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Some observations on generalized logarithmic statistical convergence of order ρ for difference sequences via ideals

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ABSTRACT: This paper explores various forms of logarithmic summability and statistical convergence for real sequences using generalized difference sequences and ideals. First, we introduce the concepts of logarithmic$(\Delta^m, \mathcal{I})$-statistical convergence of order ρ and logarithmic strong $\left(\Delta_{\mathfrak{p}}^m, \mathcal{I}\right)$-Cesàro summability of order ρ, analyzing their relationship. These notions are then extended to logarithmic $\Delta^m(f, \mathcal{I})$-statistical convergence of order ρ and logarithmic strong $\Delta^m(f, \mathcal{I})$-Cesàro summability of order ρ, with fundamental connections established.

Key Words: Logarithmic density, statistical convergence, logarithmic statistical convergence, difference sequence, statistical summability $(H, 1)$.

Contents

1. Introduction

Móricz [24] recently introduced the idea of statistical summability $(H, 1)$, a generalization of statistical convergence first proposed by Fast [13]. The researchers then investigated it further from the standpoint of sequence spaces and linked it to summability theory [15,16,17,18,22,23,25,26,27,28,29].

Gadjiev and Orhan introduced the idea of the order of statistical convergence for a sequence of numbers in [14]. Subsequently, Çolak [5] investigated strong p-Cesàro summability of order α and statistical convergence of order α.

Kostyrko et al. [20] initially introduced the concept of $\mathcal{I}$-convergence for real sequences. Subsequently, this notion was further explored by various researchers [6].

The concept of difference sequence spaces was first introduced by Kızmaz [19] and later extended by Et [7], Et et al. [8,9], Et and Başarır [10], Et and Çolak [11], Et and Gidemen [12]. A generalized form is given as

$$\Delta^m(X) := \{\varpi = (\varpi_{\mathfrak{u}}) : (\Delta^m \varpi_{\mathfrak{u}}) \in X\},$$

for $X = l_\infty, c$ or c_0, where $m \in \mathbb{N}$, $\Delta^0 \varpi = (\varpi_{\mathfrak{u}})$, $\Delta^m \varpi = \left(\Delta^{m-1}\varpi_{\mathfrak{u}} - \Delta^{m-1}\varpi_{\mathfrak{u}+1}\right)$, and so $\Delta^m \varpi_{\mathfrak{u}} = \sum_{j=1}^{m} (-1)^j \binom{m}{j} \varpi_{\mathfrak{u}+j}$.

A modulus function, as described by Maddox [21], is a real-valued function f on $(0, \infty)$ that meets the following criteria:

1. $f(y) = 0$ iff $y = 0$,
2. For every $y, z \in \mathbb{R}^+$, the subadditivity property $f(y+z) \leq f(y) + f(z)$ holds,
3. The function f is increasing,
4. f is continuous from the right at 0.

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Using an unbounded modulus function, Aizpuru et al. [1] proposed the concepts of f-density and f-statistical convergence for real number sequences (also see [3]).

When the limit is present, the f-density of a subset $E \subset \mathbb{N}$ is determined by

$$d^f(E) := \lim_{\mathfrak{s}\to\infty} \frac{f(|\{\mathfrak{u} \le \mathfrak{s} : \mathfrak{u} \in E\}|)}{f(\mathfrak{s})},$$

where f is an unbounded modulus function. When $f(\varpi) = \varpi$, the concept of f-density coincides with the usual natural density. Although it is commonly known that natural density satisfies the characteristic $d(E) + d(\mathbb{N}\diagdown E) = 1$, f-density, or $d^f(E) + d^f(\mathbb{N}\diagdown E) \neq 1$, does not always follow the same rules.

If for each $\varrho > 0$,

$$d^f(\mathfrak{u} \in \mathbb{N} : |\varpi_{\mathfrak{u}} - \varpi_0| \ge \varrho) = 0,$$

where f is an unbounded modulus function, then a number sequence $\varpi = (\varpi_{\mathfrak{u}})$ is f-statistically convergent to ϖ_0 (or S^f-convergent to ϖ_0) .

The notions of f_α-density and f-statistical convergence of order α for real number sequences employing an unbounded modulus function were more recently developed by Bhardwaj and Dhawan [3].

A class $\mathcal{I}$ of Y is an ideal in Y provided that: (i) $\emptyset \in \mathcal{I}$, (ii) $S \cup T \in \mathcal{I}$ for $S, T \in \mathcal{I}$, (iii) $T \in \mathcal{I}$ for $S \in \mathcal{I}$ and $T \subset S$, where Y is a non-empty set. The ideal $\mathcal{I}$ is a non-trivial ideal when $Y \notin \mathcal{I}$ and the non-trivial ideal $\mathcal{I}$ is a admissible ideal when $\{x\} \in \mathcal{I}$ for each $x \in Y$.

A class $\mathcal{F}$ of Y is a filter in Y provided that: (i) $\emptyset \notin \mathcal{F}$, (ii) $S \cap T \in \mathcal{F}$ for $S, T \in \mathcal{F}$, (iii) $T \in \mathcal{F}$ for $S \in \mathcal{F}$ and $S \subset T$, where Y is a non-empty set. The class $\mathcal{F}(\mathcal{I}) = \{M \subset X : M = X \backslash S \text{ for } \exists S \in \mathcal{I}\}$ is the filter associated to $\mathcal{I}$ in Y when $\mathcal{I}$ is the nontrivial ideal.

Alghamdi et al. [2] utilized logarithmic density to introduce the concept of logarithmic statistical convergence. They explored its connections with statistical convergence, as well as statistical summability $(H, 1)$, which was previously established by Móricz.

Let us recall the notion of logarithmic density and logarithmic statistical convergence.

Let χ_J denote the characteristic function of $J \subset \mathbb{N}$, and let $\mathbb{N}$ be the set of all natural numbers and Enter $d_{\mathfrak{s}}(J) = \frac{1}{\mathfrak{s}}\sum_{\mathfrak{u}=1}^{\mathfrak{s}} \chi_J(\mathfrak{u})$ and $\delta_{\mathfrak{s}}(J) = \frac{1}{l_{\mathfrak{s}}}\sum_{\mathfrak{u}=1}^{\mathfrak{s}} \frac{\chi_J(\mathfrak{u})}{\mathfrak{u}}$ for $\mathfrak{s} \in \mathbb{N}$, where $l_{\mathfrak{s}} = \sum_{\mathfrak{u}=1}^{\mathfrak{s}} \frac{1}{\mathfrak{u}}$, $(\mathfrak{s} = 1, 2, ...)$. The notions $\underline{d}(J) = \liminf_{\mathfrak{s}\to\infty} d_{\mathfrak{s}}(J)$ and $\overline{d}(J) = \limsup_{\mathfrak{s}\to\infty} d_{\mathfrak{s}}(J)$ are called the lower and upper asymptotic density of J, respectively.

The lower and upper logarithmic densities of E are denoted by the numerals $\underline{\delta}(E) = \liminf_{\mathfrak{s}\to\infty} \delta_{\mathfrak{s}}(E)$ and $\overline{\delta}(E) = \limsup_{\mathfrak{s}\to\infty} \delta_{\mathfrak{s}}(E)$, respectively. The asymptotic density of E is denoted by $d(E)$ if $\underline{d}(E) = \overline{d}(E) = d(E)$, whereas the logarithmic density of E is denoted by $\underline{\delta}(E) = \overline{\delta}(E) = \delta_{\ln}(E)$. Take note that $l_{\mathfrak{s}} = \sum_{\mathfrak{u}=1}^{\mathfrak{s}} \frac{1}{\mathfrak{u}} = \mathfrak{s}$ and hence $\delta_{\ln}(E)$ reduces to $d(E)$ for $\mathfrak{u} = 1$.

A sequence $\varpi = (\varpi_{\mathfrak{u}})$ is said to be logarithmic statistically convergent to ϖ_0 if for any $\varrho > 0$, the set $\left\{\mathfrak{u} : \frac{1}{\mathfrak{u}} |\varpi_{\mathfrak{u}} - \varpi_0| \ge \varrho\right\}$ has logarithmic density zero, i.e.,

$$\lim_{\mathfrak{s}\to\infty} \frac{1}{l_{\mathfrak{s}}} \left| \left\{ \mathfrak{u} \le \mathfrak{s} : \frac{1}{\mathfrak{u}} |\varpi_{\mathfrak{u}} - \varpi_0| \ge \varrho \right\} \right| = 0. \tag{1.1}$$

In this instance, $st_{\ln} - \lim \varpi_{\mathfrak{u}} = \varpi_0$ is written.

Logarithmic statistical convergence can be considered a particular case of weighted statistical convergence in the case of $p_{\mathfrak{u}} = \frac{1}{\mathfrak{u}}$.

This is not quite accurate, though, as the definition of weighted statistical convergence states that

$$\lim_{\mathfrak{s}\to\infty} \frac{1}{l_{\mathfrak{s}}} \left| \left\{ \mathfrak{u} \le l_{\mathfrak{s}} \approx \log \mathfrak{s} : \frac{1}{\mathfrak{u}} |\varpi_{\mathfrak{u}} - \varpi_0| \ge \varrho \right\} \right| = 0.$$

for $p_{\mathfrak{u}} = \frac{1}{\mathfrak{u}}$, $P_{\mathfrak{s}} = \sum_{\mathfrak{u}=1}^{\mathfrak{s}} p_{\mathfrak{u}} = \sum_{\mathfrak{u}=1}^{\mathfrak{s}} \frac{1}{\mathfrak{u}} \approx \log \mathfrak{s}$ $(\mathfrak{s} = 1, 2, ...)$. So, one can see the difference between this and (1.1), i.e., in (1.1) the enclosed set has bigger cardinality.

Let $\tau_{\mathfrak{s}} = \frac{1}{l_{\mathfrak{s}}}\sum_{\mathfrak{u}=1}^{\mathfrak{s}} \frac{\varpi_{\mathfrak{u}}}{\mathfrak{u}}$, where $l_{\mathfrak{s}} = \sum_{\mathfrak{u}=1}^{\mathfrak{s}} \frac{1}{\mathfrak{u}} \approx \log \mathfrak{s}$ $(\mathfrak{s} = 1, 2, ...)$. We say that $\varpi = (\varpi_{\mathfrak{u}})$ is $(H, 1)$-summable to ϖ_0 if the sequence $\tau = \tau_{\mathfrak{s}}$ converges to ϖ_0, i.e., $(H, 1) \lim \varpi_{\mathfrak{u}} = \varpi_0$.

Boztepe and Dündar [4] introduced the notions of logarithmic $\mathcal{I}$-convergence and logarithmic $\mathcal{I}$-Cauchy sequences, exploring their properties and interconnections.

2. Main results

In this section, we first present the concepts of logarithmic $(\Delta^m, \mathcal{I})$-statistical convergence of order ρ and logarithmic strong $\left(\Delta^m_{\mathfrak{p}}, \mathcal{I}\right)$-Cesàro summability of order ρ for real sequences. Then, we analyze the relationship between these notions.

