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Some results in the new direction of multiplicative metric space

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ABSTRACT: Inspired by the recent work of Stojan Radenović and Bessem Samet [9], we prove some fixed point theorems in multiplicative metric space which are not equivalent to the corresponding theorems in metric space using the function $\varphi : [1, +\infty) \to [0, +\infty)$ that satisfies the condition: $\varphi(s) \geq b(\ln s)^c$, $s > 1$, where b and c are positive constants. We give an application in boundary value problem for the second-order differential equation to validate our results.

Key Words: Multiplicative metric space, fixed point, Cauchy sequence, boundary value problem for the second-order differential equation.

Contents

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1. Introduction

The concept of multiplicative calculus was first introduced by Grossman and Katz in 1972 [6]. Inspired by this, Bashirov et al. [3] developed the framework of multiplicative metric spaces. Below, we present some fundamental definitions related to these spaces.

Definition 1.1 *[3] Let Δ be a nonempty set. A function $d_m : \Delta \times \Delta \to [1, +\infty)$ is called a multiplicative metric if it satisfies the following conditions for all $u, v, w \in \Delta$:*

(d_m1) $d_{\mathfrak{m}}(u, v) = 1$ if and only if $u = v$;

(d_m2) $d_{\mathfrak{m}}(u, v) = d_{\mathfrak{m}}(v, u)$ (symmetry);

(d_m3) $d_{\mathfrak{m}}(u, v) \leq d_{\mathfrak{m}}(u, w) d_{\mathfrak{m}}(w, v)$ (multiplicative triangle inequality).

A pair (Δ, d_m) that satisfies these properties is referred to as a multiplicative metric space.

Remark 1.1 *A multiplicative metric $d_m : \Delta \times \Delta \to [1, +\infty)$ is continuous .*

To show this, let us consider two sequences $\{\xi_n\}$ and $\{\zeta_n\}$ in Δ such that $d_{\mathfrak{m}}(\xi_n, \xi) \to 1$ and $d_{\mathfrak{m}}(\zeta_n, \zeta) \to 1$. Now, by using the multiplicative triangle inequality, we get

$$d_{\mathfrak{m}}(\xi_n, \zeta_n) \leq d_{\mathfrak{m}}(\xi_n, \xi) d_{\mathfrak{m}}(\xi, \zeta) d_{\mathfrak{m}}(\zeta_n, \zeta). \tag{1.1}$$

and

$$d_{\mathfrak{m}}(\xi, \zeta) \leq d_{\mathfrak{m}}(\xi, \xi_n) d_{\mathfrak{m}}(\xi_n, \zeta_n) d_{\mathfrak{m}}(\zeta_n, \zeta). \tag{1.2}$$

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Combining (1.1) and (1.2), we get

$$\frac{d_{\mathfrak{m}}(\xi,\zeta)}{d_{\mathfrak{m}}(\xi,\xi_n)d_{\mathfrak{m}}(\zeta_n,\zeta)} \leq d_{\mathfrak{m}}(\xi_n,\zeta_n) \leq d_{\mathfrak{m}}(\xi_n,\xi)d_{\mathfrak{m}}(\xi,\zeta)d_{\mathfrak{m}}(\zeta_n,\zeta).$$

Letting $n \to +\infty$, we get $d_{\mathfrak{m}}(\xi_n,\zeta_n) \to d_{\mathfrak{m}}(\xi,\zeta)$.
This shows that d_m is continuous.

Definition 1.2 *Let (Δ, d_m) be a multiplicative metric space and $\{\xi_n\}$ be a sequence in Δ. Then $\{\xi_n\}$ is said to be multiplicative convergent to $\xi \in \Delta$ if for all $\epsilon > 1$, there exists a natural number m such that $d_{\mathfrak{m}}(\xi_n,\xi) < \epsilon$ for all $n \geq p$ i.e. $\lim_{n\to+\infty} d_{\mathfrak{m}}(\xi_n,\xi) = 1$.*

If $\{\xi_n\}$ is a convergent sequence in a multiplicative metric space, then there exists a unique $\xi \in \Delta$ such that $\lim_{n\to+\infty} d_{\mathfrak{m}}(\xi_n,\xi) = 1$.

Definition 1.3 *Let (Δ, d_m) be a multiplicative metric space and $\{\xi_n\}$ be a sequence in Δ. Then $\{\xi_n\}$ is said to be multiplicative Cauchy, if for all $\epsilon > 1$, there exists a natural number p such that $d_{\mathfrak{m}}(\xi_n,\xi_m) < \epsilon$ for all $n, m \geq p$, i.e. $\lim_{n,m\to+\infty} d_{\mathfrak{m}}(\xi_n,\xi_m) = 1$.*

Definition 1.4 *Let (Δ, d_m) be a multiplicative metric space. If every multiplicative Cauchy sequence is multiplicative convergent to a point $\xi \in \Delta$, then (Δ, d_m) is said to be a complete multiplicative metric space.*

We recall the classical versions of the following fixed-point theorems in metric spaces.

Theorem 1.1 (Banach's Contraction Principle [2]) *Let (Δ, d) be a complete metric space, and let $F : \Delta \to \Delta$ be a mapping. Suppose there exists a constant $\lambda \in (0, 1)$ such that*

$$d(F\xi, F\zeta) \leq \lambda d(\xi,\zeta), \quad \text{for all } \xi,\zeta \in \Delta. \tag{1.3}$$

Then F has a unique fixed point.

Theorem 1.2 (Kannan's Fixed Point Theorem [7]) *Let (Δ, d) be a complete metric space, and let $F : \Delta \to \Delta$ be a mapping. Assume there exists a constant $\lambda \in (0, \frac{1}{2})$ such that*

$$d(F\xi, F\zeta) \leq \lambda\left[d(\xi, F\xi) + d(\zeta, F\zeta)\right], \quad \text{for all } \xi,\zeta \in \Delta. \tag{1.4}$$

Then F has a unique fixed point.

In the year 1973, M. Geraghty proved a fixed point theorem using such functions as a generalization of Banach's fixed point theorem by using a function $\mathcal{F} = \{f : (0, +\infty) \to [0, 1) : f(t_n) \to 1 \implies t_n \to 0\}$.

Theorem 1.3 (Geraghty's Fixed Point Theorem [5]) *Let (Δ, d) be a complete metric space and $F : \Delta \to \Delta$ satisfy*

$$d(F\xi, F\zeta) \leq f(d(\xi,\zeta))d(\xi,\zeta) \quad \text{for all } \xi,\zeta(\xi \neq \zeta) \in \Delta, f \in \mathcal{F}. \tag{1.5}$$

Then F has a unique fixed point in Δ.

