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Finite-Time Synchronization Control for Generalized Reaction–Diffusion Systems

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ABSTRACT: This paper establishes a framework for achieving finite-time synchronization (FTSY) in coupled reaction-diffusion systems (RDs) configured in a master-slave arrangement. The systems are governed by partial differential equations (PDEs) with nonlinear reaction terms and homogeneous Neumann boundary conditions. Control laws are designed for the slave system to synchronize with the master system within a predetermined finite time. Using Lyapunov stability (LS) theory, we prove finite-time stability (FTS) of the synchronization error dynamics and derive an explicit settling time formula. Numerical simulations validate the theoretical results using the Degen-Harrison and Lengyel-Epstein models, demonstrating synchronization within approximately five seconds and three seconds, respectively. The approach accommodates spatial coupling and boundary conditions, providing a foundation for applications requiring precise spatiotemporal coordination in chemical, biological, and physical systems.

Key Words: Finite-time synchronization, reaction-diffusion systems, Lyapunov stability, control design

Contents

1. Introduction

Synchronization in RDs represents a widespread phenomenon observed across various physical, chemical, and biological contexts [1]. Recent studies have investigated multiple aspects of this phenomenon, including spiral synchronization via messenger waves in the Belousov-Zhabotinsky reaction [2] and the development of phase-reduction theory for analyzing spatiotemporal rhythms in infinite-dimensional RDs [3,4,5]. Researchers have also examined impulsive synchronization, establishing conditions for global solution existence and equiattractivity in impulsive RDs [6]. Furthermore, both linear and nonlinear control schemes have been proposed to achieve complete synchronization in coupled RDs, with applications demonstrated in the Lengyel-Epstein system [7]. These contributions enhance our understanding of synchronization dynamics in complex spatiotemporal systems while providing analytical and control tools. The study of synchronization in RDs has attracted considerable attention in recent years. Various synchronization types have been investigated, including complete synchronization between coupled systems [8] and phase synchronization of spatiotemporal rhythms [9]. Researchers have developed both linear and nonlinear control schemes to achieve synchronization in different RDs models, such as the Lengyel-Epstein and Degn-Harrison systems. A phase-reduction approach has been formulated to analyze synchronization properties of limit-cycle solutions in infinite-dimensional RDs. Additionally, studies have explored spiral wave synchronization using circularly polarized electric fields, demonstrating that

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spirals in various regimes can be entrained to rotate rigidly with a 1:1 frequency ratio relative to the applied field [10]. These investigations offer valuable insights into controlling and analyzing spatiotemporal patterns in RDs.

Research on asymptotic synchronization in RDs has encompassed various neural network models. [11] examined synchronization in neural networks incorporating reaction-diffusion terms and time-varying delays, deriving both control laws and sufficient conditions for asymptotic synchronization. [12] extended this work to coupled RDs neural networks with mixed delays, employing Lyapunov-Krasovskii functionals to establish sufficient conditions for asymptotic and robust synchronization. [13] investigated synchronization in networks of generalized FitzHugh-Nagumo systems, building upon previous work with specific FitzHugh-Nagumo models. [14] presented conditions for spatial uniformity in one-dimensional RD-PDEs with Neumann boundary conditions, utilizing the Jacobian matrix of the reaction term and the first Dirichlet eigenvalue of the Laplacian operator. They further derived analogous results for synchronization in diffusion-coupled ordinary differential equation networks. Collectively, these studies advance our understanding of synchronization dynamics in diverse RDs. Recent investigations have focused on FTSY in coupled RDs, proposing various control strategies. These include pinning control with adaptive coupling strength adjustment [15,16,17], intermittent control using weighted Lyapunov-Krasovskii functional methods [18], and periodically intermittent control [19,20,21]. These studies address challenges such as time-varying delays and spatial boundary coupling, establishing sufficient conditions for FTSY through techniques including Poincaré inequality and graph theory. The proposed methods have been validated via numerical examples and simulations. Furthermore, practical applications such as image encryption algorithms have demonstrated the utility of these theoretical results.

The paper aims to establish a framework for achieving FTSY in master-slave (M-S) coupled RDs with Neumann boundary conditions, designing control laws, proving stability via Lyapunov function (LF) theory, deriving an explicit settling time, and validating the approach numerically. The paper is organized as follows: Section 2 presents the model description and synchronization error dynamics. Section 3 provides the theoretical framework for FTS and synchronization, including the control design and convergence proof. Section 4 validates the theoretical results through numerical simulations of the Degen-Harrison and Lengyel-Epstein models.