Definition 2.1 *A sequence $\varpi = (\varpi_{\mathfrak{u}})$ is defined as logarithmic $(\Delta^m, \mathcal{I})$-convergent to ϖ_0 if for every $\varrho > 0$, the set $\left\{\mathfrak{u} \in \mathbb{N} : \frac{1}{\mathfrak{u}} |\Delta^m \varpi_{\mathfrak{u}} - \varpi_0| \geq \varrho\right\} \in \mathcal{I}$. In such a case, we denote this by $(\Delta^m, \mathcal{I}_{\ln}) - \lim \varpi_{\mathfrak{u}} = \varpi_0$. The collection of all sequences that are $(\Delta^m, \mathcal{I}_{\ln})$-convergent will be represented as $c\,(\Delta^m, \mathcal{I}_{\ln})$.*

Definition 2.2 *Let ρ be a real number in the interval $(0, 1]$. The sequence $\varpi = (\varpi_{\mathfrak{u}})$ is referred to as logarithmic $(\Delta^m, \mathcal{I})$-statistically convergent of order ρ (or $S^\rho\,(\Delta^m, \mathcal{I}_{\ln})$-convergent) if there exists a real number ϖ_0 such that*

$$\left\{\mathfrak{s} \in \mathbb{N} : \frac{1}{l^\rho_{\mathfrak{s}}} \left|\left\{\mathfrak{u} \leq \mathfrak{s} : \frac{1}{\mathfrak{u}} |\Delta^m \varpi_{\mathfrak{u}} - \varpi_0| \geq \varrho\right\}\right| \geq \varsigma\right\} \in \mathcal{I},$$

for any $\varrho, \varsigma > 0$.
When this condition holds, we denote it as $S^\rho\,(\Delta^m, \mathcal{I}_{\ln}) - \lim \varpi_{\mathfrak{u}} = \varpi_0$ or equivalently as $S^\rho\,(\mathcal{I}_{\ln}) - \lim \Delta^m \varpi_{\mathfrak{u}} = \varpi_0$. The collection of all sequences that are logarithmic $(\Delta^m, \mathcal{I})$-statistically convergent of order ρ is represented by $S^\rho\,(\Delta^m, \mathcal{I}_{\ln})$.

The notion of $S^\rho\,(\Delta^m, \mathcal{I}_{\ln})$-convergence is properly defined for $\rho \in (0, 1]$, however, it is generally not well-defined when $\rho > 1$. To illustrate, the sequence $\varpi = (\varpi_{\mathfrak{u}})$ is defined as follows:

$$\Delta^m \varpi_{\mathfrak{u}} = \begin{cases} 1, & \text{if } \mathfrak{u} = 2\mathfrak{s}; \\ 0, & \text{if } \mathfrak{u} \neq 2\mathfrak{s}; \end{cases} \quad \mathfrak{s} = 1, 2, ...$$

For any $\varrho > 0$ and $\rho > 1$, we have

$$\frac{1}{l^\rho_{\mathfrak{s}}} \left|\left\{\mathfrak{u} \leq \mathfrak{s} : \frac{1}{\mathfrak{u}} |\Delta^m \varpi_{\mathfrak{u}} - 1| \geq \varrho\right\}\right| \leq \frac{\mathfrak{s}}{2 l^\rho_{\mathfrak{s}}},$$

for any $\varsigma > 0$

$$\left\{\mathfrak{s} \in \mathbb{N} : \frac{1}{l^\rho_{\mathfrak{s}}} \left|\left\{\mathfrak{u} \leq \mathfrak{s} : \frac{1}{\mathfrak{u}} |\Delta^m \varpi_{\mathfrak{u}} - 1| \geq \varrho\right\}\right| \geq \varsigma\right\} \subseteq \left\{\mathfrak{s} \in \mathbb{N} : \frac{\mathfrak{s}}{2 l^\rho_{\mathfrak{s}}} \geq \varsigma\right\} \in \mathcal{I}.$$

In addition, we get

$$\frac{1}{l^\rho_{\mathfrak{s}}} \left|\left\{\mathfrak{u} \leq \mathfrak{s} : \frac{1}{\mathfrak{u}} |\Delta^m \varpi_{\mathfrak{u}} - 0| \geq \varrho\right\}\right| \leq \frac{\mathfrak{s}}{2 l^\rho_{\mathfrak{s}}},$$

we have

$$\left\{\mathfrak{s} \in \mathbb{N} : \frac{1}{l^\rho_{\mathfrak{s}}} \left|\left\{\mathfrak{u} \leq \mathfrak{s} : \frac{1}{\mathfrak{u}} |\Delta^m \varpi_{\mathfrak{u}} - 0| \geq \varrho\right\}\right| \geq \varsigma\right\} \subseteq \left\{\mathfrak{s} \in \mathbb{N} : \frac{\mathfrak{s}}{2 l^\rho_{\mathfrak{s}}} \geq \varsigma\right\} \in \mathcal{I}.$$

Thus, $S^\rho\,(\Delta^m, \mathcal{I}_{\ln}) - \lim \varpi_{\mathfrak{u}} = 1$ and $S^\rho\,(\Delta^m, \mathcal{I}_{\ln}) - \lim \varpi_{\mathfrak{u}} = 0$. But this results in a contradiction.

It is straightforward to observe that any logarithmic $(\Delta^m, \mathcal{I})$-convergent sequence is also logarithmic $(\Delta^m, \mathcal{I})$-statistically convergent of order $\rho \in (0, 1]$. The opposite isn't always true, though. To demonstrate this, consider the sequence $\varpi = (\varpi_{\mathfrak{u}})$, which is provided by

$$\Delta^m \varpi_{\mathfrak{u}} = \begin{cases} \frac{1}{\sqrt{\mathfrak{u}}}, & \text{if } \mathfrak{u} \neq \mathfrak{s}^3; \\ 1, & \text{if } \mathfrak{u} = \mathfrak{s}^3; \end{cases} \quad \mathfrak{s} = 1, 2, ... \tag{2.1}$$

Clearly, $\varpi \in S^\rho\,(\Delta^m, \mathcal{I}_{\ln})$ for $\rho \in \left(\frac{1}{3}, 1\right]$. but it $\varpi \notin c\,(\Delta^m, \mathcal{I}_{\ln})$.

Definition 2.3 *For any real number ρ in the interval $(0,1]$ and any positive real number $\mathfrak{p}$, a sequence $\varpi=(\varpi_{\mathfrak{u}})$ is called logarithmic $\left(\Delta_{\mathfrak{p}}^{m},\mathcal{I}\right)$-Cesàro summable of order ρ (also referred to as strong $w^{\rho}\left(\Delta_{\mathfrak{p}}^{m},\mathcal{I}_{\ln}\right)$-summable) if there exists a real number ϖ_0 such that*

$$\left\{\mathfrak{s}\in\mathbb{N}:\frac{1}{l_{\mathfrak{s}}^{\rho}}\sum_{\mathfrak{u}=1}^{\mathfrak{s}}\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}-\varpi_0\right|^{\mathfrak{p}}\geq\varrho\right\}\in\mathcal{I}.$$

Under these conditions, we denote it as $w^{\rho}\left(\Delta_{\mathfrak{p}}^{m},\mathcal{I}_{\ln}\right)-\lim\varpi_{\mathfrak{u}}=\varpi_0$. The collection of all sequences that are strong $\left(\Delta_{\mathfrak{p}}^{m},\mathcal{I}_{\ln}\right)$-summable sequences of order ρ will be indicates by $w^{\rho}\left(\Delta_{\mathfrak{p}}^{m},\mathcal{I}_{\ln}\right)$. For $\rho=1$, this reduces to $w\left(\Delta_{\mathfrak{p}}^{m},\mathcal{I}_{\ln}\right)$.

Theorem 2.1 *Let $\rho\in(0,1]$ be each real number and $\varsigma>0$. Assume that*

$$S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)-\lim\varpi_{\mathfrak{u}}=\varpi_1 \text{ and } S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)-\lim t_{\mathfrak{u}}=t_1.$$

Then, the following properties hold
(i) $S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)-\lim\lambda\varpi_{\mathfrak{u}}=\lambda\varpi_1$, *where* $\lambda\in\mathbb{R}$;
(ii) $S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)-\lim\left(\varpi_{\mathfrak{u}}+t_{\mathfrak{u}}\right)=\varpi_1+t_1$.

Proof: *(i)* If $\lambda=0$, we have nothing to prove. So, we assume that $\lambda\neq0$. Let $S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)-\lim\varpi_{\mathfrak{u}}=\varpi$. Then

$$\left\{\mathfrak{s}\in\mathbb{N}:\frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}-\varpi_1\right|\geq\frac{\varrho}{|\lambda|}\right\}\right|\geq\varsigma\right\}\in\mathcal{I}. \tag{2.2}$$

Since $\left|\Delta^{m}\left(\lambda\varpi_{\mathfrak{u}}\right)-\lambda\varpi_1\right|=|\lambda|\left|\Delta^{m}\varpi_{\mathfrak{u}}-\varpi_1\right|$, we have

$$\begin{aligned}&\left\{\mathfrak{s}\in\mathbb{N}:\tfrac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\tfrac{1}{\mathfrak{u}}\left|\Delta^{m}\left(\lambda\varpi_{\mathfrak{u}}\right)-\lambda\varpi_1\right|\geq\varrho\right\}\right|\geq\varsigma\right\}\\&=\left\{\mathfrak{s}\in\mathbb{N}:\tfrac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\tfrac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}-\varpi_1\right|\geq\tfrac{\varrho}{|\lambda|}\right\}\right|\geq\varsigma\right\}.\end{aligned}$$

Hence, by invoking the equality in (2.2), it follows that $S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)-\lim\lambda\varpi_{\mathfrak{u}}=\lambda\varpi_1$.
(ii) Let $S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)-\lim\varpi_{\mathfrak{u}}=\varpi_1$ and $S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)-\lim t_{\mathfrak{u}}=t_1$. Then, $\delta_{\mathcal{I}}\left(U_1\left(\varrho,\varsigma\right)\right)=0$ and $\delta_{\mathcal{I}}\left(U_2\left(\varrho,\varsigma\right)\right)=0$, where

$$S_1\left(\varrho,\varsigma\right):=\left\{\mathfrak{s}\in\mathbb{N}:\frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}-\varpi_1\right|\geq\varrho\right\}\right|\geq\frac{\varsigma}{2}\right\},$$

and

$$S_2\left(\varrho,\varsigma\right):=\left\{\mathfrak{s}\in\mathbb{N}:\frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^{m}t_{\mathfrak{u}}-t_1\right|\geq\varrho\right\}\right|\geq\frac{\varsigma}{2}\right\}.$$

Let

$$S\left(\varrho,\varsigma\right):=\left\{\mathfrak{s}\in\mathbb{N}:\frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^{m}\left(\varpi_{\mathfrak{u}}+t_{\mathfrak{u}}\right)-\left(\varpi_1+t_1\right)\right|\geq\varrho\right\}\right|\geq\varsigma\right\}.$$

To prove that $\delta_{\mathcal{I}}\left(S\left(\varrho,\varsigma\right)\right)=0$, it suffices to show that $S\left(\varrho,\varsigma\right)\subseteq S_1\left(\varrho,\varsigma\right)\cup S_2\left(\varrho,\varsigma\right)$. Suppose $\mathfrak{u}_0\in S\left(\varrho,\varsigma\right)$. Then

$$\frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^{m}\left(\varpi_{\mathfrak{u}_0}+t_{\mathfrak{u}_0}\right)-\left(\varpi_1+t_1\right)\right|\geq\varrho\right\}\right|\geq\varsigma. \tag{2.3}$$