Theorem 1.4 (Ćirić-Reich-Rus Fixed Point Theorem [10]) *Let (Δ, d) be a complete metric space and $F : \Delta \to \Delta$ be a mapping. If there exist non-negative constants p, q, r satisfying $p + q + r < 1$ such that*

$$d(F\xi, F\zeta) \leq pd(\xi,\zeta) + qd(\xi, F\xi) + rd(\zeta, F\zeta),$$

for all $\xi,\zeta \in \Delta$. Then, F has a unique fixed point.

Özavşar and Çevikel [8] were the first to present the following multiplicative versions of classical fixed point theorems.

Theorem 1.5 (Multiplicative Version of Banach's Contraction Principle [8]) *Let (Δ, d_m) be a complete multiplicative metric space, and let $F : \Delta \to \Delta$ be a mapping. Suppose there exists a constant $\lambda \in (0,1)$ such that*

$$d_{\mathfrak{m}}(F\xi, F\zeta) \leq [d_{\mathfrak{m}}(\xi,\zeta)]^{\lambda}, \quad \text{for all } \xi,\zeta \in \Delta. \tag{1.6}$$

Then F has a unique fixed point.

Theorem 1.6 (Multiplicative Version of Kannan's Fixed Point Theorem [8]) *Let (Δ, d_m) be a complete multiplicative metric space, and let $F : \Delta \to \Delta$ be a mapping. Assume there exists a constant $\lambda \in (0, \frac{1}{2})$ such that*

$$d_{\mathfrak{m}}(F\xi, F\zeta) \leq [d_{\mathfrak{m}}(\xi, F\xi) d_{\mathfrak{m}}(\zeta, F\zeta)]^{\lambda}, \quad \text{for all } \xi,\zeta \in \Delta. \tag{1.7}$$

Then F has a unique fixed point.

Now, define $\mathcal{G} = \{g : (0,+\infty) \to [0,1) \in \mathcal{G} : g(t_n) \to 1 \implies t_n \to 1\}$

Theorem 1.7 (Multiplicative Version of Geraghty's Fixed Point Theorem [4]) *Let (Δ, d_m) be a complete multiplicative metric space, and let $F : \Delta \to \Delta$ be a mapping. Assume there exists a function $g \in \mathcal{G}$ such that*

$$d_{\mathfrak{m}}(F\xi, F\zeta) \leq d_{\mathfrak{m}}(\xi,\zeta)^{g(d_{\mathfrak{m}}(\xi,\zeta))}, \quad \text{for all } \xi,\zeta \in \Delta. \tag{1.8}$$

Then F has a unique fixed point.

Theorem 1.8 (Multiplicative Version of Ćirić-Reich-Rus Theorem [4]) *Let (Δ, d_m) be a complete multiplicative metric space, and let $F : \Delta \to \Delta$ be a self-mapping. Assume there exist non-negative constants p, q, r with $p + q + r < 1$ satisfying*

$$d_{\mathfrak{m}}(F\xi, F\zeta) \leq (d_{\mathfrak{m}}(\xi,\zeta))^{p} (d_{\mathfrak{m}}(\xi, F\xi))^{q} (d_{\mathfrak{m}}(\zeta, F\zeta))^{r}, \quad \text{for all } \xi,\zeta \in \Delta. \tag{1.9}$$

Then F has a unique fixed point.

However, in 2016, Agrawal et al. [1] and S. Shukla [11] put a critical remark on multiplicative metric spaces that a multiplicative metric space can be deduced to metric space.

Theorem 1.9 *([1], [11]) Let (Δ, d_m) be a multiplicative metric space. Then the pair (Δ, d) is a metric space where $d(\xi,\zeta) = \ln d_{\mathfrak{m}}(\xi,\zeta)$ for all $\xi,\zeta \in \Delta$.*

In a survey, Došenović et al. [4] generalized Theorem 1.9 by proving the converse of Theorem 1.9.

Theorem 1.10 *[4] Let (Δ, d_m) be a multiplicative metric space. Then the pair (Δ, d) is a metric space where $d(\xi,\zeta) = \ln d_{\mathfrak{m}}(\xi,\zeta)$ for all $\xi,\zeta \in \Delta$. Conversely, if (Δ, d) is a metric space then (Δ, d_m) is a multiplicative metric space where $d_{\mathfrak{m}}(\xi,\zeta) = e^{d(\xi,\zeta)}$ for all $\xi,\zeta \in \Delta$.*

The following results are obvious but very important.

1. If d_m is a multiplicative metric on Δ, then d is a metric on Δ;
2. If (Δ, d_m) is a complete multiplicative metric space, then (Δ, d) is a complete metric space;
3. If d is a metric on Δ, then d_m is a multiplicative metric on Δ;
4. If (Δ, d) is a complete metric space, then (Δ, d_m) is a complete multiplicative metric space.

And, consequently, proved that the multiplicative versions of most of the fixed point theorems are actually equivalent to the corresponding classical versions in metric space. Some of the results are given below.

Theorem 1.11 *[4] Theorems 1.1 and 1.5 are equivalent.*

Theorem 1.12 *[4] Theorems 1.2 and 1.6 are equivalent.*

Theorem 1.13 *[4] Theorems 1.3 and 1.7 are equivalent.*

Theorem 1.14 *[4] Theorems 1.4 and 1.8 are equivalent.*

After this interesting survey in [4], it was a natural question for the researchers that

Problem 1 *Is there any such version of Banach, Kannan, Chatterjea, etc., fixed point theorems in multiplicative metric space which are not equivalent to those in metric space?*

This problem remained unsolved until recently, Stojan Radenović and Bessem Samet [9] proved the following Theorem 1.15 and 1.16. This problem was solved with the help of the following function Φ.

Φ is defined as the collection of functions $\varphi : [1, +\infty) \to [0, +\infty)$ that satisfy the condition:

$$\varphi(s) \geq b(\ln s)^c, \quad s > 1, \tag{1.10}$$

where b and c are positive constants.

Remark 1.2 *[9] From (1.10), it follows that*

$$\varphi(s) > 0, \quad s > 1.$$

Additionally, they defined the subset $\Phi^{\uparrow}$ and Φ_0 as:

$$\Phi^{\uparrow} = \{\varphi \in \Phi : \varphi \text{ is non-decreasing}\}$$

and

$$\Phi_0 = \{\varphi \in \Phi : \varphi(0) = 0\} \quad \text{respectively.}$$

Theorem 1.15 *[9] Let (Δ, d_m) be a complete multiplicative metric space, and let $\digamma : \Delta \to \Delta$ be a mapping satisfying the following conditions:*

(i) there exist $\lambda \in (0, 1)$ and a function $\varphi \in \Phi$ such that

$$\varphi(d_{\mathfrak{m}}(\digamma\xi, \digamma\zeta)) \leq \lambda\varphi(d_{\mathfrak{m}}(\xi, \zeta)), \tag{1.11}$$

for all $\xi, \zeta \in \Delta$ with $d_{\mathfrak{m}}(\xi, \zeta) > 1$, and

(ii) for any $u, v \in \Delta$, if

$$\lim_{n\to+\infty} d_{\mathfrak{m}}(\digamma^n u, v) = 1,$$

then there exists a subsequence $\{\digamma^{n_k} u\}$ of $\{\digamma^n u\}$ such that

$$\lim_{k\to+\infty} d_{\mathfrak{m}}(\digamma^{n_k+1} u, \digamma v) = 1.$$

Under these conditions, the mapping $\digamma$ has a unique fixed point.

