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A Novel Approach using Residual Power series Method for solving nonlinear fractional partial differential equation

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ABSTRACT: This paper aims to introduce a modified method for the residual power series method (RPSM) by combining it with a novel transformation, namely the g_{mn} transformation, which is generalized for many integral transformations such as Laplace, Fourier, Elzaki and others. Moreover, a new approach is based on the residual power series method and the proposed formula for residual power series. MRPSM stands for a novel approach that reduces the steps of the RPSM method and improves the accuracy. Also by using g_{mn} transformation, MRPSM is considered to be generalized for many methods using various transformations such as Laplace, Elzaki and other transformations. We deal with an important fractional pde equation, the Newell-Whithead-Segel equation. In this paper, we provide a general solution of the general form for this equation and also some theorems, example and theorem are given.

Key Words: integral transformation, transformation, residual, power series, nonlinear differential equation.

Contents

1. Introduction

Many researchers are working on the development of the residual power series method RPSM [11] . In [13], M. I. Liaqat introduced a mixture of an integral transformation and the residual power series method RPSM to develop this method, namely the Aboohd transformation. Also in [12], M. I. Liaqat introduces a study with Laplace transform and RPSM. In [7], Geem introduced a novel integral transform, namely theg_{mn} transform, which is considered to be generalized for many integral transforms. In this paper, we introduce a combination of g_{mn} transform and RPSM. The combination of a general integral transform and RPSM is introduced step by step, starting with some important theorems of fractional integration and the effect of this transform on them, and then going into the method and finding the elements of the residuals and finally arriving at the general solution. As an important application of this method, we have discussed the nonlinear fractional partial differential equation and the Newell-Whithead-Segel equation.

2. Fundamental Concepts

Definition 2.1 *[2]: g-transformation $g_{mn}(f(X))$ for a function $f(x)$ where $x \in [0,\infty[$ is defined by the following integral:*

$$g_{mn}(f(X)) = S^m \int_0^\infty e^{-s^n x} f(x)dx = F_{mn}(s)$$

Such that the integral is convergent, s is positive constant.

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Table 1: g_{mn}-transformation for selected functions

NO	Functions $f(x)$	$g_{mn}(f(X)) = S^m \int_0^\infty e^{-s^n x} f(x)dx = F_{mn}(s)$
1	K k constant	ks^{m-n}
2	$Sin(ax)$	$\frac{as^m}{s^{2n}+a^2}$
3	$cos(ax)$	$\frac{s^{m+n}}{s^{2n}+a^2}$
4	$Sinh(ax)$	$\frac{as^m}{(s^n-a)(s^n+a)}$
5	$cosh(ax)$	$\frac{as^{n+m}}{(s^n-a)(s^n+a)}$
6	x^k	$k!s^{m-(k+1)n}$

Proposition 2.1 *[6]*

1. $g_{mn}(f'(x)) = s^n g_{mn}(f(x)) - s^m f(0)$
2. $g_{mn}(f''(x)) = s^{2n} g_{mn}(f(x)) - s^m[s^n f(0) + f'(0)]$
3. $g_{mn}(f^k(x)) = s^{kn} g_{mn}(f(x)) - s^m \sum_{i=0}^{k-1} s^{(k-i-1)n} f^{(i)}(0)$

Definition 2.2 *[13]: Let $n \in R^+$, then the operator J_a^n defined on $L_1[a,b]$ as following that*

$$J_a^n f(X) = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t)dt$$

for $a \le x \le b$ is called the Riemann-Liouville fractional integral operator of order n.
For n=0 we set $J_a^n = I$, the identity operator.

Theorem 2.1 (13) *: Let $m, n \ge 0$ and $\phi \in L_1[a,b]$ then:*

$$J_a^m J_a^n \phi(X) = J_a^n J_a^m \phi = J_a^{m+n} \phi$$

Definition 2.3 *(Riemann-Liouville Derivatives)[14]: Let $n \in R^+$ and $m = [n]$, then the Riemann-Liouville fractional differential operator define as following*

$$D_a^n f(X) = D^m J_a^{m-n} f(X)$$

Proposition 2.2 *:*

$$g_{mn}(J^\alpha f(x)) = s^{-\alpha n} g_{mn}(f(x))$$

Proof:

By using definition (2.2)

$$g_{mn}(J^\alpha f(x)) = g_{mn}\left(\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt\right)$$

$$= \frac{1}{\Gamma(\alpha)} g_{mn}(h * f)(x) = \frac{s^{-m}}{\Gamma(\alpha)} g_{mn}(h(x)) g_{mn}(f(x))$$

where $h(x) = x^{\alpha-1}$. We note that $g_{mn}(x^{\alpha-1}) = \Gamma(\alpha)s^{m-\alpha n}$
Thus

$$g_{mn}(J^\alpha f(x)) = \frac{s^{-m}}{\Gamma(\alpha)} g_{mn}(h(x)) g_{mn}(f(x)) = \frac{s^{-m}}{\Gamma(\alpha)} \Gamma(\alpha) s^{m-\alpha n} g_{mn}(f(x))$$

$$g_{mn}(J^\alpha f(x)) = s^{-\alpha n} g_{mn}(f(x))$$

Proposition 2.3 *: Let $\alpha \geq 0, k-1 < \alpha < k, k \in N$*

$$g_{mn}(D^{\alpha}f(x)) = s^{\alpha n}g_{mn}(f(x)) - s^{m}\sum_{i=0}^{k-1} s^{n(\alpha-1-i)}(s)f^{(i)}(0)$$

Proof:

$$g_{mn}(D^{\alpha}f(x)) = g_{mn}(J^{\alpha}f^{(k)}(x)) = s^{\alpha n - kn}g_{mn}(f^{(k)}(x))$$

since $g_{mn}(f^{(k)}(x)) = s^{kn}g_{mn}(f(x)) - s^{m}\sum_{i=0}^{k-1} s^{n(k-1-i)}(s)f^{(i)}(0)$ *we have*

$$g_{mn}(D^{\alpha}f(x)) = s^{\alpha n - kn}\left[s^{kn}g_{mn}(f(x)) - s^{m}\sum_{i=0}^{k-1} s^{n(k-1-i)}(s)f^{(i)}(0)\right]$$

$$g_{mn}(D^{\alpha}f(x)) = s^{\alpha n}g_{mn}(f(x)) - s^{m}\sum_{i=0}^{k-1} s^{n(\alpha-1-i)}(s)f^{(i)}(0)$$

Definition 2.4 *[6]: From Definition(2.1) we can define:*

$$g_{mn}(u(x,t)) = s^{m}\int_{0}^{\infty} e^{-s^{n}x}u(x,t)dx = U_{mn}(s)$$

such that $u(x,t)$ is a function of x,t.

Proposition 2.4 *[6]*

1. $g_{mn}(u_t(x,t)) = s^{n}U_{mn}(x,s) - s^{m}u(x,0)$
2. $g_{mn}(u_t^{k}(x,t)) = s^{kn}U_{mn}(x,s) - s^{m}\sum_{i=0}^{k-1} s^{(k-i-1)n}u_t^{(i)}(x,0)$
3. $g_{mn}(u_x(x,t)) = \frac{\partial}{\partial x}U_{mn}(x,s)$
4. $g_{mn}(u_x^{k}(x,t)) = \frac{\partial}{\partial x^{k}}U_{mn}(x,s)$

Definition 2.5 *[11]: Let $0 < \alpha \leq 1$ then the power series representation in the following form:*

$$\sum_{k=0}^{\infty} a_r(x)(t-t_0)^{k\alpha} = a_0(t-t_0)^{0} + a_1(t-t_0)^{\alpha} + a_2(t-t_0)^{2\alpha} + ...$$

where $x \in R$, it is called a multiple fractional power series (MFPS) about t_0

Proposition 2.5 *: Let $u(x,t)$ be a function and $g_{mn}(u(x,t)) = U_{mn}(x,s)$ then*

$$g_{mn}(D^{r\alpha}u(x,t)) = s^{r\alpha n}U_{mn} - s^{m}\sum_{i=0}^{r-1} s^{n(\alpha(r-1)-1-i)}(s)D^{i\alpha}u(x,0) \tag{2.1}$$

Proof:
By using mathematical induction, we get:
i)if r=1 then by Proposition(2.3) we get the result.
ii)Suppose the result is true at r=k, i.e.