2. Model Description

Recent research has significantly advanced our understanding of generalized RDs. In particular, [22] established the existence of global weak solutions for a broad class of RDs, extending previous theoretical results and demonstrating applications to image restoration problems. Building on this foundation, [23] developed a generalized master equation framework for non-extensive RDs under pressure effects, incorporating Tsallis statistics and introducing novel nonlinear terms. Further theoretical progress was made by [24], who conducted a comprehensive analysis of a generalized Degn-Harrison system, proving not only global existence and boundedness of solutions but also deriving conditions for asymptotic stability and the non-existence of non-constant steady states. A particularly important theoretical advancement was presented by [25], who introduced a gradient structure formulation for RDs with reversible mass-action kinetics. This work extended the Wasserstein metric approach to diffusion equations and demonstrated its applicability to systems involving complex interactions, including electrostatic effects and energy balance considerations, as exemplified by semiconductor equations. These collective contributions have substantially enhanced both the theoretical foundations and practical implementations of generalized RDs across multiple scientific disciplines.

We consider the following RDs as the master system:

$$\begin{cases} \dfrac{\partial U\left(r,v\right)}{\partial v} = d_1\,\Delta U + a\,U + \varkappa_1\big(U,W\big), & r\in\Omega,\ v>0,\\ \dfrac{\partial W\left(r,v\right)}{\partial v} = d_2\,\Delta W + b\,W + \varkappa_2\big(U,W\big), & r\in\Omega,\ v>0,\\ \dfrac{\partial U}{\partial \eta} = \dfrac{\partial W}{\partial \eta} = 0, & r\in\partial\Omega,\ v>0,\\ U\left(r,0\right) = U_0\left(r\right), \quad W\left(r,0\right) = W_0\left(r\right), & r\in\Omega. \end{cases} \tag{2.1}$$

Here:

- $d_1, d_2 > 0$ are the diffusion coefficients for U and W, respectively.
- $a, b \in \mathbb{R}$ are constant linear reaction rates.
- $\varkappa_j(U, W)$ $(j = 1, 2)$ are sufficiently smooth nonlinear interaction functions.
- The homogeneous Neumann boundary conditions $\partial U/\partial\eta = \partial W/\partial\eta = 0$ ensure no-flux across $\partial\Omega$.
- The initial data $U_0, W_0 \in H^1(\Omega)$ are given and satisfy the compatibility conditions with the boundary.

We impose the following Lipschitz continuity assumption on the nonlinearities:

$$\left|\varkappa_j\left(U_1, W_1\right) - \varkappa_j\left(U_2, W_2\right)\right| \leq \sigma_j \left|U_1 - U_2\right| + \mu_j \left|W_1 - W_2\right|, \quad j = 1, 2, \tag{2.2}$$

for all $(U_1, W_1), (U_2, W_2) \in \mathbb{R}^2$, where $\sigma_j, \mu_j \geq 0$ are given constants.

The control system in the slave system is crucial for achieving synchronization with the master system. It ensures that the slave system's behavior aligns with the master system within a finite time, which is essential for applications like secure communication, system coordination, and pattern formation. By applying carefully designed control inputs, the system can mitigate deviations and drive the synchronization error to zero, enabling precise and predictable dynamics. This capability is particularly valuable in real-world scenarios where timely and accurate synchronization is required, such as in chemical reactions, biological processes, and neural networks. The control system thus serves as a bridge to enforce desired behaviors and maintain stability in complex spatiotemporal systems.

To achieve synchronization between systems, we introduce a control strategy applied to a slave system. The control inputs (C_1, C_2) are critical for driving the slave system to track the master system's behavior, which is essential for applications like secure communication and system coordination. The slave system is defined as:

$$\begin{cases} \dfrac{\partial P\left(r,v\right)}{\partial v} = d_1 \Delta P + aP + \varkappa_1\left(P, Q\right) + C_1, & r \in \Omega,\ v > 0, \\ \dfrac{\partial Q\left(r,v\right)}{\partial v} = d_2 \Delta Q + bQ + \varkappa_2\left(P, Q\right) + C_2, & r \in \Omega,\ v > 0, \\ \dfrac{\partial P}{\partial \eta} = \dfrac{\partial Q}{\partial \eta} = 0, & r \in \partial\Omega,\ v > 0, \\ P\left(r,0\right) = P_0\left(r\right), \quad Q\left(r,0\right) = Q_0\left(r\right), & r \in \Omega. \end{cases} \tag{2.3}$$

Define the synchronization error between the slave and master systems as:

$$e(r,v) = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} P - U \\ Q - W \end{pmatrix}. \tag{2.4}$$

The error dynamics are derived by subtracting (2.1) from (2.3):

$$\begin{cases} \dfrac{\partial e_1\left(r,v\right)}{\partial v} = d_1 \Delta e_1 + a e_1 + \varkappa_1(P, Q) - \varkappa_1\left(U, W\right) + C_1, & r \in \Omega,\ v > 0, \\ \dfrac{\partial e_2\left(r,v\right)}{\partial v} = d_2 \Delta e_2 + b e_2 + \varkappa_2(P, Q) - \varkappa_2\left(U, W\right) + C_2, & r \in \Omega,\ v > 0, \\ \dfrac{\partial e_1}{\partial \eta} = \dfrac{\partial e_2}{\partial \eta} = 0, & r \in \partial\Omega, v > 0, \\ e_1\left(r,0\right) = P_0\left(r\right) - U_0\left(r\right), \quad e_2\left(r,0\right) = Q_0\left(r\right) - W_0\left(r\right), & r \in \Omega. \end{cases} \tag{2.5}$$