Theorem 1.16 *[9] Let (Δ, d_m) be a complete multiplicative metric space, and let $\digamma : \Delta \to \Delta$ be a mapping satisfying the following conditions:*

(i) there exist $\lambda \in (0, \frac{1}{2})$ and a function $\varphi \in \Phi$ such that

$$\varphi(d_{\mathfrak{m}}(\digamma\xi, \digamma\zeta)) \leq \lambda[\varphi(d_{\mathfrak{m}}(\xi, \digamma\xi)) + \varphi(d_{\mathfrak{m}}(\zeta, \digamma\zeta))], \tag{1.12}$$

for all $\xi, \zeta \in \Delta$ with $d_{\mathfrak{m}}(\xi, \zeta) > 1$, and

(ii) for any $u, v \in \Delta$, if

$$\lim_{n\to+\infty} d_{\mathfrak{m}}(\digamma^n u, v) = 1,$$

then there exists a subsequence $\{\digamma^{n_k} u\}$ of $\{\digamma^n u\}$ such that

$$\lim_{k\to+\infty} d_{\mathfrak{m}}(\digamma^{n_k+1} u, \digamma v) = 1.$$

Under these conditions, the mapping F has at least one fixed point.

The importance of the function Φ lies in the fact that it does not allow the deduction of (1.11) and (1.12) to (1.3) and (1.4) respectively.The property $\varphi(s) \geq b(\ln s)^c, \quad s > 1$ acts as a bridge from the inequality (1.15) to the Cauchy sequence in the proof of Theorem 1.15 and 1.16. This means that this property of Φ plays an important role in making any sequence satisfying (1.11) and (1.12) in Δ Cauchy. Moreover, under different special conditions on Φ, some other interesting results can be established.

2. Geraghty type fixed point theorem in multiplicative metric space

At first, we give an affirmative answer to the problem 1 by proving the Geraghty-type fixed point theorem in multiplicative metric space, which is not equivalent to Theorem 1.3.

Theorem 2.1 *Let (Δ, d_m) be a complete multiplicative metric space, and let $F : \Delta \to \Delta$ be a mapping satisfying the following conditions:*

(i) there exist $g \in \mathcal{G}$ and a function $\varphi \in \Phi$ such that

$$\varphi(d_{\mathfrak{m}}(F\xi, F\zeta)) \leq g(d_{\mathfrak{m}}(\xi,\zeta))\varphi(d_{\mathfrak{m}}(\xi,\zeta)), \tag{2.1}$$

for all $\xi, \zeta \in \Delta$ with $d_{\mathfrak{m}}(\xi,\zeta) > 1$, and

(ii) for any $u, v \in \Delta$, if

$$\lim_{n\to+\infty} d_{\mathfrak{m}}(F^n u, v) = 1,$$

then there exists a subsequence $\{F^{n_k}u\}$ of $\{F^n u\}$ such that

$$\lim_{k\to+\infty} d_{\mathfrak{m}}(F^{n_k+1}u, Fv) = 1.$$

Under these conditions, the mapping F has a unique fixed point.

Proof: For an arbitrary initial point $\xi_0 \in \Delta$, define the sequence $\{\xi_n\}$ by

$$\xi_n = F^n \xi_0, \quad n \geq 0.$$

If there exists some $n_0 \geq 0$ such that $\xi_{n_0} = \xi_{n_0+1}$, then ξ_{n_0} is a fixed point of F. Otherwise, assuming $\xi_n \neq \xi_{n+1}$ for all $n \geq 0$, it follows that

$$d_{\mathfrak{m}}(\xi_n, \xi_{n+1}) > 1, \text{ for all } n \geq 0.$$

Setting $(\xi,\zeta) = (\xi_n, \xi_{n+1})$ and using (2.1), we get

$$\begin{aligned}\varphi(d_{\mathfrak{m}}(\xi_n,\xi_{n+1})) &= \varphi(d_{\mathfrak{m}}(F\xi_{n-1}, F\xi_n)) \\ &\leq g(d_{\mathfrak{m}}(\xi_{n-1},\xi_n))\varphi(d_{\mathfrak{m}}(\xi_{n-1},\xi_n)),\end{aligned} \tag{2.2}$$

which is equivalent to

$$\varphi(d_{\mathfrak{m}}(\xi_n,\xi_{n+1})) \leq \varphi(d_{\mathfrak{m}}(\xi_{n-1},\xi_n)).$$

This shows that $\{\varphi(d_{\mathfrak{m}}(\xi_n,\xi_{n+1}))\}$ is a non-increasing sequence. So, there exists r such that $\lim_{n\to+\infty} \varphi(d_{\mathfrak{m}}(\xi_n,\xi_{n+1})) = r \geq 0$. Suppose that $r > 0$.

Using (2.2), we get

$$\frac{\varphi(d_{\mathfrak{m}}(\xi_n,\xi_{n+1}))}{\varphi(d_{\mathfrak{m}}(\xi_{n-1},\xi_n))} \leq g(d_{\mathfrak{m}}(\xi_{n-1},\xi_n)) < 1.$$

Letting $n \to +\infty$ in above inequality, $g(\varphi(d_{\mathfrak{m}}(\xi_{n-1},\xi_n))) \to 1$ and since $g \in \mathcal{G}$, we get $d_{\mathfrak{m}}(\xi_{n-1},\xi_n) \to 1$. This is a contradiction as $d_{\mathfrak{m}}(\xi_{n-1},\xi_n) > 1$. Thus, $\lim_{n\to+\infty} \varphi(d_{\mathfrak{m}}(\xi_n,\xi_{n+1})) = 0$.

