our theorem.

Lemma 2.1 *If σ is a generalized Jordan bi-derivation on $\mathfrak{P}$ possessing Jordan bi-derivation $\mathfrak{J}$ such that $\sigma(e, e) = 0$ and $\sigma(e, b) = 0$, then $\sigma(a, b) = 0 = \sigma(b, a)$ for all $a \in \mathfrak{A}, b \in \mathfrak{B}$ and $\mathfrak{J}(m, n) = 0$ for all $m, n \in \mathfrak{M}$.*

Proof: A generalized Jordan bi-derivation satisfies $\sigma(a, b) = \sigma(b, a)$ and $\sigma(e, e) = 0$. By bilinearity, we have

$$\sigma(a, b) = \sigma(ea, b) = \sigma(e, b)a + e\sigma(a, b).$$

Since $\sigma(e, b) = 0$, this simplifies to

$$\sigma(a, b) = e\sigma(a, b).$$

It follows that $\sigma(a, b) = 0$. Consequently, we also have, by definition
$\mathfrak{J}(m, n) = 0$ for every $m, n \in \mathfrak{M}$. □

Lemma 2.2 *If $\sigma : \mathfrak{P}\times\mathfrak{P}\to\mathfrak{P}$ is a generalized Jordan biderivation associated with a Jordan biderivation $\mathfrak{J} : \mathfrak{P}\times\mathfrak{P}\to\mathfrak{P}$, then*

$$\sigma(x,y)[u,v] = [x,y]\mathfrak{J}(u,v)$$

for all $x,y,u,v\in\mathfrak{P}$.

Proof: Consider $\sigma(xu,yv)$ for arbitrary $x,y,u,v\in\mathfrak{P}$. Since σ is a derivation in the first argument, we have

$$\sigma(xu,yv) = \sigma(x,yv)u + x\mathfrak{J}(u,yv).$$

And since it is also a derivation in the second argument, it follows that

$$\begin{aligned}\sigma(xu,yv) &= \sigma(x,yv)u + xy\mathfrak{J}(u,v) + x\mathfrak{J}(u,y)v\\ &= \sigma(x,y)vu + y\mathfrak{J}(x,v)u + xy\mathfrak{J}(u,v) + x\mathfrak{J}(u,y)v.\end{aligned} \tag{2.1}$$

Also,

$$\sigma(xu,yv) = \sigma(x,y)uv + x\mathfrak{J}(u,y)v + y\mathfrak{J}(x,v)u + yx\mathfrak{J}(u,v). \tag{2.2}$$

By subtracting (2.1) from (2.2), we obtain the following relation

$$\sigma(x,y)[u,v] = [x,y]\mathfrak{J}(u,v).$$

□

Lemma 2.3 *If $\sigma : \mathfrak{P}\times\mathfrak{P}\to\mathfrak{P}$ is a generalized Jordan bi-derivation. Then for all $x\in\mathfrak{P}$:*

1. $\sigma(x,1) = \sigma(1,1)x = \sigma(1,x)$;
2. $\sigma(x,0) = 0 = \sigma(0,x)$;
3. $\sigma(x,y) = e\sigma(x,y)f + f\sigma(x,y)f$.

Proof: Consider for (1)

$$\sigma(1\cdot x,1) = \sigma(1,1)x + 1\mathfrak{J}(x,1).$$

By assumption $\mathfrak{J}(x,1) = 0$, we get

$$\sigma(1\cdot x,1) = \sigma(1,1)x. \tag{2.3}$$

Since σ is symmetric, we have

$$\sigma(x,1) = \sigma(1,x). \tag{2.4}$$

From equations (2.3) and (2.4), we obtain

$$\sigma(x,1) = \sigma(1,1)x = \sigma(1,x).$$

Consider for (2)

$$\sigma(0^2,x) = \sigma(0,x)x + 0\mathfrak{J}(0,x) = 0.$$

By assumption $\mathfrak{J}(0,x) = 0$, we get $\sigma(0^2,x) = \sigma(0,x) = 0$.
Since σ is symmetric, we have

$$\sigma(x,0) = 0 = \sigma(0,x).$$

Let $x,y\in\mathfrak{P}$. Since $\mathfrak{P}$ is a triangular algebra, we can decompose any element $x\in\mathfrak{P}$ using the idempotents e and $f = 1-e$ as follows:

$$x = exe + exf + fe + faf.$$

Therefore, the image of $\sigma(x,y)\in\mathfrak{P}$ can also be written in terms of these blocks

$$\sigma(x,y) = e\sigma(x,y)e + e\sigma(x,y)f + f\sigma(x,y)e + f\sigma(x,y)f.$$

We aim to show that

$$e\sigma(x,y)e = 0, \quad f\sigma(x,y)e = 0.$$

For $e\sigma(x,y)e = 0$: By Lemma 2.1, we are given

$$\sigma(e,e) = 0 = \sigma(f,f).$$

This means σ vanishes on diagonal components. Therefore, the block $e\sigma(x,y)e$ lies in $e\mathfrak{P}e$, and since $\sigma(x,y)$ maps to the off-diagonal or lower block, this diagonal part must vanish:

$$e\sigma(x,y)e = 0.$$

For $f\sigma(x,y)e = 0$: This component lies in $f\mathfrak{P}e$, but since $\mathfrak{P}$ is triangular, $f\mathfrak{P}e = 0$. Therefore, we obtain

$$f\sigma(x,y)e = 0.$$

Hence,

$$\sigma(x,y) = e\sigma(x,y)f + f\sigma(x,y)f.$$

Thus (3) proved. □

Theorem 2.1 *Let $\mathfrak{P} = Tri(\mathfrak{A}, \mathfrak{M}, \mathfrak{B})$ be a triangular algebra, Then:*

1. $\pi_{\mathfrak{A}}(Z(\mathfrak{P})) = Z(\mathfrak{A}) \quad , \pi_{\mathfrak{B}}(Z(\mathfrak{P})) = Z(\mathfrak{B})$;
2. *At least one of the algebras $\mathfrak{A}$ and $\mathfrak{B}$ is non-commutative;*
3. *If $\mathfrak{M}a = 0$, $\mathfrak{M} \in Z(\mathfrak{P})$, $0 \neq a \in \mathfrak{P}$, then $\mathfrak{M} = 0$;*
4. *If each derivation of $\mathfrak{A}$ is inner, then every generalized Jordan bi-derivation $\sigma : \mathfrak{P} \times \mathfrak{P} \to \mathfrak{P}$ satisfying $\sigma(e,e) = 0 = \sigma(f,f)$ is an inner derivation.*

Proof: The center of $\mathfrak{P}$, denoted $Z(\mathfrak{P})$, consists of all elements that commute with every element of $\mathfrak{P}$.

If $z \in Z(\mathfrak{P})$, then z must commute with any element in $\mathfrak{P}$, including the block structure. Specifically, we have

$$\pi_{\mathfrak{A}}(Z(\mathfrak{P})) = Z(\mathfrak{A}) \quad \text{and} \quad \pi_{\mathfrak{B}}(Z(\mathfrak{P})) = Z(\mathfrak{B}),$$

where $\pi_{\mathfrak{A}}$ and $\pi_{\mathfrak{B}}$ are the projections onto $\mathfrak{A}$ and $\mathfrak{B}$, respectively.

Assume that both $\mathfrak{A}$ and $\mathfrak{B}$ are commutative. Then their elements commute within $\mathfrak{P}$, which has the triangular form

$$\mathfrak{P} = \begin{bmatrix} \mathfrak{A} & \mathfrak{M} \\ 0 & \mathfrak{B} \end{bmatrix}.$$

However, the presence of $\mathfrak{M}$ introduces non-commutativity unless $\mathfrak{M} = 0$, which contradicts the structure of $\mathfrak{P}$. Therefore, at least one of $\mathfrak{A}$ or $\mathfrak{B}$ must be non-commutative.