$$g_{mn}(D^{k\alpha}u(x,t)) = s^{k\alpha n}U_{mn} - s^{m}\sum_{i=0}^{k-1} s^{n(\alpha(k-1)-1-i)}(s)D^{i\alpha}u(x,0)$$

iii) if r=k+1 then by taking g_{mn}-transform for $D^{(k+1)\alpha}$ we obtain

$$g_{mn}\left(D^{\alpha}\left(D^{k\alpha}u(x,t)\right)\right)$$

Now suppose that $D^{k\alpha}u(x,t)=h(x,t)$ then we get

$$g_{mn}\left(D^{\alpha}\left(D^{k\alpha}u(x,t)\right)\right)=g_{mn}\left(D^{\alpha}h(x,t)\right)=s^{\alpha n}g_{mn}\left(h(x,t)\right)-s^{m-(\alpha-1)n}h(x,0)$$

$$=s^{n\alpha}g_{mn}\left(D^{k\alpha}u(x,t)\right)-s^{m-(\alpha-1)n}D^{k\alpha}u(x,0)$$

$$=s^{n\alpha}\left(s^{k\alpha n}U_{mn}-s^{m}\sum_{i=0}^{k-1}s^{n(\alpha(k-i)-1)}(s)D^{i\alpha}u(x,0)\right)-s^{m-(\alpha-1)n}D^{k\alpha}u(x,0)$$

$$=s^{(k+1)\alpha n}U_{mn}-s^{m}\sum_{i=0}^{k-1}s^{n(\alpha((k+1)-i)-1)}(s)D^{i\alpha}u(x,0))-s^{m-(\alpha-1)n}D^{k\alpha}u(x,0)$$

$$=s^{(k+1)\alpha n}U_{mn}-s^{m}\sum_{i=0}^{k+1-1}s^{n(\alpha((k+1)-i)-1)}(s)D^{i\alpha}u(x,0))$$

Therefore the result is hold for all $k\in Z^{+}$

Lemma 2.1 *Let $u(x,t)$ has multiple fractional Taylor's series (MFTS) representation as follows:*

$$u(x,t)=a_0(x)+a_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)}+a_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+... \tag{2.2}$$

then

$$g_{mn}(u(x,t))=a_0(x)+a_1(x)s^{m-(\alpha+1)n}+...=\sum_{k=0}^{\infty}a_k(x)s^{m-(k\alpha+1)n} \tag{2.3}$$

Proof: *since $u(x,t)=a_0(x)+a_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)}+a_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+...$*
then by taking g_{mn}-transform for Eq.(2.2) we obtain:

$$g_{mn}(u(x,t))=a_0(x)+a_1(x)\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)}s^{m-(\alpha+1)n}+...=\sum_{k=0}^{\infty}a_k(x)s^{m-(k\alpha+1)n}$$

Proposition 2.6 *: Let $u(x,t)$ be a function and $g_{mn}(u(x,t))=U_{mn}(x,s)$ has (MFTS) then*

$$a_0(x)=\lim_{s\to\infty}s^{n-m}g_{mn}(u(x,t))=u(x,0)$$

$$a_1(x)=\lim_{s\to\infty}s^{(\alpha+1)n-m}g_{mn}(u(x,t))-s^{\alpha n}a_0(x)=D^{\alpha}u(x,0)$$

$$a_2(x)=\lim_{s\to\infty}s^{(2\alpha+1)n-m}g_{mn}(u(x,t))-s^{2\alpha n}a_0(x)=D^{2\alpha}u(x,0)$$

$$\vdots$$

$$a_k(x)=D^{k\alpha}u(x,0)$$

Proof:

From Eq.(2.3) we have $g_{mn}(u(x,t))=a_0(x)+a_1(x)s^{m-(\alpha+1)n}+...=\sum_{k=0}^{\infty}a_k(x)s^{m-(k\alpha+1)n}$