3. Finite-Time Stability and Synchronization

FTS and FTSY are critical concepts in the study of RDs, particularly for ensuring that systems achieve desired behaviors within a specified time frame. The following definitions and theoretical framework provide the foundation for analyzing these properties in the context of the M-S systems described in Section 2.

Definition 3.1 *The zero solution of (3.1) is said to be globally FTS if it is LS, and for any initial condition e_0, the solution $e(r, v)$ of the system (3.1):*

$$\begin{cases} \dfrac{\partial e(r,v)}{\partial v} = D\Delta e + \mathscr{F}(r, v, e), & r \in \Omega,\ v > 0, \\ \dfrac{\partial e}{\partial \eta} = 0, & r \in \partial\Omega, \\ e(r, 0) = e_0(r), \end{cases} \tag{3.1}$$

becomes identically zero in finite time, i.e., there exists $0 \leq v^* < +\infty$ such that $e(r, v) = 0$. The function $v^*(e_0) = \inf\{v^* : e(r, v) = 0$ for all $r \in \Omega$ and $v \geq v^*\}$ is called the settling time function.

Definition 3.2 ([26]) *e^* is called the finite-time EP of the system (2.5) if there is a $v^* \in \mathbb{R}^+$ such that $e \neq e^*$, $v \in \mathbb{R}^+$, and $e \equiv e^*$, $\forall v \in \mathbb{R}^+$.*

Definition 3.3 *The system given by (2.5) is FTS with respect to $\{\delta, \varepsilon, J\}$, $\delta < \varepsilon$, if $\|e_0\| < \delta$ and $\forall v \in J$, implies $\|L(v)\| < \varepsilon, \forall v \in J$.*

Lemma 3.1 ([27]) *Let $e(r) \in H_0^1(\Omega)$ be a function such that $\left.\frac{\partial e(r)}{\partial \eta}\right|_{\partial\Omega} = 0$. Then,*

$$\nu \int_\Omega |e(r)|^2 \, dr \leq \int_\Omega |\nabla e(r)|^2 \, dr, \tag{3.2}$$

where $\nu > 0$ is an eigenvalue of the following problem:

$$\begin{cases} \nu e(r) = -\Delta e(r), & r \in \Omega, \\ \dfrac{\partial e(r)}{\partial \eta} = 0, & r \in \partial\Omega. \end{cases} \tag{3.3}$$

Theorem 3.1 ([28]) *$e^* = (0, 0)$ is a FTS-EP of the nonlinear system (2.5) if there exists a positive definite LF $L : [0, +\infty) \times \Omega \to \mathbb{R}_+$, three class $\mathcal{K}$ functions η, β, Λ, and a $\delta > 0$ such that*

1. $\eta \|e(v)\| \leq L(v, e(v)) \leq \beta \|e(v)\|,$
2. $\dfrac{\partial L(v, e(v))}{\partial v} < -\Lambda L(v, e(v)),$
3. $\displaystyle\int_0^\varepsilon \frac{de}{\Lambda(e)} < +\infty, \quad (\forall \varepsilon : 0 < \varepsilon \leq \delta).$

Definition 3.4 ([28]) *If there exists a setting time $v^* > 0$ such that*

$$\lim_{v \to \mathfrak{v}^*} \|e_1(v)\| + \|e_2(v)\| = 0, \tag{3.4}$$

and

$$\|e_1(v)\| + \|e_2(v)\| \equiv 0, \quad \forall v \geq v^*, \tag{3.5}$$

then the M-S systems (2.1)-(2.3) are FTSY.

Theorem 3.2 *The M-S systems (2.1)-(2.3) achieve synchronization within a finite time v^* under the control laws:*

$$\begin{cases} C_1(r, v) = -(2a + \sigma_1) e_1 - \mu_1 e_2, \\ C_2(r, v) = -\sigma_2 e_1 - (2b + \mu_2) e_2. \end{cases} \tag{3.6}$$

Specifically, for a fixed time horizon $v_{\max} > 0$*, the settling time* v^* *satisfies:*

$$v^* \;=\; \frac{L_2(0)}{2\min\{d_1\nu_1 + a,\;\; d_2\nu_2 + b\}\, L_2(v_{\max})}, \tag{3.7}$$

where

$$L_2(v_{\max}) = \frac{1}{2}\left(\|e_1(v_{\max})\|_2^2 + \|e_1(v_{\max})\|_2^2\right). \tag{3.8}$$