On the other hand, Using (1.10), we derive

$$\ln(d_{\mathfrak{m}}(\xi_n, \xi_{n+1})) \leq \left[\frac{\varphi(d_{\mathfrak{m}}(\xi_n, \xi_{n+1}))}{b}\right]^{\frac{1}{c}}, \quad n \geq 0. \tag{2.3}$$

And since $\ln(s) > 0$ as $s > 1$, we get

$$0 < \ln(d_{\mathfrak{m}}(\xi_n, \xi_{n+1})) \leq \left[\frac{\varphi(d_{\mathfrak{m}}(\xi_n, \xi_{n+1}))}{b}\right]^{\frac{1}{c}}.$$

Letting $n \to +\infty$, we get $\lim_{n\to+\infty} \ln(d_{\mathfrak{m}}(\xi_n, \xi_{n+1})) = 0$. Now, we show that $\{\xi_n\}$ is a Cauchy sequence. By the multiplicative triangle inequality, we get

$$d_{\mathfrak{m}}(\xi_n, \xi_{n+m}) \leq d_{\mathfrak{m}}(\xi_n, \xi_{n+1}) \cdots d_{\mathfrak{m}}(\xi_{n+m-1}, \xi_{n+m}).$$

Since $\ln(\xi)$ is non decreasing

$$\ln(d_{\mathfrak{m}}(\xi_n, \xi_{n+m})) \leq \ln(d_{\mathfrak{m}}(\xi_n, \xi_{n+1})) + \cdots + \ln(d_{\mathfrak{m}}(\xi_{n+m-1}, \xi_{n+m}))$$

As $n \to +\infty$, $\ln(d_{\mathfrak{m}}(\xi_n, \xi_{n+m})) \to 0$ i.e. $d_{\mathfrak{m}}(\xi_n, \xi_{n+m}) \to 1$.

Thus, $\{\xi_n\}$ is a multiplicative Cauchy sequence in the complete multiplicative space (Δ, d_m), ensuring the existence of a limit point $\xi^* \in \Delta$ such that

$$\lim_{n\to+\infty} d_{\mathfrak{m}}(\xi_n, \xi^*) = 1. \tag{2.4}$$

By condition (ii), this sequence admits a subsequence $\{\xi_{n_k}\}$ satisfying:

$$\lim_{k\to+\infty} d_{\mathfrak{m}}(\xi_{n_k+1}, F\xi^*) = 1. \tag{2.5}$$

From (2.4), (2.5) and the continuity of d_m, we conclude that ξ^* is a fixed point of F.

To prove the uniqueness of the fixed point of F, suppose that there exists two different fixed points ξ^+ and $\xi^\times$ of F, i.e.,

$$\xi^+ = F\xi^+, \quad \xi^\times = F\xi^\times, \quad \text{and} \quad d_{\mathfrak{m}}(\xi^+, \xi^\times) > 1.$$

Applying inequality (2.1) with $(\xi^+, \xi^\times)$, we get:

$$\varphi(d_{\mathfrak{m}}(F\xi^+, F\xi^\times)) \leq g(d_{\mathfrak{m}}(\xi^+, \xi^\times))\varphi(d_{\mathfrak{m}}(\xi^+, \xi^\times)). \tag{2.6}$$

Since $d_{\mathfrak{m}}(\xi^+, \xi^\times) > 1$, $\varphi(d_{\mathfrak{m}}(\xi^+, \xi^\times)) > 0$. Dividing both sides of (2.6) by $\varphi(d_{\mathfrak{m}}(\xi^+, \xi^\times))$, we get $g(d_{\mathfrak{m}}(\xi^+, \xi^\times)) = 1$ i.e. $d_{\mathfrak{m}}(\xi^+, \xi^\times) = 1$. This is a contradiction as $d_{\mathfrak{m}}(\xi^+, \xi^\times) > 1$. This completes the proof. □

We can show that Theorem 2.1 is not equivalent to Theorem 1.3.

The function $d_{\varphi(d_m)} : \Delta \times \Delta \to [0, +\infty)$ is defined as

$$d_{\varphi(d_m)}(\xi, \zeta) = \begin{cases} 0, & \text{if } \xi = \zeta, \\ \ln\left[\varphi(d_m(\xi, \zeta))\right], & \text{if } \xi \neq \zeta. \end{cases}$$

However, this function does not necessarily define a metric on Δ. Therefore, in general, contractive condition (2.1) cannot be transformed into (1.5) in a metric space. As a result, the approach used in [4] is not applicable in this case. The following example demonstrates this.

Example 2.1 *Consider the set $\Delta = \{a_1, a_2, a_3\}$ and a multiplicative metric $d_m : \Delta \times \Delta \to [1, +\infty)$ defined as follows.*

$$d_m = \begin{bmatrix} d_{\mathfrak{m}}(a_1,a_1) & d_{\mathfrak{m}}(a_1,a_2) & d_{\mathfrak{m}}(a_1,a_3) \\ d_{\mathfrak{m}}(a_2,a_1) & d_{\mathfrak{m}}(a_2,a_2) & d_{\mathfrak{m}}(a_2,a_3) \\ d_{\mathfrak{m}}(a_3,a_1) & d_{\mathfrak{m}}(a_3,a_2) & d_{\mathfrak{m}}(a_3,a_3) \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 1 \end{bmatrix}.$$

Notice that

$$\begin{aligned} d_{\mathfrak{m}}(a_1,a_2)d_{\mathfrak{m}}(a_2,a_3) =& 3\times 5 = 15 > 4 = d_{\mathfrak{m}}(a_1,a_3), \\ d_{\mathfrak{m}}(a_1,a_3)d_{\mathfrak{m}}(a_3,a_2) =& 4\times 5 = 20 > 3 = d_{\mathfrak{m}}(a_1,a_2), \\ d_{\mathfrak{m}}(a_2,a_1)d_{\mathfrak{m}}(a_1,a_3) =& 3\times 4 = 12 > 5 = d_{\mathfrak{m}}(a_2,a_3) \end{aligned}$$

which confirms that d_m is a multiplicative metric on Δ.
Now, consider the function $\varphi : [1,+\infty) \to [0,+\infty)$ given by

$$\varphi(s) = \begin{cases} \frac{4s}{3}, & \text{if } 1 \le s \le 3, \\ \frac{5s}{4}, & \text{if } 3 < s \le 4, \\ \frac{s}{5}, & \text{if } s > 4. \end{cases}$$

For all $s > 1$, we observe that

$$\varphi(s) \ge \frac{1}{5}\ln s,$$

which implies that $\varphi \in \Phi$ (indeed, φ satisfies condition (1.10) with $b = \frac{1}{5}$ and $c = 1$).
On the other hand, we find that

$$\varphi(4) = 5 > 4 = 4 \times 1 = \varphi(3)\varphi(5),$$

which leads to

$$\ln\varphi(4) > \ln\varphi(3) + \ln\varphi(5).$$

Rewriting this inequality, we obtain

$$\ln\varphi(d_{\mathfrak{m}}(a_1,a_3)) > \ln\varphi(d_{\mathfrak{m}}(a_1,a_2)) + \ln\varphi(d_{\mathfrak{m}}(a_2,a_3)),$$

which simplifies to

$$d_{\varphi(d_m)}(a_1,a_3) > d_{\varphi(d_m)}(a_1,a_2) + d_{\varphi(d_m)}(a_2,a_3).$$

This confirms that $d_{\varphi(d_m)}$ does not define a metric on Δ.