Thus

$$a_0(x) = s^{n-m} g_{mn}(u(x,t)) - s^{\alpha n} a_1(x) - ...$$

Hence

$$a_0(x) = \lim_{s\to\infty} s^{n-m} g_{mn}(u(x,t)) = u(x,0)$$

Similarly we can prove that

$$a_1(x) = \lim_{s\to\infty} s^{(\alpha+1)n-m} g_{mn}(u(x,t)) - s^{\alpha n} a_0(x)$$

$$= \lim_{s\to\infty} s^{(\alpha+1)n-m} g_{mn}(u(x,t)) - s^{\alpha n} u(x,0)$$

$$= \lim_{s\to\infty} s^{(n-m)} \left(s^{\alpha} g_{mn}(u(x,t)) - s^{m+(\alpha-1)n} u(x,0) \right)$$

$$a_1(x) = \lim_{s\to\infty} s^{(\alpha+1)n-m} g_{mn}(u(x,t)) - s^{\alpha n} a_0(x) = D^{\alpha} u(x,0)$$

Also we can prove that $a_k(x) = D^{k\alpha} u(x,0)$ *by the same process*

3. Outline method of MSuad Residual Power Series

In this section we take the form of fractional partial differential equation:

$$D_t^{r\alpha} u(t,x) + L(u(t,x)) + N(u) = f(t,x); D^{d\alpha} u(x,0) = w_d(x), d = 0,1,\dots,r-1, 0 < \alpha < 1 \tag{3.1}$$

Where L is linear partial operator with respect to t of with order q, N is a nonlinear operator , $f(t,x)$ is a function. If we take g_{mn}-transformation of both sides for Eq.(3.1) we get:

$$g_{mn}(D_t^{r\alpha}(u(t,x))) + g_{mn}(L(u(t,x))) + g_{mn}(N(u)) = g_{mn}(f(t,x))$$

$$S^{r\alpha n} g_{mn}(u(t,x)) - S^m \sum_{i=0}^{r-1} S^{(\alpha(r-i)-1)n} D^{i\alpha} u(x,0) + g_{mn}(L(u(t,x)) + N(u) - f(x,t)) = 0$$

$$g_{mn}(u(t,x)) = S^{m-r\alpha n} \sum_{i=0}^{r-1} S^{(\alpha(r-i)-1)n} D^{i\alpha} u_j(x,0) - S^{-r\alpha n} g_{mn}(L(u(t,x)) + N(u) - f(x,t)) \tag{3.2}$$

By Eq.(2.3) we have

$$U(x,s) = g_{mn}(u(x,t)) = \sum_{k=0}^{\infty} a_k(x) S^{(m-(k\alpha+1)n)}, a_{kj}(x) = D^{k\alpha} u_j(x,0) \tag{3.3}$$

Obtain the pth-truncated series of

$$U_p = a_p(x) S^{(m-(p\alpha+1)n)} + a_{(p+1)}(x) S^{(m-((p+1)\alpha+1)n)}, p = 0,1,2,\dots \tag{3.4}$$

Now , from Eq.(3.2) we construct residual, namely multistep MSuaad residual function (MMSRF) as:

$$MSRes_{p+1}(x,s) = U_p - S^{(m-r\alpha n)} \sum_{i=0}^{r-1} S^{(\alpha(r-i)-1)n} D^{i\alpha} u(x,0) + S^{-r\alpha n} g_{mn}(L(u(t,x)) + N(u) - f(x,t))$$

$$= \sum_{i=0}^{p} a_i(x) S^{m-(i\alpha+1)n} - S^{(m-r\alpha n)} \sum_{i=0}^{r-1} S^{(\alpha(r-i)-1)n} D^{i\alpha} u(x,0) + S^{-r\alpha n} g_{mn}(L(u(t,x)) + N(u) - f(x,t))$$

By multiply both sides by $S^{(((p+1)\alpha+1)n-m)}$ we get

$$s^{((p+1)\alpha+1)n-m} MSRes_{p+1}(x,s) = s^{((p+1)\alpha+1)n-m} \left(U_p - S^{m-r\alpha n} \sum_{i=0}^{r-1} s^{(\alpha(r-i)-1)n} D^{i\alpha} u(x,0) \right)$$

$$+ s^{-r\alpha n} g_{mn}(L(u(t,x)) + N(u) - f(x,t)) \quad (3.5)$$

Finally, we solve the equation for $a_{p+1}(x)$:

$$\lim_{s\to\infty} S^{((p+1)\alpha+1)n-m} MSRes_{p+1}(x,s) = 0, \quad (3.6)$$

And then we can find $U(x,s)$ and by using inverse of g_{mn}-transform we get the solution $u(x,t)$.