Proof: Substituting the control laws (3.6) into the error dynamics (2.5), we obtain the controlled error system:

$$\begin{cases} \dfrac{\partial e_1\,(r,v)}{\partial v} = d_1\Delta e_1 - (a+\sigma_1)\, e_1 - \mu_1 e_2 + \varkappa_1(P,Q) - \varkappa_1\,(U,W)\,, \\ \dfrac{\partial e_2\,(r,v)}{\partial v} = d_2\Delta e_2 - \sigma_2 e_1 - (b+\mu_2)\, e_2 + \varkappa_2(P,Q) - \varkappa_2\,(U,W)\,. \end{cases} \tag{3.9}$$

To analyze the FTS of the error system, we define the LF:

$$L_2(v) = \frac{1}{2}\int_\Omega e_1^2 + e_2^2\, dr, \tag{3.10}$$

which corresponds to the squared $\mathcal{L}^2$-norm of the synchronization error. Thus, we obtain:

$$\begin{aligned} \frac{\partial L_2(v)}{\partial v} &= \frac{1}{2}\int_\Omega \frac{\partial e_1^2}{\partial v}\, dr + \frac{1}{2}\int_\Omega \frac{\partial e_2^2}{\partial v}\, dr \\ &= \int_\Omega e_1 \frac{\partial e_1}{\partial v}\, dr + \int_\Omega e_2 \frac{\partial e_2}{\partial v}\, dr \\ &= \int_\Omega e_1\left[d_1\Delta e_1 - (a+\sigma_1)e_1 - \mu_1 e_2 + \varkappa_1(P,Q) - \varkappa_1(U,W)\right] dr \\ &\quad + \int_\Omega e_2\left[d_2\Delta e_2 - \sigma_2 e_1 - (b+\mu_2)e_2 + \varkappa_2(P,Q) - \varkappa_2(U,W)\right] dr \\ &\le \int_\Omega e_1\left[d_1\Delta e_1 - (a+\sigma_1)e_1 - \mu_1 e_2 + |\varkappa_1(P,Q) - \varkappa_1(U,W)|\right] dr \\ &\quad + \int_\Omega e_2\left[d_2\Delta e_2 - \sigma_2 e_1 - (b+\mu_2)e_2 + |\varkappa_2(P,Q) - \varkappa_2(U,W)|\right] dr \\ &\le \int_\Omega e_1\left[d_1\Delta e_1 - (a+\sigma_1)e_1 - \mu_1 e_2 + \sigma_1 e_1 + \mu_1 e_2\right] dr \\ &\quad + \int_\Omega e_2\left[d_2\Delta e_2 - \sigma_2 e_1 - (b+\mu_2)e_2 + \sigma_2 e_1 + \mu_2 e_2\right] dr \\ &\le d_1\int_\Omega e_1\Delta e_1\, dr + d_2\int_\Omega e_2\Delta e_2\, dr - a\int_\Omega e_1^2\, dr - b\int_\Omega e_2^2\, dr. \end{aligned} \tag{3.11}$$

Applying Green's formula and Lemma 3.1, and using the Poincaré inequality (3.2), we derive:

$$\begin{aligned} \frac{\partial L_2(v)}{\partial v} &\le -d_1\int_\Omega |\nabla e_1|^2\, dr - d_2\int_\Omega |\nabla e_2|^2\, dr - a\int_\Omega e_1^2\, dr - b\int_\Omega e_2^2\, dr \\ &\le -\,(d_1\nu_1 + a)\int_\Omega e_1^2\, dr - (d_2\nu_2 + b)\int_\Omega e_2^2\, dr \\ &\le -2\min\{d_1\nu_1 + a,\;\; d_2\nu_2 + b\}\, L_2(v). \end{aligned} \tag{3.12}$$

Setting $\Lambda\,(e) = 2\min\{d_1\nu_1 + a, d_2\nu_2 + b\}$, we observe that $\Lambda(e)$ is a positive constant. The integral condition in Theorem 3.1 is satisfied:

$$\int_0^\varepsilon \frac{de}{\Lambda\,(e)} = \frac{\varepsilon}{2\min\{d_1\nu_1 + a, d_2\nu_2 + b\}} < +\infty, \tag{3.13}$$

for any $\varepsilon > 0$. Therefore, by Theorem 3.1, the equilibrium point $(e_1^*, e_2^*) = (0, 0)$ is FTS. Since $L_2(v)$ is positive and strictly decreasing for $0 \le v < v^* \le v_{\max}$, we have $L_2(s) \ge L_2(v_{\max}) > 0$ for all $s \in [0, v_{\max}]$. Integrating the derivative inequality yields:

$$\begin{aligned} L_2(v) &\le L_2(0) - 2\min\{d_1\nu_1 + a, d_2\nu_2 + b\} \int_0^v L_2(s)\, ds \\ &\le L_2(0) - 2\min\{d_1\nu_1 + a, d_2\nu_2 + b\} \int_0^v L_2(v_{\max})\, ds \\ &= L_2(0) - 2\min\{d_1\nu_1 + a, d_2\nu_2 + b\} L_2(v_{\max})\, v. \end{aligned} \tag{3.14}$$