Suppose to the contrary, that $\mathfrak{u}_0\notin S_1\left(\varrho,\varsigma\right)\cup S_2\left(\varrho,\varsigma\right)$. Then, $\mathfrak{u}_0\notin S_1\left(\varrho,\varsigma\right)$ and $\mathfrak{u}_0\notin S_2\left(\varrho,\varsigma\right)$. If $\mathfrak{u}_0\notin S_1\left(\varrho,\varsigma\right)$ and $\mathfrak{u}_0\notin S_2\left(\varrho,\varsigma\right)$, we have

$$\frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}_0}-\varpi_1\right|\geq\varrho\right\}\right|<\frac{\varsigma}{2}$$

and

$$\frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^{m}t_{\mathfrak{u}_0}-t_1\right|\geq\varrho\right\}\right|<\frac{\varsigma}{2}.$$

Then, we obtain

$$\begin{aligned}&\tfrac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\tfrac{1}{\mathfrak{u}}\left|\Delta^{m}\left(\varpi_{\mathfrak{u}_0}+t_{\mathfrak{u}_0}\right)-\left(\varpi_1+t_1\right)\right|\geq\varrho\right\}\right|\\&\quad\leq\tfrac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\tfrac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}_0}-\varpi_1\right|\geq\varrho\right\}\right|+\tfrac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\tfrac{1}{\mathfrak{u}}\left|\Delta^{m}t_{\mathfrak{u}_0}-t_1\right|\geq\varrho\right\}\right|\\&\quad<\tfrac{\varsigma}{2}+\tfrac{\varsigma}{2}=\varsigma,\end{aligned}$$

which contradicts (2.2). Hence, $\mathfrak{u}_0\in S_1(\varrho,\varsigma)\cup S_2(\varrho,\varsigma)$, that is $S(\varrho,\varsigma)\subseteq S_1(\varrho,\varsigma)\cup S_2(\varrho,\varsigma)$. This gives that $S^{\rho}(\Delta^m,\mathcal{I}_{\ln})-\lim(\varpi_{\mathfrak{u}}+t_{\mathfrak{u}})=\varpi_1+t_1$. □

Theorem 2.2 *For any real number $\rho\in(0,1]$, the limit of an $S^{\rho}(\Delta^m,\mathcal{I}_{\ln})$-convergent sequence is uniquely defined.*

Proof: Assume, for the sake of contradiction, that the sequence $\varpi=(\varpi_{\mathfrak{u}})$ is logarithmic $(\Delta^m,\mathcal{I})$-statistically convergent of order ρ to two distinct values ϖ_0 and ϖ_1, with $\varpi_0\neq\varpi_1$. That is, for every $\varrho>0$ and $\varsigma>0$, the sets

$$A_1=\left\{\mathfrak{s}\in\mathbb{N}:\frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^m\varpi_{\mathfrak{u}}-\varpi_0\right|\geq\varrho\right\}\right|<\varsigma\right\}$$

and

$$A_2=\left\{\mathfrak{s}\in\mathbb{N}:\frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^m\varpi_{\mathfrak{u}}-\varpi_1\right|\geq\varrho\right\}\right|<\varsigma\right\}$$

belong to the filter associated with the ideal $\mathcal{I}$, i.e., $A_1,A_2\in\mathcal{F}(\mathcal{I})$. Since filters are closed under finite intersections, we have $A_1\cap A_2\in\mathcal{F}(\mathcal{I})$, so in particular, $A_1\cap A_2\neq\emptyset$. Let $\mathfrak{t}\in A_1\cap A_2$ and choose $\varrho=\frac{|\varpi_0-\varpi_1|}{3}>0$. Then, by the definitions of A_1 and A_2, we have

$$\frac{1}{l_{\mathfrak{t}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{t}:\frac{1}{\mathfrak{u}}\left|\Delta^m\varpi_{\mathfrak{u}}-\varpi_0\right|\geq\varrho\right\}\right|<\varsigma$$

and

$$\frac{1}{l_{\mathfrak{t}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{t}:\frac{1}{\mathfrak{u}}\left|\Delta^m\varpi_{\mathfrak{u}}-\varpi_1\right|\geq\varrho\right\}\right|<\varsigma.$$

This implies that, for maximum $\mathfrak{u}\leq\mathfrak{t}$, the inequalities $\frac{1}{\mathfrak{u}}\left|\Delta^m\varpi_{\mathfrak{u}}-\varpi_0\right|<\varrho$ and $\frac{1}{\mathfrak{u}}\left|\Delta^m\varpi_{\mathfrak{u}}-\varpi_1\right|<\varrho$ hold simultaneously for a very small $\varsigma>0$.
However, since these neighborhoods are disjoint due to the choice $\varrho=\frac{|\varpi_0-\varpi_1|}{3}$, the intersection

$$\left\{\mathfrak{u}\leq\mathfrak{t}:\frac{1}{\mathfrak{u}}\left|\Delta^m\varpi_{\mathfrak{u}}-\varpi_0\right|\geq\varrho\right\}\cap\left\{\mathfrak{u}\leq\mathfrak{t}:\frac{1}{\mathfrak{u}}\left|\Delta^m\varpi_{\mathfrak{u}}-\varpi_1\right|\geq\varrho\right\}$$

must be non-empty, leading to a contradiction. Therefore, the assumption that, $\varpi=(\varpi_{\mathfrak{u}})$ is logarithmic $(\Delta^m,\mathcal{I})$-statistically convergent of order ρ to two different values is invalid. The evidence of uniqueness is now complete. □

Theorem 2.3 *Let $\varpi=(\varpi_{\mathfrak{u}})$, $t=(t_{\mathfrak{u}})$ and $z=(z_{\mathfrak{u}})$ be any real sequences such that $\Delta^m\varpi_{\mathfrak{u}}\leq\Delta^m t_{\mathfrak{u}}\leq\Delta^m z_{\mathfrak{u}}$. If $S^{\rho}(\Delta^m,\mathcal{I}_{\ln})-\lim\varpi_{\mathfrak{u}}=\varpi_1=S^{\rho}(\Delta^m,\mathcal{I}_{\ln})-\lim z_{\mathfrak{u}}$, then $S^{\rho}(\Delta^m,\mathcal{I}_{\ln})-\lim t_{\mathfrak{u}}=\varpi_1$.*

Proof: Let $\varrho,\varsigma>0$ be arbitrary. Since

$$S^{\rho}(\Delta^m,\mathcal{I}_{\ln})-\lim\varpi_{\mathfrak{u}}=\varpi_1,$$

it follows from the definition that the set

$$A(\varrho,\varsigma)=\left\{\mathfrak{s}\in\mathbb{N}:\frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^m\varpi_{\mathfrak{u}}-\varpi_1\right|\geq\varrho\right\}\right|\geq\varsigma\right\}$$

belongs to the ideal $\mathcal{I}$.
Similarly, the condition $S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)-\lim z_{\mathfrak{u}}=\varpi_{1}$ implies that

$$B\left(\varrho,\varsigma\right)=\left\{\mathfrak{s}\in\mathbb{N}:\frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^{m}z_{\mathfrak{u}}-\varpi_{1}\right|\geq\varrho\right\}\right|\geq\varsigma\right\}\in\mathcal{I}.$$

Now, for each $\mathfrak{u}\in\mathbb{N}$, the inequality $\Delta^{m}\varpi_{\mathfrak{u}}\leq\Delta^{m}t_{\mathfrak{u}}\leq\Delta^{m}z_{\mathfrak{u}}$ implies that

$$\left|\Delta^{m}t_{\mathfrak{u}}-\varpi_{1}\right|\leq\max\left\{\left|\Delta^{m}\varpi_{\mathfrak{u}}-\varpi_{1}\right|,\left|\Delta^{m}z_{\mathfrak{u}}-\varpi_{1}\right|\right\}.$$

Therefore, if $\frac{1}{\mathfrak{u}}\left|\Delta^{m}t_{\mathfrak{u}}-\varpi_{1}\right|\geq\varrho$, then either $\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}-\varpi_{1}\right|\geq\varrho$ or $\frac{1}{\mathfrak{u}}\left|\Delta^{m}z_{\mathfrak{u}}-\varpi_{1}\right|\geq\varrho$.
It follows that

$$\begin{aligned}\frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^{m}t_{\mathfrak{u}}-\varpi_{1}\right|\geq\varrho\right\}\right| &\leq\frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}-\varpi_{1}\right|\geq\varrho\right\}\right|\\ &+\frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^{m}z_{\mathfrak{u}}-\varpi_{1}\right|\geq\varrho\right\}\right|.\end{aligned}$$

Thus, the set

$$C\left(\varrho,\varsigma\right)=\left\{\mathfrak{s}\in\mathbb{N}:\frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^{m}t_{\mathfrak{u}}-\varpi_{1}\right|\geq\varrho\right\}\right|\geq\varsigma\right\}$$

is contained in the union $A\left(\varrho,\varsigma\right)\cup B\left(\varrho,\varsigma\right)$, and hence $C\left(\varrho,\varsigma\right)\in\mathcal{I}$.
Since $\varrho,\varsigma>0$ is arbitrary, it follows from the definition that $S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)-\lim t_{\mathfrak{u}}=\varpi_{1}$. □

Theorem 2.4 *For any real number ρ in the interval $(0,1]$, the set $S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)\cap l_{\infty}\left(\Delta^{m}\right)$ is closed subsets of $l_{\infty}\left(\Delta^{m}\right)$.*

Proof: Let $\left(\varpi^{\mathfrak{h}}\right)_{\mathfrak{h}\in\mathbb{N}}\in S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)\cap l_{\infty}\left(\Delta^{m}\right)$ be a sequence that converges to $\varpi\in l_{\infty}\left(\Delta^{m}\right)$. We must demonstrate that $\varpi\in S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)\cap l_{\infty}\left(\Delta^{m}\right)$. Assume that $\varpi^{\mathfrak{h}}\rightarrow L_{\mathfrak{h}}\left(S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)\right)$, $\forall\mathfrak{h}\in\mathbb{N}$. For a given $\varrho>0$, consider a positive strictly decreasing sequence $\left\{\varrho_{\mathfrak{h}}\right\}_{\mathfrak{h}\in\mathbb{N}}$ where $\varrho_{\mathfrak{h}}=\frac{\varrho}{2^{\mathfrak{h}}}$. Hence $\left\{\varrho_{\mathfrak{h}}\right\}_{\mathfrak{h}\in\mathbb{N}}$ converges to 0. Select an integer $\mathfrak{h}$ that is positive so that $\left\|\varpi-\varpi^{\mathfrak{h}}\right\|_{\infty}<\frac{\varrho_{\mathfrak{h}}}{4}$. Let $0<\varsigma<1$. Then

$$A=\left\{\mathfrak{s}\in\mathbb{N}:\frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}^{\mathfrak{h}}-L_{\mathfrak{h}}\right|\geq\frac{\varrho_{\mathfrak{h}}}{4}\right\}\right|<\frac{\varsigma}{3}\right\}\in\mathcal{F}\left(\mathcal{I}\right)$$

and

$$B=\left\{\mathfrak{s}\in\mathbb{N}:\frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}+1}^{\mathfrak{h}+1}-L_{\mathfrak{h}+1}\right|\geq\frac{\varrho_{\mathfrak{h}+1}}{4}\right\}\right|<\frac{\varsigma}{3}\right\}\in\mathcal{F}\left(\mathcal{I}\right).$$