4. Solving the fractional Newell-Whithead-Segel equation

$$D_t^{\alpha} u - k u_{xx} + cu + eu^c = 0, u(x,0) = \phi(x), c > 2, c, e, k > 0 \quad (4.1)$$

By taking g_{mn}-transform for both sides we get:

$$S^{\alpha n} U - S^{m+(\alpha-1)n} u(x,0) - k \frac{\partial^2 U}{\partial x^2} + cU + e g_{mn} \left(g_{mn}^{-1}(U)^c \right) = 0$$

Thus

$$U = s^{m-n} \phi(x) + s^{-\alpha} k \frac{\partial^2 U}{\partial x^2} - c s^{-\alpha} U - e s^{-\alpha} g_{mn}(\left[g_{mn}^{-1}(U) \right]^c) = 0$$

Now by using Eq.(3.4) we get:

$$U_p = a_p(x) s^{m-(p\alpha+1)n} + a_{p+1}(x) s^{m-((p+1)\alpha+1)n}, a_0(x) = u(x,0)$$

Hence

$$MSRes_{p+1}(x,s) = U_p - s^{m-n}\phi(x) + s^{-\alpha} g_{mn} \left(-k \frac{\partial^2}{\partial x^2} g_{mn}^{-1}(U_p) + c g_{mn}^{-1}(U_p) + e(g_{mn}^{-1}(U_p))^c \right)$$

By multiply both sides by $S^{((p+1)\alpha+1)n-m}$ and using the relation :

$$\lim_{s\to\infty} s^{((p+1)\alpha+1)n-m} MSRes_{(p+1)}(x,s) = 0$$

We get $a_{p+1}(x)$. If we continue with this process we get another elements of $a_i(x)$. that means we can find $U(x,s)$ in Eq.(3.3) and by using inverse of g_{mn}-transform we have $u(x,t)$.

Example 4.1

$$D_t^{\alpha} u - u_{xx} - 2u + 3u^2 = 0, u(x,0) = \lambda, 0 < \alpha \le 1$$

We know that

$$U_p = a_p(x) s^{m-(p\alpha+1)n} + a_{p+1}(x) s^{m-((p+1)\alpha+1)n}$$

Since

$$MSRes_{(p+1)}(x,s) = U_p - s^{m-n}\phi(x) + s^{-\alpha} g_{mn} \left(\frac{\partial^2}{\partial x^2} g_{mn}^{-1}(U_p) - 2 g_{mn}^{-1}(U_p) + 3(g_{mn}^{-1}(U_p))^2 \right)$$

Thus we have :

$$MSRes_{(p+1)}(x,s) = a_p(x)s^{m-(p\alpha+1)n} + a_{p+1}(x)s^{m-((p+1)\alpha+1)n} - s^{m-n}\phi(x)$$

$$+s^{-\alpha}g_{mn}(\frac{\partial^2}{\partial x^2}g_{mn}^{-1}(a_p(x)s^{m-(p\alpha+1)n} + a_{p+1}(x)s^{m-((p+1)\alpha+1)n})$$
$$-2g_{mn}^{-1}(a_p(x)s^{m-(p\alpha+1)n} + a_{p+1}(x)s^{m-((p+1)\alpha+1)n}))$$
$$+3((g_{mn}^{-1}(a_p(x)s^{m-(p\alpha+1)n} + a_{p+1}(x)s^{m-((p+1)\alpha+1)n}))^2)$$

$$= a_p(x)s^{m-(p\alpha+1)n} + a_{(p+1)}(x)s^{m-((p+1)\alpha+1)n}$$
$$-s^{m-n}\phi(x) + (a_p^{''}(x)s^{m-((p+1)\alpha+1)n} + a_{(p+1)}^{''}(x)s^{m-((p+2)\alpha+1)n})$$
$$-2(a_p(x)s^{m-((p+1)\alpha+1)n} + a_{(p+1)}(x)s^{m-((p+2)\alpha+1)n}) + 3a_p^2\frac{\Gamma(2p\alpha+1)}{\Gamma^2(p\alpha+1)}s^{m-(2p\alpha+1)n}$$
$$+6\frac{\Gamma((2p+1)\alpha+1)}{\Gamma((p+1)\alpha+1)}s^{m-(2(p+1)\alpha+1)n} + 3a_{p+1}^2\frac{\Gamma(2(p+1)\alpha+1)}{\Gamma^2((p+1)\alpha+1)}s^{m-((2p+3)\alpha+1)n}$$