We provide below an example to illustrate Theorem 2.1.

Example 2.2 *Let $\Delta = \{a_1,a_2,a_3\}$ and define the mapping $\digamma : \Delta \to \Delta$ as follows:*

$$\digamma a_1 = a_1, \quad \digamma a_2 = a_3, \quad \digamma a_3 = a_1.$$

Consider the multiplicative metric $d_m : \Delta\times\Delta \to [1,+\infty)$ as in Example 2.1.
It is important to note that there is no function $g \in \mathcal{G}$ satisfies

$$d_{\mathfrak{m}}(\digamma a_1, \digamma a_2) \le [d_{\mathfrak{m}}(a_1,a_2)]^{g(d_{\mathfrak{m}}(a_1,a_2))}.$$

If such a g were to exist, applying $\ln$ would yield

$$\ln d_{\mathfrak{m}}(a_1,a_3) \le g(d_{\mathfrak{m}}(a_1,a_2))\ln[d_{\mathfrak{m}}(a_1,a_2)].$$

Substituting the values, this reduces to

$$\ln 4 \le g(3)\ln 3,$$

which can be rewritten as

$$g(3) \ge \frac{\ln 4}{\ln 3} > 1.$$

Since this contradicts the assumption that $g(\xi) \in (0,1)$, Theorem 1.7 does not apply in this case.
Now, define the function $\varphi : [1,+\infty) \to [0,+\infty)$ as

$$\varphi(s) = \begin{cases} 0, & s = 1, \\ 3s, & 1 < s \leq 3, \\ s, & 3 < s \leq 4, \\ 3s, & s > 4. \end{cases}$$

For all $s > 1$, it holds that

$$\varphi(s) \geq \ln s,$$

which implies that $\varphi \in \Phi$ (as φ satisfies (1.10) with $b = c = 1$).
Using this definition and taking $g(\xi) = \ln(\xi + e - 1)$, we compute
Case 1: $(i,j) = (1,2)$

$$\frac{\varphi(d_{\mathfrak{m}}(\digamma a_1, \digamma a_2))}{\varphi(d_{\mathfrak{m}}(a_1,a_2))} = \frac{\varphi(d_{\mathfrak{m}}(a_1,a_3))}{\varphi(d_{\mathfrak{m}}(a_1,a_2))} = \frac{\varphi(4)}{\varphi(3)} = \frac{4}{9} < \ln(3 + e - 1) = g(d_{\mathfrak{m}}(a_1,a_2)).$$

Case 2: $(i,j) = (1,3)$, we obtain

$$\varphi(d_{\mathfrak{m}}(\digamma a_1, \digamma a_3)) = \varphi(d_{\mathfrak{m}}(a_1,a_1)) = \varphi(1) = 0.$$

Case 3: $(i,j) = (2,3)$, we have

$$\frac{\varphi(d_{\mathfrak{m}}(\digamma a_2, \digamma a_3))}{\varphi(d_{\mathfrak{m}}(a_2,a_3))} = \frac{\varphi(d_{\mathfrak{m}}(a_3,a_1))}{\varphi(d_{\mathfrak{m}}(a_2,a_3))} = \frac{\varphi(4)}{\varphi(5)} = \frac{4}{15} < \ln(5 + e - 1) = g(d_{\mathfrak{m}}(a_2,a_3)).$$

Thus, $\digamma$ satisfies the contractive condition (2.1) for all $\xi, \zeta \in \Delta$.
Furthermore, for any $u \in \Delta$, the definition of $\digamma$ ensures that

$$\digamma^n u = a_1, \quad n \geq 2.$$

Therefore, if $\lim_{n\to+\infty} d_{\mathfrak{m}}(\digamma^n u, v) = 1$, then $v = a_1$ and

$$\lim_{n\to+\infty} d_{\mathfrak{m}}(\digamma^{n+1}u, \digamma v) = d_{\mathfrak{m}}(\digamma a_1, \digamma a_1) = 1.$$

This satisfies condition (ii) of Theorem 2.1, allowing us to apply the theorem. Consequently, a_1 is the unique fixed point of $\digamma$, validating our result.

We can obtain the condition (ii) of Theorem 2.1 if we use $\Phi^{\uparrow}$ in place of Φ.

Theorem 2.2 *Let (Δ, d_m) be a complete multiplicative metric space, and let $\digamma : \Delta \to \Delta$ be a mapping satisfying*

$$\varphi(d_{\mathfrak{m}}(\digamma\xi, \digamma\zeta)) \leq g(d_{\mathfrak{m}}(\xi,\zeta))\varphi(d_{\mathfrak{m}}(\xi,\zeta)),$$

for all $\xi, \zeta \in \Delta$, $g \in \mathcal{G}$ and a function $\varphi \in \Phi^{\uparrow}$ with $d_{\mathfrak{m}}(\xi,\zeta) > 1$. Then, the mapping $\digamma$ satisfies condition (ii) of Theorem 2.1.

Proof: Consider elements $u, v \in \Delta$ such that

$$\lim_{n\to+\infty} d_{\mathfrak{m}}(\digamma^n u, v) = 1. \tag{2.7}$$

If there exists some integer $n_0 \geq 0$ such that $\digamma^n u = v$ for all $n \geq n_0$, then

$$\lim_{n\to+\infty} d_{\mathfrak{m}}(\digamma^{n+1}u, \digamma v) = d_{\mathfrak{m}}(\digamma v, \digamma v) = 1.$$

Otherwise, there exists a subsequence $\{F^{n_k}u\}$ of $\{F^n u\}$ such that

$$F^{n_k}u \neq v, \quad k \geq 0,$$

which implies that

$$d_{\mathfrak{m}}(F^{n_k}u, v) > 1, \quad k \geq 0.$$

Applying contractive condition with $(\xi, \zeta) = (F^{n_k}u, v)$, we obtain

$$\begin{aligned}\varphi(d_{\mathfrak{m}}(F^{n_k+1}u, Fv)) &\leq g(d_{\mathfrak{m}}(F^{n_k}u, v))\varphi(d_{\mathfrak{m}}(F^{n_k}u, v)), \quad k \geq 0 \\ &< \varphi(d_{\mathfrak{m}}(F^{n_k}u, v))\end{aligned}$$

which, due to the non-decreasing property of φ, leads to

$$1 \leq d_{\mathfrak{m}}(F^{n_k+1}u, Fv) \leq d_{\mathfrak{m}}(F^{n_k}u, v), \quad k \geq 0.$$

Letting $n_k \to +\infty$ and using (2.7), we get

$$d_{\mathfrak{m}}(F^{n_k+1}u, Fv) \to 1.$$

So, the condition (ii) of Theorem 2.1 is satisfied. □

Corollary 2.1 *Let (Δ, d_m) be a complete multiplicative metric space, and let $F : \Delta \to \Delta$ be a mapping satisfying*

$$\varphi(d_{\mathfrak{m}}(F\xi, F\zeta)) \leq g(d_{\mathfrak{m}}(\xi, \zeta))\varphi(d_{\mathfrak{m}}(\xi, \zeta)),$$

for all $\xi, \zeta \in \Delta$, $g \in \mathcal{G}$ and a function $\varphi \in \Phi^{\uparrow}$ with $d_{\mathfrak{m}}(\xi, \zeta) > 1$. Then, the mapping F has a unique fixed point.