Given that $A\cap B\in\mathcal{F}\left(\mathcal{I}\right)$ and $\emptyset\notin\mathcal{F}\left(\mathcal{I}\right)$, we may select $\mathfrak{u}\in A\cap B$. Then

$$\frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}^{\mathfrak{h}}-L_{\mathfrak{h}}\right|\geq\frac{\varrho_{\mathfrak{h}}}{4}\right\}\right|<\frac{\varsigma}{3}$$

and

$$\frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}+1}^{\mathfrak{h}+1}-L_{\mathfrak{h}+1}\right|\geq\frac{\varrho_{\mathfrak{h}+1}}{4}\right\}\right|<\frac{\varsigma}{3}$$

and so

$$\frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}^{\mathfrak{h}}-L_{\mathfrak{h}}\right|\geq\frac{\varrho_{\mathfrak{h}}}{4}\vee\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}+1}^{\mathfrak{h}+1}-L_{\mathfrak{h}+1}\right|\geq\frac{\varrho_{\mathfrak{h}+1}}{4}\right\}\right|<\varsigma<1.$$

Hence, there exists a $\mathfrak{u}\leq\mathfrak{s}$ for which $\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}^{\mathfrak{h}}-L_{\mathfrak{h}}\right|<\frac{\varrho_{\mathfrak{h}}}{4}$ and $\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}+1}^{\mathfrak{h}+1}-L_{\mathfrak{h}+1}\right|<\frac{\varrho_{\mathfrak{h}+1}}{4}$.
Then, we can write

$$\begin{aligned}\left|L_{\mathfrak{h}}-L_{\mathfrak{h}+1}\right| &\leq\frac{1}{\mathfrak{u}}\left|L_{\mathfrak{h}}-\Delta^{m}\varpi_{\mathfrak{u}}^{\mathfrak{h}}\right|+\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}^{\mathfrak{h}}-\Delta^{m}\varpi_{\mathfrak{u}+1}^{\mathfrak{h}+1}\right|+\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}+1}^{\mathfrak{h}+1}-L_{\mathfrak{h}+1}\right|\\ &\leq\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}^{\mathfrak{h}}-L_{\mathfrak{h}}\right|+\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}+1}^{\mathfrak{h}+1}-L_{\mathfrak{h}+1}\right|+\left\|\varpi-\varpi^{\mathfrak{h}}\right\|_{\infty}+\left\|\varpi-\varpi^{\mathfrak{h}+1}\right\|_{\infty}\\ &\leq\frac{\varrho_{\mathfrak{h}}}{4}+\frac{\varrho_{\mathfrak{h}+1}}{4}+\frac{\varrho_{\mathfrak{h}}}{4}+\frac{\varrho_{\mathfrak{h}+1}}{4}\leq\varrho_{\mathfrak{h}}.\end{aligned}$$

This implies that $(L_{\mathfrak{h}})_{\mathfrak{h}\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, and thus there is a real number L such that $L_{\mathfrak{h}} \to L$, as $\mathfrak{h} \in \mathbb{N}$. We must demonstrate that $\varpi \to L\left(S^{\rho}\left(\Delta^{m}, \mathcal{I}_{\ln}\right)\right)$. For each $\varrho > 0$, select $\mathfrak{h} \in \mathbb{N}$ so that $\varrho_{\mathfrak{h}} < \frac{\varrho}{4}$, $\left\|\varpi - \varpi^{\mathfrak{h}}\right\|_{\infty} < \frac{\varrho}{4}$, $|L_{\mathfrak{h}} - L| < \frac{\varrho}{4}$. Then

$$\begin{aligned}
&\tfrac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u} \leq \mathfrak{s} : \tfrac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}} - L\right| \geq \varrho\right\}\right| \\
&\leq \tfrac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u} \leq \mathfrak{s} : \tfrac{1}{\mathfrak{u}}\left(\left|\Delta^{m}\varpi_{\mathfrak{u}}^{\mathfrak{h}} - L_{\mathfrak{h}}\right| + \left\|\Delta^{m}\varpi_{\mathfrak{u}} - \Delta^{m}\varpi_{\mathfrak{u}}^{\mathfrak{h}}\right\|_{\infty} + |L_{\mathfrak{h}} - L|\right) \geq \varrho\right\}\right| \\
&\leq \tfrac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u} \leq \mathfrak{s} : \tfrac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}^{\mathfrak{h}} - L_{\mathfrak{h}}\right| + \tfrac{\varrho}{4} + \tfrac{\varrho}{4} \geq \varrho\right\}\right| \\
&\leq \tfrac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u} \leq \mathfrak{s} : \tfrac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}^{\mathfrak{h}} - L_{\mathfrak{h}}\right| \geq \tfrac{\varrho}{2}\right\}\right|.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\left\{\mathfrak{s} \in \mathbb{N} : \tfrac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u} \leq \mathfrak{s} : \tfrac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}} - L_{\mathfrak{h}}\right| \geq \varrho\right\}\right| < \varsigma\right\} \\
&\supseteq \left\{\mathfrak{s} \in \mathbb{N} : \tfrac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u} \leq \mathfrak{s} : \tfrac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}^{\mathfrak{h}} - L_{\mathfrak{h}}\right| \geq \tfrac{\varrho}{2}\right\}\right| < \varsigma\right\} \in \mathcal{F}(\mathcal{I}).
\end{aligned}$$

Hence

$$\left\{\mathfrak{s} \in \mathbb{N} : \frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u} \leq \mathfrak{s} : \frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}} - L_{\mathfrak{h}}\right| \geq \varrho\right\}\right| < \varsigma\right\} \in \mathcal{F}(\mathcal{I}),$$

and so

$$\left\{\mathfrak{s} \in \mathbb{N} : \frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u} \leq \mathfrak{s} : \frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}} - L_{\mathfrak{h}}\right| \geq \varrho\right\}\right| \geq \varsigma\right\} \in \mathcal{I}.$$

This concludes the theorem's proof by indicating that $\varpi \to L\left(S^{\rho}\left(\Delta^{m}, \mathcal{I}_{\ln}\right)\right)$. □

Theorem 2.5 *For fixed real numbers ρ and κ satisfying $0 < \rho \leq \kappa \leq 1$, and for any positive real number $\mathfrak{p}$, then the inclusion $w^{\rho}\left(\Delta_{\mathfrak{p}}^{m}, \mathcal{I}_{\ln}\right) \subset S^{\kappa}\left(\Delta^{m}, \mathcal{I}_{\ln}\right)$ holds, and this inclusion is strict.*

Proof: For any $\varrho > 0$ and given that $w^{\rho}\left(\Delta_{\mathfrak{p}}^{m}, \mathcal{I}_{\ln}\right) - \lim \varpi_{\mathfrak{u}} = \varpi_{1}$, the following inequality holds:

$$\begin{aligned}
\sum_{\mathfrak{u}=1}^{\mathfrak{s}} \tfrac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}} - \varpi_{1}\right|^{\mathfrak{p}} &\geq \varrho^{\mathfrak{p}} \sum_{\mathfrak{u}=1, \left|\Delta^{m}\varpi_{\mathfrak{u}} - \varpi_{1}\right| \geq \varrho}^{\mathfrak{s}} \tfrac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}} - \varpi_{1}\right|^{\mathfrak{p}} \\
&\geq \varrho^{\mathfrak{p}}\left|\left\{\mathfrak{u} \leq \mathfrak{s} : \tfrac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}} - \varpi_{1}\right| \geq \varrho\right\}\right|.
\end{aligned}$$

Thus, we obtain

$$\frac{1}{\varrho^{\mathfrak{p}}}\frac{1}{l_{\mathfrak{s}}^{\rho}}\sum_{\mathfrak{u}=1}^{\mathfrak{s}}\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}} - \varpi_{1}\right|^{\mathfrak{p}} \geq \frac{1}{l_{\mathfrak{s}}^{\kappa}}\left|\left\{\mathfrak{u} \leq \mathfrak{s} : \frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}} - \varpi_{1}\right| \geq \varrho\right\}\right|.$$

Then, for any $\varsigma > 0$, we get

$$\begin{aligned}
&\left\{\mathfrak{s} \in \mathbb{N} : \tfrac{1}{l_{\mathfrak{s}}^{\kappa}}\left|\left\{\mathfrak{u} \leq \mathfrak{s} : \tfrac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}} - \varpi_{1}\right| \geq \varrho\right\}\right| \geq \varsigma\right\} \\
&\subseteq \left\{\mathfrak{s} \in \mathbb{N} : \tfrac{1}{l_{\mathfrak{s}}^{\rho}}\sum_{\mathfrak{u}=1}^{\mathfrak{s}}\tfrac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}} - \varpi_{1}\right|^{\mathfrak{p}} \geq \varrho^{\mathfrak{p}}\varsigma\right\} \in \mathcal{I}.
\end{aligned}$$

As a result, we have $S^{\kappa}\left(\Delta^{m}, \mathcal{I}_{\ln}\right) - \lim \varpi_{\mathfrak{u}} = \varpi_{1}$. So, $w^{\rho}\left(\Delta_{\mathfrak{p}}^{m}, \mathcal{I}_{\ln}\right) \subset S^{\kappa}\left(\Delta^{m}, \mathcal{I}_{\ln}\right)$.
Setting $\rho = \kappa$, we establish the strictness of the inclusion $w^{\rho}\left(\Delta_{\mathfrak{p}}^{m}, \mathcal{I}_{\ln}\right) \subset S^{\rho}\left(\Delta^{m}, \mathcal{I}_{\ln}\right)$ in a particular case. To demonstrate this, consider the sequence $\varpi = (\varpi_{\mathfrak{u}})$ defined by

$$\Delta^{m}\varpi_{\mathfrak{u}} = \begin{cases} 1, & \text{if } \mathfrak{u} = \mathfrak{s}^{2}; \\ 0, & \text{if } \mathfrak{u} \neq \mathfrak{s}^{2}; \end{cases} \quad \mathfrak{s} = 1, 2, \ldots$$

For any $\varrho > 0$ and $\rho \in \left(\frac{1}{2}, 1\right]$, we have

$$\frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u} \leq \mathfrak{s} : \frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}} - 0\right| \geq \varrho\right\}\right| \leq \frac{\sqrt{\mathfrak{s}}}{\mathfrak{s}^{\rho}} = \frac{1}{\mathfrak{s}^{\rho - \frac{1}{2}}},$$

and for any $\varsigma > 0$,we get

$$\left\{\mathfrak{s}\in\mathbb{N}:\frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}-0\right|\geq\varrho\right\}\right|\geq\varsigma\right\}\subseteq\left\{\mathfrak{s}\in\mathbb{N}:\frac{\left[\sqrt{\mathfrak{s}}\right]}{\mathfrak{s}^{\rho}}\geq\varsigma\right\}\in\mathcal{I}.$$

Since the set on the right-hand side is finite, it belongs to $\mathcal{I}$. Consequently $S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)-\lim\varpi_{\mathfrak{u}}=0$ for $\rho\in\left(\frac{1}{2},1\right]$. On the other hand, for $\rho\in\left(0,\frac{1}{2}\right)$, the following inequality holds:

$$\frac{\sqrt{\mathfrak{s}}-1}{l_{\mathfrak{s}}^{\rho}}\leq\frac{1}{l_{\mathfrak{s}}^{\rho}}\sum_{\mathfrak{u}=1}^{\mathfrak{s}}\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}\right|^{\mathfrak{p}}=\frac{1}{l_{\mathfrak{s}}^{\rho}}\sum_{\mathfrak{u}=1}^{\mathfrak{s}}\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}-0\right|^{\mathfrak{p}}.$$