By multiply both sides by $s^{(p+1)\alpha-m}$ *and using the relation* $\lim_{s\to\infty} s^{((p+1)\alpha+1)n-m}MSRes_{(p+1)}(x,s) = 0$ *We get*

$$a_{p+1}(x) + a_p^{''}(x) - 2a_p(x) = 0$$

Therefore

$$a_{(p+1)}(x) = -a_p^{''}(x) + 2a_p(x)$$

By above relation we obtain

$$a_0(x) = \lambda, a_1(x) = 2\lambda, a_2(x) = 2^2\lambda, \ldots, a_n(x) = 2^n\lambda$$

Hence

$$U(x,s) = g_{mn}(u(x,t)) = \sum_{k=0}^{\infty} a_k(x)s^{m-(k\alpha+1)n} = \sum_{k=0}^{\infty} 2^k\lambda s^{m-(k\alpha+1)n}$$

By using inverse of gmn-transform we have $u(x,t)$

$$u(x,t) = \lambda\sum_{k=0}^{\infty} 2^k\frac{t^{k\alpha}}{\Gamma(k\alpha+1)} = \lambda\sum_{k=0}^{\infty}\frac{2t^{k\alpha}}{\Gamma(k\alpha+1)} = \lambda E_\alpha(2t^{k\alpha})$$

To show the rapid of convergence of the MRPSM , we compute the error between the exact solution and a new approach . We make a Table-2 when $\alpha = 1$ *and* $\lambda = 0.001$ *and then we compute the consecutive absolute errors. Fig-1 Show the difference between the surface of exact solution and our approach.*

Table 2: The difference between exact solution and Approach solution where $\alpha = 1, \lambda = 0.001$

t	u(x,t)-Exact	u(x,t)-Approach	Absolute Error
0.1	0.001220997	0.001221403	4.05498E-07
0.2	0.001490725	0.001491825	1.09976E-06
0.3	0.001819875	0.001822119	2.24423E-06
0.4	0.002221457	0.002225541	4.08373E-06
0.5	0.002711294	0.002718282	6.98815E-06
0.6	0.003308602	0.003320117	1.15145E-05
0.7	0.004036701	0.0040552	1.84994E-05
0.8	0.004923836	0.004953032	2.91961E-05
0.9	0.006004169	0.006049647	4.54784E-05

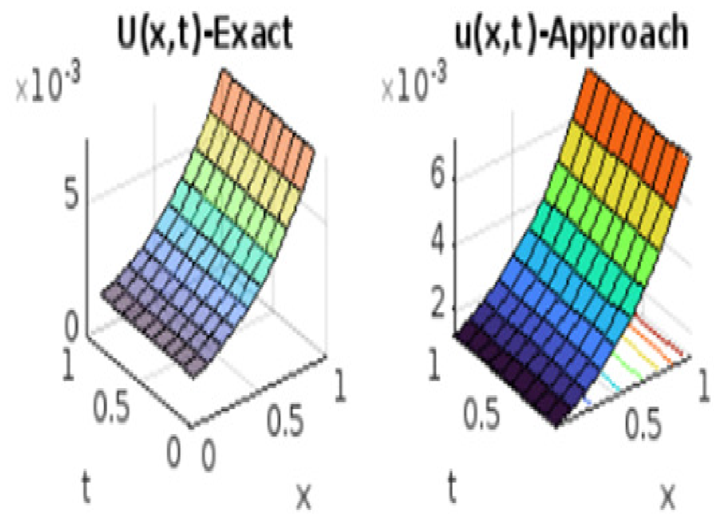

Figure 1: This cat is a eps file

Conclusion:
This work has saved the effort of many researchers such that the application of this g_{mn}-transformation is a general representation of many integral transformations, and the analytical results confirm this, as the results of integrating a transformation such as Laplace can be derived by taking values m=0 , n=1. This paper introduces a novel approach developed through a new transformation, gmn-transform and the residual power series are introduced to provide a solution. The value of A novel method aims to minimize the amount of computational effort required.

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