Proof: Using Theorem 2.1 and Theorem 2.2, we can easily prove this corollary. □

3. Ćirić-Reich-Rus fixed point theorem in multiplicative metric space

Now we prove the Ćirić-Reich-Rus fixed point theorem in multiplicative metric space, which is not equivalent to Theorem 1.4.

Theorem 3.1 *Let (Δ, d_m) be a complete multiplicative metric space, and let $F : \Delta \to \Delta$ be a mapping satisfying the following conditions:*

(i) there exist non-negative numbers p, q, r satisfying $p + q + r < 1$ and a function $\varphi \in \Phi$ such that

$$\varphi(d_{\mathfrak{m}}(F\xi, F\zeta)) \leq p\varphi(d_{\mathfrak{m}}(\xi, \zeta)) + q\varphi(d_{\mathfrak{m}}(\xi, F\xi)) + r\varphi(d_{\mathfrak{m}}(\zeta, F\zeta)), \tag{3.1}$$

for all $\xi, \zeta \in \Delta$ with $d_{\mathfrak{m}}(\xi, \zeta) > 1$, and

(ii) for any $u, v \in \Delta$, if

$$\lim_{n\to+\infty} d_{\mathfrak{m}}(F^n u, v) = 1,$$

then there exists a subsequence $\{F^{n_k}u\}$ of $\{F^n u\}$ such that

$$\lim_{k\to+\infty} d_{\mathfrak{m}}(F^{n_k+1}u, Fv) = 1.$$

Under these conditions, the mapping F has at least one fixed point.

Proof: For an arbitrary initial point $\xi_0 \in \Delta$, define the sequence $\{\xi_n\}$ by

$$\xi_n = \digamma^n \xi_0, \quad n \geq 0.$$

If there exists some $n_0 \geq 0$ such that $\xi_{n_0} = \xi_{n_0+1}$, then ξ_{n_0} is a fixed point of $\digamma$. Otherwise, assuming $\xi_n \neq \xi_{n+1}$ for all $n \geq 0$, it follows that

$$d_{\mathfrak{m}}(\xi_n, \xi_{n+1}) > 1, \quad \text{for all } n \geq 0.$$

Applying (3.1) with $(\xi, \zeta) = (\xi_0, \xi_1)$, we obtain:

$$\begin{aligned}\varphi(d_{\mathfrak{m}}(\digamma\xi_0, \digamma\xi_1)) &\leq p\varphi(d_{\mathfrak{m}}(\xi_0, \xi_1)) + q\varphi(d_{\mathfrak{m}}(\xi_0, \digamma\xi_0)) + r\varphi(d_{\mathfrak{m}}(\xi_1, \digamma\xi_1)) \\ &\leq p\varphi(d_{\mathfrak{m}}(\xi_0, \xi_1)) + q\varphi(d_{\mathfrak{m}}(\xi_0, \xi_1)) + r\varphi(d_{\mathfrak{m}}(\xi_1, \xi_2)),\end{aligned}$$

which is equivalent to

$$\varphi(d_{\mathfrak{m}}(\xi_1, \xi_2)) \leq \Big(\frac{p+q}{1-r}\Big)\varphi(d_{\mathfrak{m}}(\xi_0, \xi_1)).$$

Again applying (3.1) with $(\xi, \zeta) = (\xi_1, \xi_2)$, we obtain:

$$\begin{aligned}\varphi(d_{\mathfrak{m}}(\digamma\xi_1, \digamma\xi_2)) &\leq p\varphi(d_{\mathfrak{m}}(\xi_1, \xi_2)) + q\varphi(d_{\mathfrak{m}}(\xi_1, \digamma\xi_1)) + r\varphi(d_{\mathfrak{m}}(\xi_2, \digamma\xi_2)) \\ &\leq p\varphi(d_{\mathfrak{m}}(\xi_1, \xi_2)) + q\varphi(d_{\mathfrak{m}}(\xi_1, \xi_2)) + r\varphi(d_{\mathfrak{m}}(\xi_2, \xi_3)) \\ &\leq \frac{p+q}{1-r}\varphi(d_{\mathfrak{m}}(\xi_1, \xi_2)),\end{aligned}$$

which is equivalent to

$$\varphi(d_{\mathfrak{m}}(\xi_1, \xi_2)) \leq \Big(\frac{p+q}{1-r}\Big)^2\varphi(d_{\mathfrak{m}}(\xi_0, \xi_1)).$$

By induction and putting $\kappa = \frac{p+q}{1-r} \in (0,1)$, we get

$$\varphi(d_{\mathfrak{m}}(\xi_n, \xi_{n+1})) \leq \kappa^n \varphi(d_{\mathfrak{m}}(\xi_0, \xi_1)), \quad n \geq 0. \tag{3.2}$$

Using (1.10), we derive

$$\ln(d_{\mathfrak{m}}(\xi_n, \xi_{n+1})) \leq \left[\frac{\varphi(d_{\mathfrak{m}}(\xi_n, \xi_{n+1}))}{b}\right]^{\frac{1}{c}}, \quad n \geq 0. \tag{3.3}$$

From (3.2) and (3.3), we get

$$\begin{aligned}\ln(d_{\mathfrak{m}}(\xi_n, \xi_{n+1})) &\leq \left[\frac{\kappa^n\varphi(d_{\mathfrak{m}}(\xi_0, \xi_1))}{b}\right]^{\frac{1}{c}} \\ &= \left[\frac{\varphi(d_{\mathfrak{m}}(\xi_0, \xi_1))}{b}\right]^{\frac{1}{c}} \alpha^n,\end{aligned} \tag{3.4}$$

where $\alpha = \kappa^{\frac{1}{c}} \in (0,1)$.