This leads to

$$\begin{aligned}\left\{\mathfrak{s}_0,\mathfrak{s}_0+1,\mathfrak{s}_0+2,...\right\}&=\left\{\mathfrak{s}\in\mathbb{N}:\tfrac{\sqrt{\mathfrak{s}}-1}{\mathfrak{s}^{\rho}}\geq1\right\}\\&\subset\left\{\mathfrak{s}\in\mathbb{N}:\tfrac{1}{l_{\mathfrak{s}}^{\rho}}\sum_{\mathfrak{u}=1}^{\mathfrak{s}}\tfrac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}-0\right|^{\mathfrak{p}}\geq1\right\},\end{aligned}$$

for some $\mathfrak{s}_0\in\mathbb{N}$ that belongs to $\mathcal{F}(\mathcal{I})$, since $\mathcal{I}$ is an admissible ideal. Hence, $\varpi_{\mathfrak{u}}\nrightarrow0$ in the sense of $w^{\rho}\left(\Delta_{\mathfrak{p}}^{m},\mathcal{I}_{\ln}\right)$. □

The converse of Theorem 2.5 does not always hold. To illustrate this, we construct a sequence that is Δ^{m}-bounded and $S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)$-convergent but not necessarily $w^{\rho}\left(\Delta_{\mathfrak{p}}^{m},\mathcal{I}_{\ln}\right)$-summable. Consider the sequence $\varpi=(\varpi_{\mathfrak{u}})$ given by (2.1). It can be verified that $\varpi\in l_{\infty}\left(\Delta^{m}\right)$ and $\varpi\in S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)$ for $\rho\in\left(\frac{1}{3},1\right]$ and $\varpi\notin w^{\rho}\left(\Delta_{\mathfrak{p}}^{m},\mathcal{I}_{\ln}\right)$ for $\rho\in\left(0,\frac{1}{2}\right)$. Therefore,

$$\varpi\in S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)\diagdown w^{\rho}\left(\Delta_{\mathfrak{p}}^{m},\mathcal{I}_{\ln}\right)$$

for $\rho\in\left(\frac{1}{3},\frac{1}{2}\right)$.

Theorem 2.5 directly leads to the following outcome.

Corollary 2.1 *If $w^{\rho}\left(\Delta_{\mathfrak{p}}^{m},\mathcal{I}_{\ln}\right)-\lim\varpi_{\mathfrak{u}}=\varpi_0$, then it follows that $S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)-\lim\varpi_{\mathfrak{u}}=\varpi_0$.*

Theorem 2.6 *For fixed real numbers ρ and κ satisfying $0<\rho\leq\kappa\leq1$, and a positive real number $\mathfrak{p}$, the inclusion $w^{\rho}\left(\Delta_{\mathfrak{p}}^{m},\mathcal{I}_{\ln}\right)\subset w^{\kappa}\left(\Delta_{\mathfrak{p}}^{m},\mathcal{I}_{\ln}\right)$ holds, and this inclusion is strict.*

Proof:

It is easy to understand the inclusion section of the evidence. We illustrate the strictness of the inclusion $w^{\rho}\left(\Delta_{\mathfrak{p}}^{m},\mathcal{I}_{\ln}\right)\subset w^{\kappa}\left(\Delta_{\mathfrak{p}}^{m},\mathcal{I}_{\ln}\right)$ for a specific situation by putting $\mathfrak{p}=1$. Examine the sequence $\varpi=(\varpi_{\mathfrak{u}})$, which is defined so that

$$\Delta^{m}\varpi_{\mathfrak{u}}=\left\{\begin{array}{ll}1, & \text{if } \mathfrak{u}=\mathfrak{s}^2;\\0, & \text{if } \mathfrak{u}\neq\mathfrak{s}^2;\end{array}\right.\ \mathfrak{s}=1,2,...$$

It is straightforward to verify that

$$\frac{1}{l_{\mathfrak{s}}^{\kappa}}\sum_{\mathfrak{u}=1}^{\mathfrak{s}}\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}-0\right|^{\mathfrak{p}}\leq\frac{\sqrt{\mathfrak{s}}}{\mathfrak{s}^{\rho}}=\frac{1}{\mathfrak{s}^{\kappa-\frac{1}{2}}}\to0,\text{ as }\mathfrak{s}\to\infty\text{ for }\kappa\in\left(\frac{1}{2},1\right),$$

but

$$\frac{1}{l_{\mathfrak{s}}^{\rho}}\sum_{\mathfrak{u}=1}^{\mathfrak{s}}\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}-0\right|^{\mathfrak{p}}\geq\frac{\sqrt{\mathfrak{s}}-1}{\mathfrak{s}^{\rho}}\to\infty,\text{ as }\mathfrak{s}\to\infty\text{ for }\rho\in\left(0,\frac{1}{2}\right).$$

So, $\varpi\in w^{\kappa}\left(\Delta_{\mathfrak{p}}^{m},\mathcal{I}_{\ln}\right)$ for $\kappa\in\left(\frac{1}{2},1\right)$ but $\varpi\notin w^{\rho}\left(\Delta_{\mathfrak{p}}^{m},\mathcal{I}_{\ln}\right)$ for $\rho\in\left(0,\frac{1}{2}\right)$. □

The result that follows is a consequence of Theorem 2.6.

Corollary 2.2 *For given real numbers ρ and κ such that $0<\rho\leq\kappa\leq 1$, the following properties hold,*
(i) If $\rho=\kappa$, then $w^{\rho}\left(\Delta_{\mathfrak{p}}^{m},\mathcal{I}_{\ln}\right)=w^{\kappa}\left(\Delta^{m},\mathcal{I}_{\ln}\right)$,
(ii) $w^{\rho}\left(\Delta_{\mathfrak{p}}^{m},\mathcal{I}_{\ln}\right)\subset w\left(\Delta^{m},\mathcal{I}_{\ln}\right)$ for each $\rho\in(0,1]$.

Theorem 2.7 *Let ρ and κ be fixed real numbers such that $0<\rho\leq\kappa\leq 1$, then $S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)\subset S^{\kappa}\left(\Delta^{m},\mathcal{I}_{\ln}\right)$, and the inclusion is strict.*

Proof: Assume that $\varpi\in S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)$. Take ρ and κ such that $0<\rho\leq\kappa\leq 1$. Then, we can write

$$\frac{1}{l_{\mathfrak{s}}^{\kappa}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}-\varpi_{0}\right|\geq\varrho\right\}\right|\leq\frac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}-\varpi_{0}\right|\geq\varrho\right\}\right|.$$

From this, we derive

$$\begin{aligned}&\left\{\mathfrak{s}\in\mathbb{N}:\tfrac{1}{l_{\mathfrak{s}}^{\kappa}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\tfrac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}-\varpi_{0}\right|\geq\varrho\right\}\right|\geq\varsigma\right\}\\&\qquad\subseteq\left\{\mathfrak{s}\in\mathbb{N}:\tfrac{1}{l_{\mathfrak{s}}^{\rho}}\left|\left\{\mathfrak{u}\leq\mathfrak{s}:\tfrac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}-\varpi_{0}\right|\geq\varrho\right\}\right|\geq\varsigma\right\}\in\mathcal{I},\end{aligned}$$

and this gives that $S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)\subset S^{\kappa}\left(\Delta^{m},\mathcal{I}_{\ln}\right)$.
For a specific situation, we show how stringent the inclusion $S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)\subset S^{\kappa}\left(\Delta^{m},\mathcal{I}_{\ln}\right)$ is. Take the sequence $\varpi=(\varpi_{\mathfrak{u}})$, which is defined so that

$$\Delta^{m}\varpi_{\mathfrak{u}}=\begin{cases}\mathfrak{u}, & \text{if } \mathfrak{u}=\mathfrak{s}^{2};\\ 0, & \text{if } \mathfrak{u}\neq\mathfrak{s}^{2};\end{cases}\quad\mathfrak{s}=1,2,...$$

Then, $\varpi\in S^{\kappa}\left(\Delta^{m},\mathcal{I}_{\ln}\right)$ for $\kappa\in\left(\frac{1}{2},1\right]$, but $\varpi\notin S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)$ for $\rho\in\left(0,\frac{1}{2}\right]$. □

Corollary 2.3 *Let $\rho\in(0,1]$ be a real number, then $S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)\subset S\left(\Delta^{m},\mathcal{I}_{\ln}\right)$.*

The ideas of logarithmic $\Delta^{m}(f,\mathcal{I})$-statistical convergence of order ρ and logarithmic strong $\Delta^{m}(f,\mathcal{I})$-Cesàro summability of order ρ are now presented, along with important relationships between them.

Definition 2.4 *Let $\rho\in(0,1]$ be a real number and let f be an unbounded modulus. A sequence $\varpi=(\varpi_{\mathfrak{u}})$ is said to be logarithmic $\Delta^{m}(f,\mathcal{I})$-statistically convergent of order ρ to ϖ_0 (or $S_{\rho}^{f}\left(\Delta^{m},\mathcal{I}_{\ln}\right)$-convergent to ϖ_0) if for each $\varrho,\varsigma>0$*

$$\left\{\mathfrak{s}\in\mathbb{N}:\frac{1}{f\left(l_{\mathfrak{s}}^{\rho}\right)}f\left(\left|\left\{\mathfrak{u}:\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}-\varpi_{0}\right|\geq\varrho\right\}\right|\right)\geq\varsigma\right\}\in\mathcal{I}.$$

When this condition holds, we denote it as $S_{\rho}^{f}\left(\Delta^{m},\mathcal{I}_{\ln}\right)-\lim\varpi_{\mathfrak{u}}=\varpi_0$ or equivalently as $S_{\rho}^{f}\left(\Delta^{m},\mathcal{I}_{\ln}\right)-\lim\Delta^{m}\varpi_{\mathfrak{u}}=\varpi_0$. The set of all logarithmic $\Delta^{m}(f,\mathcal{I})$-statistically convergent sequences of order ρ is represented by $S_{\rho}^{f}\left(\Delta^{m},\mathcal{I}_{\ln}\right)$. For $f(\varpi)=\varpi$, we write $S^{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)$ rather than $S_{\rho}^{f}\left(\Delta^{m},\mathcal{I}_{\ln}\right)$.

The following example shows that the logarithmic $\Delta^{m}(f,\mathcal{I})$-statistical limit of order ρ could not be unique for $\rho>1$.