By Definition 3.1-3.4, the settling time v^* is defined as the smallest time such that $L_2(v) = 0$. Setting the right-hand side to zero and solving for v, we obtain:

$$\lim_{v \to v^*} L_2(v) \le L_2(0) - 2\min\{d_1\nu_1 + a, d_2\nu_2 + b\} L_2(v_{\max})\, v^* = 0. \tag{3.15}$$

Consequently,

$$v^* = \frac{L_2(0)}{2\min\{d_1\nu_1 + a, d_2\nu_2 + b\} L_2(v_{\max})},$$

is the finite settling time. Therefore, according to Definition 3.1-3.4, the M-S systems (2.1)-(2.3) achieve FTSY. □

4. Numerical Applications

This section establishes the theoretical basis of our method and validates it through numerical simulations in MATLAB. This dual approach ensures both theoretical soundness and practical reliability. We employ two benchmark models—Degen-Harrison and Lengyel-Epstein—to demonstrate FTSY under the proposed control framework.

4.1. Degen-Harrison Model

The Degen-Harrison model describes oxygen concentration effects in *Klebsiella aerogenes* bacterial cultures. For detailed reaction schemes and biological significance, see [29,30,31]. The RDs is governed by:

$$\begin{cases} \dfrac{\partial U(r,v)}{\partial v} = d_1 \Delta U + a - U - \dfrac{UW}{1 + qU^2}, & r \in \Omega,\ v > 0, \\ \dfrac{\partial W(r,v)}{\partial v} = d_2 \Delta W + b - \dfrac{UW}{1 + qW^2}, & r \in \Omega,\ v > 0. \end{cases} \tag{4.1}$$

Let the spatial domain be defined as $\Omega = [0, 5]$, and consider the temporal interval $v \in [0, 5]$, with the following parameters:

The control inputs for synchronization are designed as:

$$\begin{cases} C_1(r,v) = -16.5811 e_1 - 1.5811 e_2, \\ C_2(r,v) = -1.8311 e_1 - 2.0811 e_2. \end{cases} \tag{4.2}$$

yielding the slave system:

$$\begin{cases} \dfrac{\partial P(r,v)}{\partial v} = d_1 \Delta P + a - P - \dfrac{PQ}{1 + qP^2} - 16.5811 e_1 - 1.5811 e_2, \\ \dfrac{\partial Q(r,v)}{\partial v} = d_2 \Delta Q + b - \dfrac{PQ}{1 + qQ^2} - 1.8311 e_1 - 2.0811 e_2. \end{cases} \tag{4.3}$$

Table 1: Model parameters used in simulations

Parameter	**Value**
d_1	0.01
d_2	0.01
a	5
b	0.25
q	0.1
σ_1	6.5811
σ_2	1.8311
$\mu_{1,2}$	1.5811
ν_1	1.5
ν_2	1.5
N	50

Initial conditions are specified as:

$$\begin{cases} U_0(r) = 1 - 0.1\cos(3r), \\ W_0(r) = 1 - 0.2\sin(4r), \\ P_0(r) = 1.5 - 0.225\sin(3r), \\ Q_0(r) = 1.5 - 0.225\cos(4r). \end{cases} \tag{4.4}$$

The $\mathcal{L}^2$-norm values and synchronization parameters yield:

$$\begin{aligned} L_2(0) = 1.2984, \quad L_2(5) = 0.0456, \\ \min\{d_1\nu_1 + a, d_2\nu_2 + b\} = 2.8750 \end{aligned} \tag{4.5}$$

with settling time:

$$v^* = \frac{L_2(0)}{2\min\{d_1\nu_1 + a, d_2\nu_2 + b\}L_2(5)} = 4.9519\text{s}. \tag{4.6}$$