Now, we show that $\{\xi_n\}$ is a Cauchy sequence. By the multiplicative triangle inequality, we get

$$d_{\mathfrak{m}}(\xi_n, \xi_{n+m}) \leq d_{\mathfrak{m}}(\xi_n, \xi_{n+1}) \cdots d_{\mathfrak{m}}(\xi_{n+m-1}, \xi_{n+m}),$$

which implies by (3.4) that

$$\begin{aligned}\ln(d_{\mathfrak{m}}(\xi_n, \xi_{n+m})) &\leq \ln(d_{\mathfrak{m}}(\xi_n, \xi_{n+1})) + \cdots + \ln(d_{\mathfrak{m}}(\xi_{n+m-1}, \xi_{n+m})) \\ &\leq \left(\alpha^n + \cdots + \alpha^{n+m-1}\right)\left[\frac{\varphi(d_{\mathfrak{m}}(\xi_0, \xi_1))}{b}\right]^{\frac{1}{c}} \\ &= \frac{\alpha^n(1-\alpha^m)}{1-\alpha}\left[\frac{\varphi(d_{\mathfrak{m}}(\xi_0, \xi_1))}{b}\right]^{\frac{1}{c}} \\ &\leq \frac{\alpha^n}{1-\alpha}\left[\frac{\varphi(d_{\mathfrak{m}}(\xi_0, \xi_1))}{b}\right]^{\frac{1}{c}}.\end{aligned}$$

This implies that

$$d_{\mathfrak{m}}(\xi_n, \xi_{n+m}) \leq e^{\frac{\alpha^n}{\alpha-1}\left[\frac{\varphi(d_{\mathfrak{m}}(\xi_0,\xi_1))}{b}\right]^{\frac{1}{c}}} \to e^0 = 1 \text{ as } n \to +\infty.$$

Thus, $\{\xi_n\}$ is a multiplicative Cauchy sequence in the complete multiplicative space (Δ, d_m), ensuring the existence of a limit point $\xi^* \in \Delta$ such that

$$\lim_{n\to+\infty} d_{\mathfrak{m}}(\xi_n, \xi^*) = 1. \tag{3.5}$$

By condition (ii), this sequence admits a subsequence $\{\xi_{n_k}\}$ satisfying:

$$\lim_{k\to+\infty} d_{\mathfrak{m}}(\xi_{n_k+1}, F\xi^*) = 1. \tag{3.6}$$

From (3.5), (3.6) and the continuity of d_m, we conclude that ξ^* is a fixed point of F. This completes the proof. □

The following example shows that Theorem 3.1 is stronger than Theorems 1.15 and 1.16.

Example 3.1 *Let $\Delta = \{a_1, a_2, a_3\}$ and consider the multiplicative metric $d_m : \Delta \times \Delta \to [1, +\infty)$ defined as follows:*

$$D_m = \begin{bmatrix} d_{\mathfrak{m}}(a_1,a_1) & d_{\mathfrak{m}}(a_1,a_2) & d_{\mathfrak{m}}(a_1,a_3) \\ d_{\mathfrak{m}}(a_2,a_1) & d_{\mathfrak{m}}(a_2,a_2) & d_{\mathfrak{m}}(a_2,a_3) \\ d_{\mathfrak{m}}(a_3,a_1) & d_{\mathfrak{m}}(a_3,a_2) & d_{\mathfrak{m}}(a_3,a_3) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{bmatrix}.$$

Consider the mapping $F : \Delta \to \Delta$ defined by

$$Fa_1 = a_1, \quad Fa_2 = a_2, \quad Fa_3 = a_2.$$

Now, define the function $\varphi : [1, +\infty) \to [0, +\infty)$ as

$$\varphi(s) = \begin{cases} 3, & s = 1, \\ \frac{s+2}{2}, & 1 < s < 4, \\ \frac{3}{4}s, & s \geq 4. \end{cases}$$

For all $s > 1$, it holds that

$$\varphi(s) \geq \frac{1}{10} \ln s,$$

which implies that $\varphi \in \Phi$ (as φ satisfies (1.10) with $b = \frac{1}{10}, c = 1$).

Now, we show that Theorem 1.15 and 1.16 is not applicable because

$$\varphi(d_{\mathfrak{m}}(Fa_1, Fa_2)) = \varphi(d_{\mathfrak{m}}(a_1, a_2))$$

and

$$\begin{aligned} \varphi(d_{\mathfrak{m}}(Fa_2, Fa_3)) &= \varphi(d_{\mathfrak{m}}(a_2, a_2)) \\ &= \varphi(1) = 3 = \frac{1}{2}(3+3) \\ &= \frac{1}{2}(\varphi(1) + \varphi(4)) \\ &= \frac{1}{2}(\varphi(d_{\mathfrak{m}}(a_2, Fa_2)) + \varphi(d_{\mathfrak{m}}(a_3, Fa_3))). \end{aligned}$$

However, F satisfies the condition (i) of Theorem 3.1 for all $\xi, \zeta \in \Delta$ when $p = \frac{2}{5}, q = \frac{3}{10}$ and $r = \frac{233}{1000}$.

Furthermore, for any $u \in \Delta$, the definition of $\digamma$ ensures that

$$\digamma^n a_1 = a_1, \digamma^n a_2 = a_2, \digamma^n a_3 = a_2 \quad n \geq 2.$$

Therefore, if $\lim_{n\to+\infty} d_{\mathfrak{m}}(\digamma^n u, v) = 1$ and $\digamma^n a_1 = a_1$, then $v = a_1$ and

$$\lim_{n\to+\infty} d_{\mathfrak{m}}(\digamma^{n+1}u, \digamma v) = d_{\mathfrak{m}}(\digamma a_1, \digamma a_1) = 1.$$

If $\lim_{n\to+\infty} d_{\mathfrak{m}}(\digamma^n u, v) = 1$ and $\digamma^n a_2 = a_2$, then $v = a_2$ and

$$\lim_{n\to+\infty} d_{\mathfrak{m}}(\digamma^{n+1}u, \digamma v) = d_{\mathfrak{m}}(\digamma a_2, \digamma a_2) = 1.$$

Lastly, if $\lim_{n\to+\infty} d_{\mathfrak{m}}(\digamma^n u, v) = 1$ and $\digamma^n a_3 = a_2$, then $v = a_2$ and

$$\lim_{n\to+\infty} d_{\mathfrak{m}}(\digamma^{n+1}u, \digamma v) = d_{\mathfrak{m}}(\digamma a_2, \digamma a_2) = 1.$$

This satisfies condition (ii) of Theorem 3.1, allowing us to apply the theorem. Consequently, a_1, a_2 are the unique fixed points of $\digamma$, validating our result.

Corollary 3.1 *(Theorem 2.5 of [9]) Let (Δ, d_m) be a complete multiplicative metric space, and let $\digamma : \Delta \to \Delta$ be a mapping satisfying the following conditions:*

(i) there exist $\lambda \in (0,1)$ and a function $\varphi \in \Phi$ such that

$$\varphi(d_{\mathfrak{m}}(\digamma\xi, \digamma\zeta)) \leq \lambda\varphi(d_{\mathfrak{m}}(\xi, \zeta)),$$

for all $\xi, \zeta \in \Delta$ with $d_{\mathfrak{m}}(\xi, \zeta) > 1$, and

(ii) for any $u, v \in \Delta$, if

$$\lim_{n\to+\infty} d_{\mathfrak{m}}(\digamma^n u, v) = 1,$$

then there exists a subsequence $\{\digamma^{n_k} u\}$ of $\{\digamma^n u\}$ such that

$$\lim_{k\to+\infty} d_{\mathfrak{m}}(\digamma^{n_k+1}u, \digamma v) = 1.$$

Under these conditions, the mapping $\digamma$ has a unique fixed point.