Let $\mathfrak{M} \in Z(\mathfrak{P})$ and suppose there exists a nonzero element $x \in \mathfrak{P}$ such that $\mathfrak{M}x = 0$. Write the elements as

$$x = \begin{bmatrix} x_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathfrak{M} = \begin{bmatrix} m_1 & m \\ 0 & m_2 \end{bmatrix}.$$

Then,

$$\mathfrak{M}x = \begin{bmatrix} m_1x_1 & mx_1 \\ 0 & 0 \end{bmatrix}.$$

Given that $\mathfrak{M} \in Z(\mathfrak{P})$, we must have $\mathfrak{M}x = x\mathfrak{M}$, and both lie in $Z(\mathfrak{P})$. Since $x \neq 0$, and $\mathfrak{M}x = 0$, it follows that

$$m_1x_1 = 0, \quad mx_1 = 0.$$

Because $x_1 \neq 0$, this implies $m_1 = 0$ and $m = 0$.
Now take

$$y = \begin{bmatrix} 0 & 0 \\ 0 & y_2 \end{bmatrix} \in \mathfrak{P}.$$

Then,

$$\mathfrak{M}y = \begin{bmatrix} 0 & my_2 \\ 0 & m_2y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & m_2y_2 \end{bmatrix}.$$

Using the same argument, for arbitrary y_2, and since $\mathfrak{M}y = y\mathfrak{M}$, we get $m_2 = 0$. Hence:

$$\mathfrak{M} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which implies $\mathfrak{M} = 0$.

Let $\sigma : \mathfrak{P} \times \mathfrak{P} \to \mathfrak{P}$ be a generalized Jordan bi-derivation such that $\sigma(e, e) = \sigma(f, f) = 0$, and assume every derivation of $\mathfrak{A}$ is inner.

The diagonal idempotents in the triangular algebra $\mathfrak{P} = \mathrm{Tri}(\mathfrak{A}, \mathfrak{M}, \mathfrak{B})$ are

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, any element $x \in \mathfrak{P}$ can be decomposed as

$$x = exe + exf + fxe + fxf.$$

Since $\sigma(e, e) = \sigma(f, f) = 0$, it follows that $\sigma(x, x) = 0$ when $x = e$ or $x = f$, meaning σ vanishes on the diagonal blocks $e\mathfrak{P}e$ and $f\mathfrak{P}f$.

Hence, the behavior of σ is determined entirely by its action on the off-diagonal component $e\mathfrak{P}f \cong \mathfrak{M}$.

Now, since every derivation $\Gamma : \mathfrak{A} \to \mathfrak{A}$ is inner, there exists $p_{\mathfrak{A}} \in \mathfrak{A}$ such that:

$$\Gamma(a) = [p_{\mathfrak{A}}, a], \quad \forall a \in \mathfrak{A}.$$

Using the standard form of generalized Jordan bi-derivations and their structure on triangular algebras, we conclude that σ acts like an inner generalized derivation.

Therefore, there exists an element $p \in \mathfrak{P}$ such that:

$$\sigma(x, y) = [p, xy - yx], \quad \forall x, y \in \mathfrak{P},$$

which shows that σ is an inner derivation. □

3. Discussion

We raised a point of discussion that is it always true that such inner derivation exists in the existing hypothesis of our main theorem? We also consider this question as a limitation of the structure of innerness. We pose the following example to give an answer to such a question:

Example 3.1 *Let A be an algebra with identity that has a non-inner derivation $\delta : A \to A$. Let $\mathcal{A} = Tri(A, A, A)$. Then the map $d : \mathcal{A} \to A$ defined by*

$$d\left(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} \delta(a) & \delta(m) \\ 0 & \delta(b) \end{pmatrix}$$

is a non-inner derivation.

Hence we conclude that the innerness of the existing derivation is necessary to validate the obtained theorem 2.1 in the setting of a triangular structure.

4. Conclusion

The current study aims to describe the structure of generalized Jordan biderivation on triangular algebra. We derive the following fact as an application: all generalized Jordan biderivations on triangular algebra will have the inner derivation form. Seeing how such mappings affect triangular matrix algebra, nest algebra, block matrices (upper triangular), etc., will be interesting for future research. In addition to the established results, our findings are useful in ternary mapping, generalized triangular matrix algebra, and related fields. We advance our knowledge of how these mappings interact with the underlying algebraic structures by exploring the idea to permuting n-derivation, skew n-derivation, Jordan skew derivations, and higher derivation in respect of [17], etc. Our findings lay the basis for future research into how these maps behave in various other settings. We successfully bridge gaps between classical ring theory, operator algebra and functional analysis.

Conflict of interest. The authors declare that there are no conflicts of interest.

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