Example 2.1 *Define a sequence $\varpi=(\varpi_{\mathfrak{u}})$ by*

$$\Delta^{m}\varpi_{\mathfrak{u}}=\begin{cases}1, & \textit{if } \mathfrak{u}=2\mathfrak{s};\\ 0, & \textit{if } \mathfrak{u}\neq 2\mathfrak{s};\end{cases}\quad\mathfrak{s}=1,2,...$$

and let f be an unbounded modulus such that $\lim_{\mathfrak{s}\to\infty}\frac{f(\mathfrak{s})}{f(l_{\mathfrak{s}}^{\rho})}\geq 0$. Since $\lim_{\mathfrak{s}\to\infty}\frac{f(\mathfrak{s})}{f(l_{\mathfrak{s}}^{\rho})}\geq 0$, we have

$$\left\{\mathfrak{s}\in\mathbb{N}:\frac{1}{f\left(l_{\mathfrak{s}}^{\rho}\right)}f\left(\left|\left\{\mathfrak{u}:\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}-1\right|\geq\varrho\right\}\right|\right)\geq\varsigma\right\}\subseteq\left\{\mathfrak{s}\in\mathbb{N}:\frac{f\left(\frac{\mathfrak{s}}{2}\right)}{f\left(l_{\mathfrak{s}}^{\rho}\right)}\geq\varsigma\right\}\in\mathcal{I},$$

and

$$\left\{\mathfrak{s}\in\mathbb{N}:\frac{1}{f\left(l_{\mathfrak{s}}^{\rho}\right)}f\left(\left|\left\{\mathfrak{u}:\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}-0\right|\geq\varrho\right\}\right|\right)\geq\varsigma\right\}\subseteq\left\{\mathfrak{s}\in\mathbb{N}:\frac{f\left(\frac{\mathfrak{s}}{2}\right)}{f\left(l_{\mathfrak{s}}^{\rho}\right)}\geq\varsigma\right\}\in\mathcal{I},$$

for any $\varsigma>0$ *and* $\rho>1$. *As a result,* $S_{\rho}^{f}\left(\Delta^{m},\mathcal{I}_{\ln}\right)-\lim\varpi_{\mathfrak{u}}=0$ *and* $S_{\rho}^{f}\left(\Delta^{m},\mathcal{I}_{\ln}\right)-\lim\varpi_{\mathfrak{u}}=1$ *are not conceivable.*
For any unbounded modulus f *and* $\rho\in(0,1]$, *a sequence is* $S_{\rho}^{f}\left(\Delta^{m},\mathcal{I}_{\ln}\right)$*-convergent if it is logarithmic* $\Delta^{m}\left(\mathcal{I}\right)$*-convergent; however, the opposite is not true. To do this, select* $f\left(\varpi\right)=\varpi^{\mathfrak{p}}$ *for* $\mathfrak{p}\in(0,1]$ *and take into account a sequence* ϖ *that is described by*

$$\Delta^{m}\varpi_{\mathfrak{u}}=\begin{cases}1, & \text{if } \mathfrak{u}=\mathfrak{s}^{2};\\ 0, & \text{if } \mathfrak{u}\neq\mathfrak{s}^{2};\end{cases}\quad \mathfrak{s}=1,2,... \tag{2.4}$$

.

Clearly ϖ *is not logarithmic* $\Delta^{m}\left(\mathcal{I}\right)$*-convergent, but it is* $S_{\rho}^{f}\left(\Delta^{m},\mathcal{I}_{\ln}\right)$*-convergent for* $\rho\in\left(\frac{1}{2},1\right]$ *since*

$$\left\{\mathfrak{s}\in\mathbb{N}:\frac{1}{f\left(l_{\mathfrak{s}}^{\rho}\right)}f\left(\left|\left\{\mathfrak{u}:\mathfrak{u}\leq\mathfrak{s}:\frac{1}{\mathfrak{u}}\left|\Delta^{m}\varpi_{\mathfrak{u}}-0\right|\geq\varrho\right\}\right|\right)\geq\varsigma\right\}\subseteq\left\{\mathfrak{s}\in\mathbb{N}:\frac{f\left(\sqrt{\mathfrak{s}}\right)}{f\left(l_{\mathfrak{s}}^{\rho}\right)}\geq\varsigma\right\}\in\mathcal{I},$$

for $\rho\in\left(\frac{1}{2},1\right]$.

Theorem 2.8 *Let* ρ *and* κ *be fixed real numbers such that* $0<\rho\leq\kappa\leq1$, *then* $S_{\rho}^{f}\left(\Delta^{m},\mathcal{I}_{\ln}\right)\subset S_{\kappa}^{f}\left(\Delta^{m},\mathcal{I}_{\ln}\right)$, *and the inclusion is strict.*

Proof: The proof's inclusion section comes right after. To establish the strictness of $S_{\rho}^{f}\left(\Delta^{m},\mathcal{I}_{\ln}\right)\subset S_{\kappa}^{f}\left(\Delta^{m},\mathcal{I}_{\ln}\right)$, choose $f\left(\varpi\right)=\varpi^{\mathfrak{p}}$, for $\mathfrak{p}\in(0,1]$ and take the sequence ϖ defined by

$$\Delta^{m}\varpi_{\mathfrak{u}}=\begin{cases}1, & \text{if } \mathfrak{u}=\mathfrak{s}^{3};\\ 0, & \text{if } \mathfrak{u}\neq\mathfrak{s}^{3};\end{cases}\quad \mathfrak{s}=1,2,...$$

It can be verified that $\varpi\in S_{\kappa}^{f}\left(\Delta^{m},\mathcal{I}_{\ln}\right)$ for $\kappa\in\left(\frac{1}{3},1\right)$, but $\varpi\notin S_{\rho}^{f}\left(\Delta^{m},\mathcal{I}_{\ln}\right)$ for $\rho\in\left(0,\frac{1}{3}\right]$. □

Theorem 2.8 immediately yields the following result.

Corollary 2.4 *Let* f *be an unbounded modulus and* $\rho\in(0,1]$. *Then,* $S_{\rho}^{f}\left(\Delta^{m},\mathcal{I}_{\ln}\right)\subset S^{f}\left(\Delta^{m},\mathcal{I}_{\ln}\right)$ *and the inclusion is strict.*

Theorem 2.9 *Let* f *be an unbounded modulus and* $\rho\in(0,1]$. *Then*
(i) $S_{\rho}^{f}\left(\Delta^{m},\mathcal{I}_{\ln}\right)\subset S_{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)$ *and the inclusion is strict,*
(ii) $S_{\rho}^{f}\left(\Delta^{m},\mathcal{I}_{\ln}\right)\subset S\left(\Delta^{m},\mathcal{I}_{\ln}\right)$ *and the inclusion is strict.*

Proof: The proof's inclusion arguments are simple to understand. Let $f\left(\varpi\right)=\log\left(\varpi+1\right)$ to illustrate how stringent the inclusion is. and Take the sequence given by

$$\Delta^{m}\varpi_{\mathfrak{u}}=\begin{cases}1, & \text{if } \mathfrak{u}=\mathfrak{s}^{2};\\ 0, & \text{if } \mathfrak{u}\neq\mathfrak{s}^{2};\end{cases}\quad \mathfrak{s}=1,2,...$$

It follows that $\varpi\in S_{\rho}\left(\Delta^{m},\mathcal{I}_{\ln}\right)$ for $\rho\in\left(\frac{1}{2},1\right]$ implying that $\varpi\in S\left(\Delta^{m},\mathcal{I}_{\ln}\right)$. However $\varpi\notin S_{\rho}^{f}\left(\Delta^{m},\mathcal{I}_{\ln}\right)$. □

Now, we define the concept of logarithmic strong $\Delta^{m}\left(f,\mathcal{I}\right)$-Cesàro summability of order ρ and establish certain relationships between logarithmic strong $\Delta^{m}\left(f,\mathcal{I}\right)$-Cesàro summability of order ρ and $\Delta^{m}\left(f,\mathcal{I}\right)$-Cesàro summability of order κ, where κ and ρ are fixed real numbers satisfying $\kappa\geq\rho>0$.

Definition 2.5 *Let ρ be a positive real number and f be a modulus.*

$$w^f_{\rho,0}\left(\Delta^m,\mathcal{I}_{\ln}\right):=\left\{\varpi\in w:\left\{\mathfrak{s}\in\mathbb{N}:\tfrac{1}{l^\rho_\mathfrak{s}}\sum_{\mathfrak{u}=1}^{\mathfrak{s}}\tfrac{1}{\mathfrak{u}}f\left(|\Delta^m\varpi_\mathfrak{u}|\right)\geq\varrho\right\}\in\mathcal{I}\right\},$$
$$w^f_{\rho}\left(\Delta^m,\mathcal{I}_{\ln}\right):=\left\{\varpi\in w:\left\{\mathfrak{s}\in\mathbb{N}:\tfrac{1}{l^\rho_\mathfrak{s}}\sum_{\mathfrak{u}=1}^{\mathfrak{s}}\tfrac{1}{\mathfrak{u}}f\left(|\Delta^m\varpi_\mathfrak{u}-\varpi_0|\right)\geq\varrho\right\}\in\mathcal{I}\right\},$$
$$w^f_{\rho,\infty}\left(\Delta^m,\mathcal{I}_{\ln}\right):=\left\{\varpi\in w:K>0\text{ such that }\left\{\mathfrak{s}\in\mathbb{N}:\tfrac{1}{l^\rho_\mathfrak{s}}\sum_{\mathfrak{u}=1}^{\mathfrak{s}}\tfrac{1}{\mathfrak{u}}f\left(|\Delta^m\varpi_\mathfrak{u}|\right)\geq K\right\}\in\mathcal{I}\right\}.$$

is what we define.

Logarithmically strong $\Delta^m\left(f,\mathcal{I}\right)$-Cesàro summable of order ρ to ϖ_0 with regard to the modulus f is the term used to describe the sequence $\varpi=(\varpi_\mathfrak{u})$ if $\varpi\in w^f_\rho\left(\Delta^m,\mathcal{I}_{\ln}\right)$. This property is also known as strong $\Delta^m\left(f,\mathcal{I}_{\ln}\right)$-Cesàro summability of order ρ.

Different spaces can be derived by selecting specific values for f and ρ.

1. When $f(\varpi)=\varpi$; the notations simplify as follows:
(*i*) $w_{\rho,0}\left(\Delta^m,\mathcal{I}_{\ln}\right)$ instead of $w^f_{\rho,0}\left(\Delta^m,\mathcal{I}_{\ln}\right)$,
(*ii*) $w_{\rho}\left(\Delta^m,\mathcal{I}_{\ln}\right)$ instead of $w^f_{\rho}\left(\Delta^m,\mathcal{I}_{\ln}\right)$ and
(*iii*) $w_{\rho,\infty}\left(\Delta^m,\mathcal{I}_{\ln}\right)$ instead of $w^f_{\rho,\infty}\left(\Delta^m,\mathcal{I}_{\ln}\right)$.

2. When $\rho=1$; the notations are adjusted as follows:
(*i*) $w^f_0\left(\Delta^m,\mathcal{I}_{\ln}\right)$ instead of $w^f_{\rho,0}\left(\Delta^m,\mathcal{I}_{\ln}\right)$,
(*ii*) $w^f\left(\Delta^m,\mathcal{I}_{\ln}\right)$ instead of $w^f_{\rho}\left(\Delta^m,\mathcal{I}_{\ln}\right)$ and
(*iii*) $w^f_\infty\left(\Delta^m,\mathcal{I}_{\ln}\right)$ instead of $w^f_{\rho,\infty}\left(\Delta^m,\mathcal{I}_{\ln}\right)$.

3. In the special case where $f(\varpi)=\varpi$ and $\rho=1$; the notations further simplify:
(*i*) $w_0\left(\Delta^m,\mathcal{I}_{\ln}\right)$ instead of $w^f_{\rho,0}\left(\Delta^m,\mathcal{I}_{\ln}\right)$,
(*ii*) $w\left(\Delta^m,\mathcal{I}_{\ln}\right)$ instead of $w^f_{\rho}\left(\Delta^m,\mathcal{I}_{\ln}\right)$ and
(*iii*) $w_\infty\left(\Delta^m,\mathcal{I}_{\ln}\right)$ instead of $w^f_{\rho,\infty}\left(\Delta^m,\mathcal{I}_{\ln}\right)$.