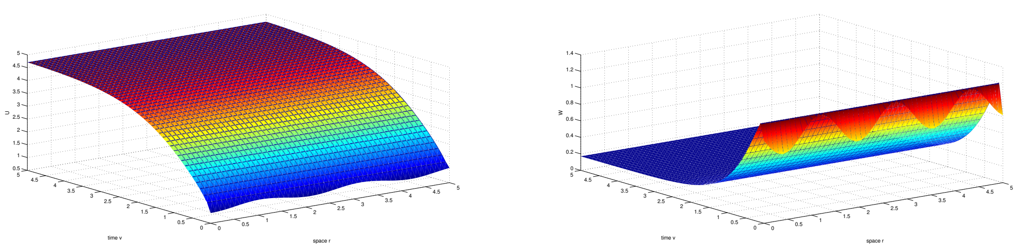

Figure 1: Master system spatiotemporal dynamics: $U(r, v)$ and $W(r, v)$

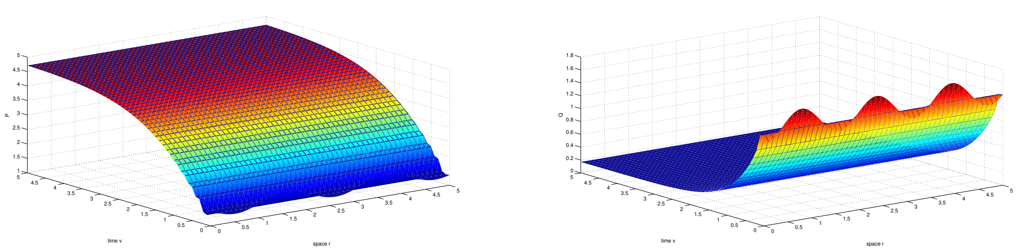

Figure 2: Slave system dynamics under control: $P(r, v)$ and $Q(r, v)$

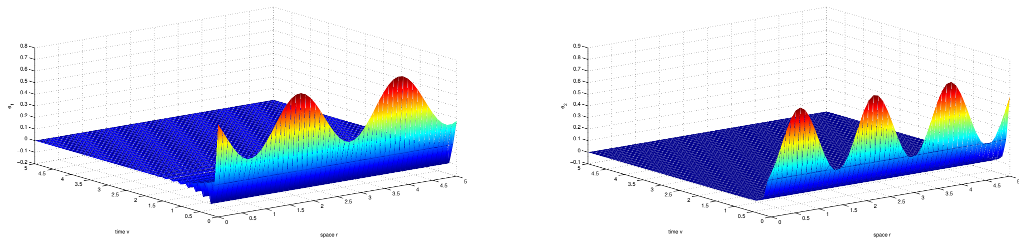

Figure 3: Synchronization error evolution: $e_1(r,v)$ and $e_2(r,v)$

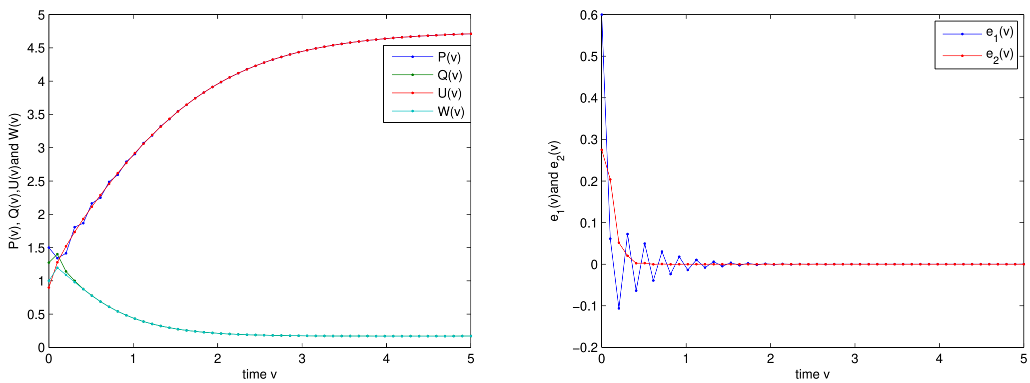

Figure 4: Spatiotemporal Dynamics in 2D

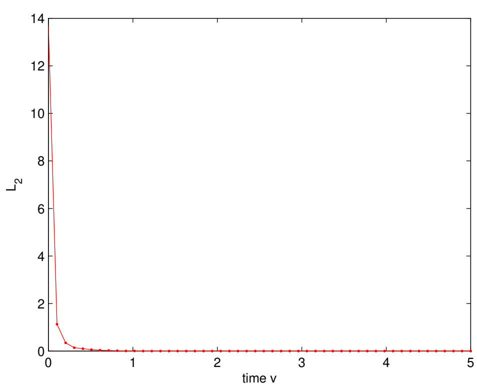

Figure 5: $\mathcal{L}^2$-norm decay with settling time $v^* = 4.9519$s

Figures 1–5 demonstrate progressive error reduction in both spatial and temporal domains. The $\mathcal{L}^2$-norm decays to zero at $v^* = 4.9519$s (Fig. 5), confirming FTSY. Spatiotemporal solutions (Fig. 4) validate convergence to identical states.