The following theorem may be stated without evidence.

Theorem 2.10 (*i*) $w^f_{\rho,0}\left(\Delta^m,\mathcal{I}_{\ln}\right)\subset w^f_{\rho,\infty}\left(\Delta^m,\mathcal{I}_{\ln}\right)$ *for every positive ρ and modulus f,*
(*ii*) $w^f_{\rho}\left(\Delta^m,\mathcal{I}_{\ln}\right)\subset w^f_{\rho,\infty}\left(\Delta^m,\mathcal{I}_{\ln}\right)$ *for every $\rho\geq1$ and modulus f.*

Theorem 2.11 *We have the following statements:*
(*i*) $w_{\rho}\left(\Delta^m,\mathcal{I}_{\ln}\right)\subset w^f_{\rho}\left(\Delta^m,\mathcal{I}_{\ln}\right)$;
(*ii*) $w_{\rho,0}\left(\Delta^m,\mathcal{I}_{\ln}\right)\subset w^f_{\rho,0}\left(\Delta^m,\mathcal{I}_{\ln}\right)$;
(*iii*) $w_{\rho,\infty}\left(\Delta^m,\mathcal{I}_{\ln}\right)\subset w^f_{\rho,\infty}\left(\Delta^m,\mathcal{I}_{\ln}\right)$
for every modulus f and $\rho\geq1$

Proof: We focus on the last inclusion, as the others can be proven in the same manner. Let $\varpi\in w_{\rho,\infty}\left(\Delta^m,\mathcal{I}_{\ln}\right)$, then there exists a number $K>0$ such that

$$\left\{\mathfrak{s}\in\mathbb{N}:\frac{1}{l^\rho_\mathfrak{s}}\sum_{\mathfrak{u}=1}^{\mathfrak{s}}\frac{1}{\mathfrak{u}}\left|\Delta^m\varpi_\mathfrak{u}\right|\geq K\right\}\in\mathcal{I}.$$

Let $\varrho>0$ and choose μ with $0<\mu<1$ such that $f(t)<\varrho$ for $0<t<\mu$. So, we have

$$\frac{1}{l^\rho_\mathfrak{s}}\sum_{\mathfrak{u}=1}^{\mathfrak{s}}\frac{1}{\mathfrak{u}}f\left(|\Delta^m\varpi_\mathfrak{u}|\right)\leq\frac{1}{l^\rho_\mathfrak{s}}\sum_{|\Delta^m\varpi_\mathfrak{u}|\leq\mu}\frac{1}{\mathfrak{u}}f\left(|\Delta^m\varpi_\mathfrak{u}|\right)+\frac{1}{l^\rho_\mathfrak{s}}\sum_{|\Delta^m\varpi_\mathfrak{u}|>\mu}\frac{1}{\mathfrak{u}}f\left(|\Delta^m\varpi_\mathfrak{u}|\right).$$

Then, we have

$$\frac{1}{l^\rho_\mathfrak{s}}\sum_{|\Delta^m\varpi_\mathfrak{u}|\leq\mu}\frac{1}{\mathfrak{u}}f\left(|\Delta^m\varpi_\mathfrak{u}|\right)\leq\frac{1}{l^\rho_\mathfrak{s}}\sum\varrho\leq\frac{\mathfrak{s}\varrho}{\mathfrak{s}^\rho}=\frac{\varrho}{\mathfrak{s}^{\rho-1}},\tag{2.5}$$

given $|\Delta^m \varpi_{\mathfrak{u}}| > \mu$ and $|\Delta^m \varpi_{\mathfrak{u}}| < \frac{|\Delta^m \varpi_{\mathfrak{u}}|}{\mu} < 1 + \left[\frac{|\Delta^m \varpi_{\mathfrak{u}}|}{\mu}\right]$, where $[q]$ represents the integral part of q, we may write

$$f(|\Delta^m \varpi_{\mathfrak{u}}|) < \left(1 + \left[\frac{|\Delta^m \varpi_{\mathfrak{u}}|}{\mu}\right]\right) f(1) \leq 2f(1) \frac{|\Delta^m \varpi_{\mathfrak{u}}|}{\mu}$$

using the modulus function definition. Hence

$$\frac{1}{l_{\mathfrak{s}}^{\rho}} \sum_{|\Delta^m \varpi_{\mathfrak{u}}| > \mu} \frac{1}{\mathfrak{u}} f(|\Delta^m \varpi_{\mathfrak{u}}|) \leq 2f(1) \mu^{-1} \frac{1}{l_{\mathfrak{s}}^{\rho}} \sum_{\mathfrak{u}=1}^{\mathfrak{s}} \frac{1}{\mathfrak{u}} |\Delta^m \varpi_{\mathfrak{u}}| \tag{2.6}$$

From Equations (2.5) and (2.6), we have

$$\frac{1}{l_{\mathfrak{s}}^{\rho}} \sum_{\mathfrak{u}=1}^{\mathfrak{s}} \frac{1}{\mathfrak{u}} f(|\Delta^m \varpi_{\mathfrak{u}}|) \leq \frac{\varrho}{\mathfrak{s}^{\rho-1}} + 2f(1) \mu^{-1} \frac{1}{l_{\mathfrak{s}}^{\rho}} \sum_{\mathfrak{u}=1}^{\mathfrak{s}} \frac{1}{\mathfrak{u}} |\Delta^m \varpi_{\mathfrak{u}}|.$$

Since $\rho \geq 1$ and $\varpi \in w_{\rho,\infty}(\Delta^m, \mathcal{I}_{\ln})$, we have $\varpi \in w_{\rho,\infty}^{f}(\Delta^m, \mathcal{I}_{\ln})$. □

Theorem 2.12 *Let ρ be a positive real number and f be a modulus. $w_{\rho}^{f}(\Delta^m, \mathcal{I}_{\ln}) \subset w_{\rho}(\Delta^m, \mathcal{I}_{\ln})$, if $\lim_{t\to\infty} \frac{f(t)}{t} > 0$.*

Proof: Take $\lim_{t\to\infty} \frac{f(t)}{t} > 0$. So, we deduce that $\delta = \lim_{t\to\infty} \frac{f(t)}{t} = \inf\left\{\frac{f(t)}{t}; t > 0\right\}$. From δ's statement, we obtain $f(t) \geq \delta t$ for every $t > 0$. By $\delta > 0$, we get $t \leq \delta^{-1} f(t)$ for each $t > 0$. Thus, we get

$$\frac{1}{l_{\mathfrak{s}}^{\rho}} \sum_{\mathfrak{u}=1}^{\mathfrak{s}} \frac{1}{\mathfrak{u}} |\Delta^m \varpi_{\mathfrak{u}} - \varpi_0| \leq \delta^{-1} \frac{1}{l_{\mathfrak{s}}^{\rho}} \sum_{\mathfrak{u}=1}^{\mathfrak{s}} \frac{1}{\mathfrak{u}} f(|\Delta^m \varpi_{\mathfrak{u}} - \varpi_0|).$$

For each $\varrho > 0$, we get

$$\left\{\mathfrak{s} \in \mathbb{N} : \frac{1}{l_{\mathfrak{s}}^{\rho}} \sum_{\mathfrak{u}=1}^{\mathfrak{s}} \frac{1}{\mathfrak{u}} |\Delta^m \varpi_{\mathfrak{u}} - \varpi_0| \geq \delta^{-1} \varrho\right\} \subseteq \left\{\mathfrak{s} \in \mathbb{N} : \frac{1}{l_{\mathfrak{s}}^{\rho}} \sum_{\mathfrak{u}=1}^{\mathfrak{s}} \frac{1}{\mathfrak{u}} f(|\Delta^m \varpi_{\mathfrak{u}} - \varpi_0|) \geq \varrho\right\} \in \mathcal{I}.$$

The evidence is now complete. □

The following is the outcome of Theorem 2.11 and Theorem 2.12.

Theorem 2.13 *Given that $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ and $\rho \geq 1$, let f be any modulus. $w_{\rho}^{f}(\Delta^m, \mathcal{I}_{\ln}) = w_{\rho}(\Delta^m, \mathcal{I}_{\ln})$ is the next equation.*

Theorem 2.14 *Assume $\kappa \geq \rho > 0$ and f is a modulus. In such case, $w_{\rho}^{f}(\Delta^m, \mathcal{I}_{\ln}) \subset w_{\kappa}^{f}(\Delta^m, \mathcal{I}_{\ln})$ and the inclusion is strict.*

Proof: The proof's inclusion component is simple. Let f be a modulus and examine the sequence provided by Equation (2.4) to demonstrate that the inclusion is stringent. For any $\mathfrak{s} \in \mathbb{N}$, we obtain

$$\frac{1}{l_{\mathfrak{s}}^{\kappa}} \sum_{\mathfrak{u}=1}^{\mathfrak{s}} \frac{1}{\mathfrak{u}} f(|\Delta^m \varpi_{\mathfrak{u}} - 0|) \leq \frac{2\delta^{-1} f(1)}{\mathfrak{s}^{\kappa}} \sum_{\mathfrak{u}=1}^{\mathfrak{s}} \frac{1}{\mathfrak{u}} |\Delta^m \varpi_{\mathfrak{u}}| \leq \frac{2\delta^{-1} f(1) \sqrt{\mathfrak{s}}}{\mathfrak{s}^{\kappa}} \to 0, \text{ as } \mathfrak{s} \to \infty,$$

using the knowledge that $f(0) = 0$.

Hence, $\varpi \in w_{\kappa}^{f}(\Delta^m, \mathcal{I}_{\ln})$ for $\kappa > \frac{1}{2}$. Also

$$\frac{1}{l_{\mathfrak{s}}^{\rho}} \sum_{\mathfrak{u}=1}^{\mathfrak{s}} \frac{1}{\mathfrak{u}} f(|\Delta^m \varpi_{\mathfrak{u}} - \varpi_0|) \geq \frac{\sqrt{\mathfrak{s}} - 1}{\mathfrak{s}^{\rho}} f(1) \to \infty, \text{ as } \mathfrak{s} \to \infty,$$

which gives that $\varpi \notin w_{\rho}^{f}(\Delta^m, \mathcal{I}_{\ln})$ for $\rho \in \left(0, \frac{1}{2}\right)$. □

Finally, we give some relations between logarithmic strong $\Delta^m(f, \mathcal{I})$-Cesàro summability of order ρ and logarithmic $\Delta^m(f, \mathcal{I})$-statistical convergence of order ρ.