4.2. Lengyel-Epstein Model

The Lengyel-Epstein system characterizes the chlorine dioxide-iodine-malonic acid (CIMA) reaction, providing the first experimental evidence of Turing patterns. The governing equations are:

$$\begin{cases} \dfrac{\partial U(r,v)}{\partial v} = d_1 \Delta U + a - U - \dfrac{4UW}{1+U^2}, & r \in \Omega,\ v > 0, \\ \dfrac{\partial W(r,v)}{\partial v} = d_2 \Delta W + b\gamma \left(U - \dfrac{UW}{1+W^2} \right), & r \in \Omega,\ v > 0, \end{cases} \tag{4.7}$$

where U and W represent iodide and chlorite concentrations, respectively. Parameters a and b relate to feed concentrations, $d_{1,2}$ are diffusion coefficients, and γ depends on starch concentration [32].

Consider $\Omega = [0,8]$ and $v \in [0,3]$ with parameters:

Table 2: Model parameters used in the second simulation

Parameter	Value
d_1	0.05
d_2	0.05
a	5
b	1
γ	1
$\sigma_{1,2}$	1
μ_1	4
μ_2	1
ν_1	1
ν_2	1
N	50

Control laws are implemented as:

$$\begin{cases} C_1(r,v) = -11e_1 - 4e_2, \\ C_2(r,v) = -e_1 - 3e_2, \end{cases} \tag{4.8}$$

defining the slave system:

$$\begin{cases} \dfrac{\partial P(r,v)}{\partial v} = d_1 \Delta P + a - P - \dfrac{4PW}{1+P^2} - 11e_1 - 4e_2, \\ \dfrac{\partial Q(r,v)}{\partial v} = d_2 \Delta Q + b\gamma \left(P - \dfrac{PQ}{1+Q^2} \right) - e_1 - 3e_2. \end{cases} \tag{4.9}$$

Initial conditions are:

$$\begin{cases} U_0(r) = 2 - 0.2\cos(5r), \\ W_0(r) = 1.5 - 0.375\sin(5r), \\ P_0(r) = 1.5 - 0.375\sin(4r), \\ Q_0(r) = 2 - 1.5\cos(4r). \end{cases} \tag{4.10}$$

Computations yield:

$$\begin{aligned} L_2(0) = 1.4267, \quad L_2(3) = 0.2287, \\ \min\{d_1\nu_1 + a, d_2\nu_2 + b\} = 1.0500 \end{aligned} \tag{4.11}$$

with settling time:

$$v^* = \frac{L_2(0)}{2\min\{d_1\nu_1 + a, d_2\nu_2 + b\} L_2(3)} = 2.9706\text{s}. \tag{4.12}$$

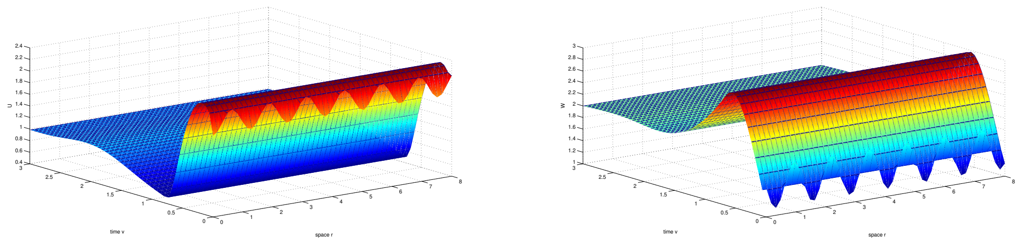

Figure 6: Master system spatiotemporal dynamics: $U(r, v)$ and $W(r, v)$

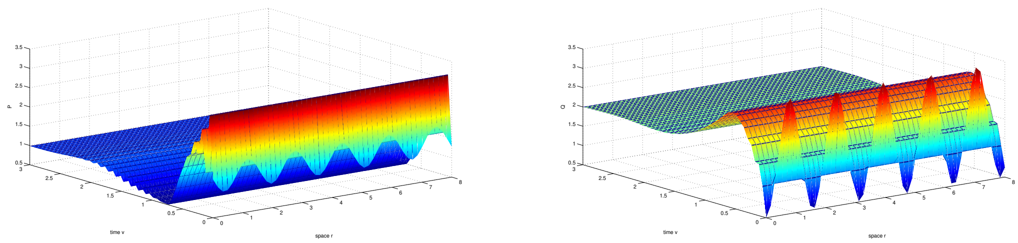

Figure 7: Slave system dynamics under control: $P(r, v)$ and $Q(r, v)$

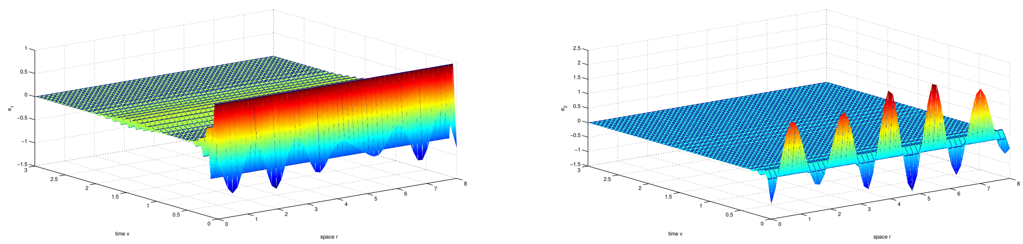

Figure 8: Synchronization error evolution: $e_1(r, v)$ and $e_2(r, v)$

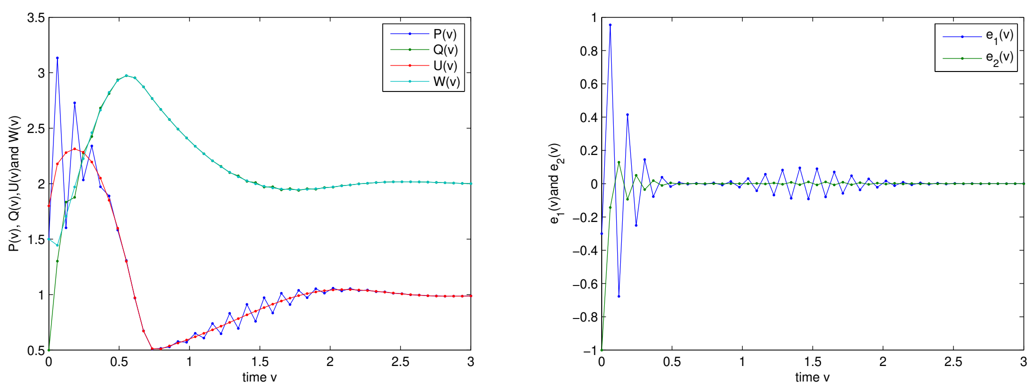

Figure 9: Spatiotemporal Dynamics in 2D

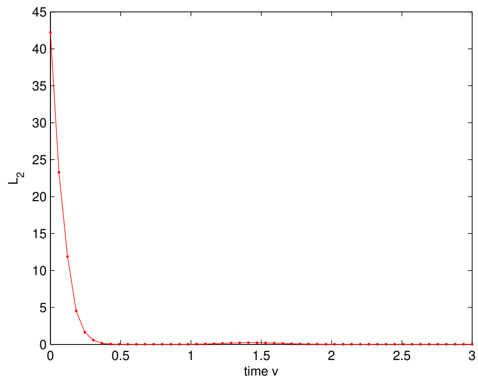

Figure 10: $\mathcal{L}^2$-norm decay with $v^* = 2.9706$s

Figures 6–10 exhibit error convergence to zero at $v^* = 2.9706$s. The $\mathcal{L}^2$-norm decay (Fig. 10) and spatiotemporal solutions (Fig. 9) confirm FTSY across the domain.

This study establishes a comprehensive framework for achieving FTSY in coupled RDs. The core theoretical contribution involves the development of control laws guaranteeing synchronization between M-S systems within a finite settling time. LS-based analysis rigorously proves that the synchronization error converges to zero under the proposed control strategy, satisfying the conditions for FTS. Numerical simulations conducted on both the Degen-Harrison and Lengyel-Epstein models confirmed these theoretical predictions. For the Degen-Harrison model, synchronization was achieved in approximately five seconds, with the error norm decaying to zero as anticipated. The Lengyel-Epstein model exhibited even faster convergence, synchronizing in under three seconds. Spatiotemporal dynamics and error evolution plots demonstrated consistent convergence across the entire spatial domain, thereby validating the efficacy of the controller. Beyond theoretical and numerical contributions, the results provide a foundation for practical applications requiring precise temporal coordination of spatiotemporal patterns, including chemical process control in RDs experiments, neural network synchronization in computational neuroscience, and secure communication systems leveraging synchronized dynamics. Methodologically, the study's strengths lie in the model-agnostic nature of the control strategy, applicable to diverse RDs; the provision of a settling time formula enabling predictable synchronization deadlines; and the explicit accommodation of boundary conditions and spatial coupling.

5. Conclusion

This study established a rigorous framework for achieving FTSY in coupled RDs. Through LS-based analysis, we proved that the proposed control laws (3.6) guarantee synchronization between M-S systems (2.1)-(2.3) within a finite settling time v^* given by (3.7). The theoretical findings were validated numerically using two benchmark models:

- Degen-Harrison model achieved synchronization at $v^* = 4.9519$s with $\mathcal{L}^2$-norm convergence
- Lengyel-Epstein model demonstrated faster convergence at $v^* = 2.9706$s

Key advantages of our approach include:

1. Model-agnostic control design applicable to diverse RDs
2. Explicit settling time formula enabling predictable synchronization deadlines
3. Explicit accommodation of Neumann boundary conditions and spatial coupling

These results provide foundational tools for applications requiring precise spatiotemporal coordination, including chemical process control, neural network synchronization, and secure communication systems. Future work will investigate extensions to fractional-order systems and networks with heterogeneous coupling topologies.

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