Theorem 2.15 *Let ρ and κ be fixed real numbers such that $0<\rho\leq\kappa\leq 1$, f be an unbounded modulus such that $f(uv)\geq cf(u)f(v)$ for all $u>0$; $v>0$; $c>0$ and $\lim_{t\to\infty}\frac{f(t)}{t}>0$, then if a sequence is logarithmic strongly $\Delta^m(f,\mathcal{I})$-Cesàro summable of order ρ to ϖ_0, then it is logarithmic $\Delta^m(f,\mathcal{I})$-statistical convergent of order ρ to ϖ_0.*

Proof: For any sequence $\varpi=(\varpi_{\mathfrak{u}})$ and $\varrho>0$; using the definition of modulus function, we obtain

$$\begin{aligned}\sum_{\mathfrak{u}=1}^{\mathfrak{s}}\tfrac{1}{\mathfrak{u}}f\left(\left|\Delta^m\varpi_{\mathfrak{u}}-\varpi_0\right|\right) &\geq f\left(\sum_{\mathfrak{u}=1}^{\mathfrak{s}}\tfrac{1}{\mathfrak{u}}\left|\Delta^m\varpi_{\mathfrak{u}}-\varpi_0\right|\right)\\ &\geq f\left(\tfrac{1}{\mathfrak{u}}\left|\{\mathfrak{u}\leq\mathfrak{s}:\left|\Delta^m\varpi_{\mathfrak{u}}-\varpi_0\right|\geq\varrho\}\right|\varrho\right)\\ &\geq cf\left(\tfrac{1}{\mathfrak{u}}\left|\{\mathfrak{u}\leq\mathfrak{s}:\left|\Delta^m\varpi_{\mathfrak{u}}-\varpi_0\right|\geq\varrho\}\right|\right)f(\varrho)\end{aligned}$$

and $\rho\leq\kappa$,

$$\begin{aligned}\frac{1}{l_{\mathfrak{s}}^{\rho}}\sum_{\mathfrak{u}=1}^{\mathfrak{s}}\tfrac{1}{\mathfrak{u}}f\left(\left|\Delta^m\varpi_{\mathfrak{u}}-\varpi_0\right|\right) &\geq \frac{cf\left(\frac{1}{\mathfrak{u}}\left|\{\mathfrak{u}\leq\mathfrak{s}:\left|\Delta^m\varpi_{\mathfrak{u}}-\varpi_0\right|\geq\varrho\}\right|\right)f(\varrho)}{l_{\mathfrak{s}}^{\rho}}\\ &\geq \frac{cf\left(\frac{1}{\mathfrak{u}}\left|\{\mathfrak{u}\leq\mathfrak{s}:\left|\Delta^m\varpi_{\mathfrak{u}}-\varpi_0\right|\geq\varrho\}\right|\right)f(\varrho)}{l_{\mathfrak{s}}^{\kappa}}\\ &= \frac{cf\left(\frac{1}{\mathfrak{u}}\left|\{\mathfrak{u}\leq\mathfrak{s}:\left|\Delta^m\varpi_{\mathfrak{u}}-\varpi_0\right|\geq\varrho\}\right|\right)f(\varrho)f(l_{\mathfrak{s}}^{\kappa})}{l_{\mathfrak{s}}^{\kappa}f(l_{\mathfrak{s}}^{\kappa})}\end{aligned}$$

Then, for any $\varsigma>0$, we get

$$\begin{aligned}&\left\{\mathfrak{s}\in\mathbb{N}:\frac{1}{f(\mathfrak{s}^{\kappa})}f\left(\tfrac{1}{\mathfrak{u}}\left|\mathfrak{u}\leq\mathfrak{s}:\left|\Delta^m\varpi_{\mathfrak{u}}-\varpi_0\right|\geq\varrho\right|\right)\geq\varsigma\right\}\\ &\subset\left\{\mathfrak{s}\in\mathbb{N}:\frac{1}{l_{\mathfrak{s}}^{\rho}}\sum_{\mathfrak{u}=1}^{\mathfrak{s}}\tfrac{1}{\mathfrak{u}}f\left(\left|\Delta^m\varpi_{\mathfrak{u}}-\varpi_0\right|\right)\geq cf(\varrho)\varsigma\right\}\in\mathcal{I},\end{aligned}$$

As a result, we get $S_{\rho}^{f}(\Delta^m,\mathcal{I}_{\ln})-\lim\varpi_{\mathfrak{u}}=\varpi_0$. □

Theorem 2.15 readily yields the following findings.

Corollary 2.5 *Let ρ and κ be fixed real numbers such that $0<\rho\leq 1$, f be an unbounded modulus such that $f(uv)\geq cf(u)f(v)$ for all $u>0$; $v>0$; $c>0$ and $\lim_{t\to\infty}\frac{f(t)}{t}>0$, then if a sequence is logarithmic strongly $\Delta^m(f,\mathcal{I})$-Cesàro summable of order ρ to ϖ_0, then it is logarithmic $\Delta^m(f,\mathcal{I})$-statistical convergent of order ρ to ϖ_0.*

Corollary 2.6 *Let f be an unbounded modulus such that $f(uv)\geq cf(u)f(v)$ for all $u>0$; $v>0$; $c>0$ and $\lim_{t\to\infty}\frac{f(t)}{t}>0$, then if a sequence is logarithmic strongly $\Delta^m(f,\mathcal{I})$-Cesàro summable of order ρ to ϖ_0, then it is logarithmic $\Delta^m(f,\mathcal{I})$-statistical convergent of order ρ to ϖ_0.*

Theorem 2.16 *Let f be a modulus function such that $\lim_{t\to\infty}\frac{f(t)}{t}>0$ and $\rho\in(0,1]$. If a sequence is logarithmic strongly $\Delta^m(f,\mathcal{I})$-Cesàro summable of order ρ to ϖ_0, then it is logarithmic $\Delta^m(\mathcal{I})$-statistical convergent of order ρ to ϖ_0.*

Applying Theorem 2.16 allows us to draw the following conclusions.

Corollary 2.7 *Let f be a modulus function such that $\lim_{t\to\infty}\frac{f(t)}{t}>0$. If a sequence is logarithmic strongly $\Delta^m(f,\mathcal{I})$-Cesàro summable to ϖ_0, then it is logarithmic $\Delta^m(\mathcal{I})$-statistical convergent to ϖ_0.*

Corollary 2.8 *If a sequence is logarithmic strongly $\Delta^m(\mathcal{I})$-Cesàro summable to ϖ_0, then it is logarithmic $\Delta^m(\mathcal{I})$-statistical convergent to ϖ_0.*

3. Conclusions

In this paper, we explored various types of logarithmic summability and statistical convergence methods for real sequences. Initially, we introduced the notions of logarithmic $(\Delta^m, \mathcal{I})$-statistical convergence of order ρ and logarithmic strong $\left(\Delta_{\mathfrak{p}}^m, \mathcal{I}\right)$-Cesàro summability of order ρ, followed by an examination of their interconnections. Subsequently, we extended these concepts to logarithmic $\Delta^m(f, \mathcal{I})$-statistical convergence of order ρ and logarithmic strong $\Delta^m(f, \mathcal{I})$-Cesàro summability of order ρ, establishing fundamental relationships between them. Furthermore, we investigated the link between logarithmic $\Delta^m(f, \mathcal{I})$-Cesàro summability of order ρ and $\Delta^m(f, \mathcal{I})$-Cesàro summability of order κ, where κ and ρ are fixed real numbers satisfying $\kappa \geq \rho > 0$. The results obtained contribute to the ongoing study of summability methods and statistical convergence, providing a broader framework for analyzing real sequences under logarithmic transformations.

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References

1. Aizpuru, A., Listán-García, M. C., Rambla-Barreno, F., *Density by moduli and statistical convergence*, Quaest. Math. 37 (4), 525-530, (2014).

2. Alghamdi, M. A., Mursaleen, M., Alotaibi, A., *Logarithmic density and logarithmic statistical convergence*, Adv. Differ. Equ., 227, 1-7, (2013).

3. Bhardwaj, V. K., Dhawan, S., *f-statistical convergence of order α and strong Cesàro summability of order α with respect to a modulus*, J. Inequal. Appl., 2015, 332:14, (2015).

4. Boztepe, A., Dündar, E., *Logarithmic ideal convergence*, AKÜ FEMÜBİD, 25, 89-94, (2025).

5. Çolak, R., *Statistical convergence of order α*, Modern Methods in Analysis and Its Applications, New Delhi, India: Anamaya Pub., 1, 121-129, (2010).

6. Das, P., Savaş, E., Ghosal, S. K., *On generalizations of certain summability methods using ideals*, Appl. Math. Lett., 24(9), 1509-1514, (2011).

7. Et, M., *Strongly almost summable difference sequences of order m defined by a modulus*, Stud. Sci. Math. Hung., 40(4), 463-476, (2003).

8. Et, M., Alotaibi, A., Mohiuddine, S. A., *On $(\Delta^m, \mathcal{I})$-statistical convergence of order α*, Sci. World J., Article ID 535419, 5 pages, (2014).

9. Et, M., Altınok, H., Altın, Y., *On generalized statistical convergence of order α of difference sequences*, J. Funct. Spaces Appl., Article ID 370271, 7 pages, (2013).

10. Et, M., Başarır, M., *On some new generalized difference sequence spaces*, Period. Math. Hung., 35, 169-175, (1997).

11. Et, M., Çolak, R., *On some generalized difference sequence spaces*, Soochow J. Math., 21, 377-386, (1995).

12. Et, M., Gidemen, H., *On $\Delta_v^m(f)$-statistical convergence of order α*, Commun. Stat. Theory Methods, 49(14), 3521-3529, (2020).

13. Fast, H., *Sur la convergence statistique*, Colloq. Math., 2, 241-244, (1951).

14. Gadjiev, A. D., Orhan, C., *Some approximation theorems via statistical convergence*, Rocky Mountain J. Math., 32(1), 129-138, (2002).

15. Gürdal, M., Şahiner, A., *Statistical approximation with a sequence of 2-Banach spaces*, Math. Comput. Model., 55(3-4), 471-479, (2012).

16. Gürdal, M., Sarı, N., Savaş, E., *A-statistically localized sequences in n-normed spaces*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., 69(2), 1484-1497, (2020).

17. Gürdal, M., Yamanci, U., *Statistical convergence and some questions of operator theory*, Dynam. Systems Appl., 24 (7), 305-311, (2015).

18. Kadak, U., Mohiuddine, S. A., *Generalized statistically almost convergence based on the difference operator which includes the p, q-gamma function and related approximation theorems*, Results Math., 73, 9, (2018).

19. Kızmaz, H., *On certain sequence spaces*, Canad. Math. Bull., 24(2), 169-176, (1981).

20. Kostyrko, P., Šalát, T., Wilczyński, W., *I-convergence*, Real Anal. Exchange, 26(2), 669-686, (2000/2001).

21. Maddox, I. J., *Sequence spaces defined by a modulus*, Math. Proc. Camb. Philos. Soc., 100, 161-166, (1986).

22. Mohiuddine, S. A., *Statistical weighted A-summability with application to Korovkin's type approximation theorem*, J. Inequal. Appl., 2016, 101, (2016).

23. Mohiuddine, S. A., Alamri, B. A. S., *Generalization of equi-statistical convergence via weighted lacunary sequence with associated Korovkin and Voronovskaya type approximation theorems*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. RACSAM, 113(3), 1955-1973, (2019).

24. Móricz, F., *Theorems relating to statistical harmonic summability and ordinary convergence of slowly decreasing or oscillating sequences*, Analysis, 24, 127-145, (2004).

25. Nabiev, A., Savaş, E., Gürdal, M., *Statistically localized sequences in metric spaces*, J. Appl. Anal. Comput., 9(2), 739-746, (2019).

26. Yamancı U., Gürdal M., *Statistical convergence and operators on Fock space*, New York J. Math., 22, 199-207, (2016).

27. Yamancı U., Gürdal M., *On lacunary ideal convergence in random n-normed space*, J. Math., 868457, 1-8, (2013).

28. Savas, E., Kisi, Ö., Gürdal, M., *On statistical convergence in credibility space*, Numer. Funct. Anal. Optim., 43(8), 987-1008, (2022).

29. Gürdal M., Şahiner A., Açık, I., *Approximation theory in 2-Banach spaces*, Nonlinear Anal., 71(5-6), 1654-1661, (2009).

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