Proof: Setting $q = r = 0$ and $p = \lambda$. □

Corollary 3.2 *(Theorem 2.12 of [9]) Let (Δ, d_m) be a complete multiplicative metric space, and let $\digamma : \Delta \to \Delta$ be a mapping satisfying the following conditions:*

(i) there exist $\lambda \in (0,1)$ and a function $\varphi \in \Phi$ such that

$$\varphi(d_{\mathfrak{m}}(\digamma\xi, \digamma\zeta)) \leq \lambda[\varphi(d_{\mathfrak{m}}(\xi, \digamma\xi)) + \varphi(d_{\mathfrak{m}}(\zeta, \digamma\zeta))],$$

for all $\xi, \zeta \in \Delta$ with $d_{\mathfrak{m}}(\xi, \zeta) > 1$, and

(ii) for any $u, v \in \Delta$, if

$$\lim_{n\to+\infty} d_{\mathfrak{m}}(\digamma^n u, v) = 1,$$

then there exists a subsequence $\{\digamma^{n_k} u\}$ of $\{\digamma^n u\}$ such that

$$\lim_{k\to+\infty} d_{\mathfrak{m}}(\digamma^{n_k+1}u, \digamma v) = 1.$$

Under these conditions, the mapping $\digamma$ has at least one fixed point.

Proof: Setting $p = 0$ and $q = r = \lambda$. □

Remark 3.1 *Unlike the Ćirić-Reich-Rus type fixed point theorem in metric space, the fixed point of $\mathcal{F}$ in Theorem 3.1 may not be unique.*

The uniqueness of the fixed point of $\mathcal{F}$ in Theorem 3.1 can be established with the imposition of additional conditions on Φ such as $\Phi^{\uparrow}$ and Φ_0.

We now establish that if $\varphi \in \Phi^{\uparrow}$ in Theorem 3.1, then $\mathcal{F}$ has exactly one fixed point.

Theorem 3.2 *If Φ in Theorem 3.1 is replaced by $\Phi^{\uparrow}$, then $\mathcal{F}$ has a unique fixed point.*

Proof: Continuing the same as in the proof of Theorem 3.1, one can prove that the set of fixed points of $\mathcal{F}$ is nonempty. To prove uniqueness, assume that there exist two different fixed points ξ^{+} and $\xi^{\times}$ of $\mathcal{F}$, i.e.,

$$\xi^{+} = \mathcal{F}\xi^{+}, \quad \xi^{\times} = \mathcal{F}\xi^{\times}, \quad \text{and} \quad d_{\mathfrak{m}}(\xi^{+}, \xi^{\times}) > 1.$$

Applying inequality (3.1) with $(\xi^{+}, \xi^{\times})$, we get:

$$\varphi(d_{\mathfrak{m}}(\mathcal{F}\xi^{+}, \mathcal{F}\xi^{\times})) \leq p\varphi(d_{\mathfrak{m}}(\xi^{+}, \xi^{\times})) + q\varphi(d_{\mathfrak{m}}(\xi^{+}, \mathcal{F}\xi^{+})) + r\varphi(d_{\mathfrak{m}}(\xi^{\times}, \mathcal{F}\xi^{\times}))$$

which simplifies to

$$\varphi(d_{\mathfrak{m}}(\xi^{+}, \xi^{\times})) \leq \frac{q + r}{1 - p}\varphi(1). \tag{3.7}$$

Since $\frac{q+r}{1-p} < 1$, we get

$$\varphi(d_{\mathfrak{m}}(\xi^{+}, \xi^{\times})) < \varphi(1).$$

Since φ is nondecreasing, it follows that

$$d_{\mathfrak{m}}(\xi^{+}, \xi^{\times}) = 1.$$

This contradicts our assumption that $d_{\mathfrak{m}}(\xi^{+}, \xi^{\times}) > 1$, proving that $\mathcal{F}$ has a unique fixed point. □

Similarly, we prove the uniqueness of $\mathcal{F}$ in Theorem 3.1 when $\varphi \in \Phi_0$.

Theorem 3.3 *If Φ in Theorem 3.1 is replaced by Φ_0 $\varphi \in \Phi_0$. Then $\mathcal{F}$ has a unique fixed point.*

Proof: From Theorem 3.1, the set of fixed points of $\mathcal{F}$ is nonempty. To establish uniqueness, assume ξ^{+} and $\xi^{\times}$ are distinct fixed points of $\mathcal{F}$. Using (3.7), we get

$$0 \leq \varphi(d_{\mathfrak{m}}(\xi^{+}, \xi^{\times})) \leq \varphi(1) = 0.$$

Thus, we can conclude.

$$\varphi(d_{\mathfrak{m}}(\xi^{+}, \xi^{\times})) = 0.$$

From the definition of Φ, we know that $\varphi(s) > 0$ for all $s > 1$. Since $d_{\mathfrak{m}}(\xi^{+}, \xi^{\times}) > 1$, this leads to a contradiction. Hence, $\mathcal{F}$ has a unique fixed point. □

4. Application to the boundary value problem for the second-order differential equation

Many researcher has applies fixed point theory in many branches of science(see [12], [3], [6]). Now, we give an application of our main theorem (Theorem 2.2) to the boundary value problem for the second-order differential equation. We also give an example to validate the application.

Consider the boundary value problem for the second-order differential equation.

$$\begin{cases} \xi''(t) = -f(t, \xi(t)), t \in I, \\ \xi(0) = \xi(1) = 0, \end{cases} \tag{4.1}$$

where $I = [0, 1]$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

It is known, and easy to check, that the problem (4.1) is equivalent to the integral equation

$$\xi(t) = \int_0^1 G(t, s) f(s, \xi(s)) ds, \quad \text{for } t \in I, \tag{4.2}$$

where G is the Green function defined by

$$G(t, s) = \begin{cases} t(1 - s) & \text{if } 0 \le t \le s \le 1; \\ s(1 - t) & \text{if } 0 \le s \le t \le 1. \end{cases}$$

That is, if $\xi \in C^2(I, \mathbb{R})$, then ξ is a solution to problem (4.1) iff ξ is a solution of the integral equation (4.2). Let $\Delta = C(I)$ be the space of all continuous functions defined on I. Consider the metric d on Δ defined by $d(\xi, \zeta) = \|\xi - \zeta\|_{+\infty} = \max_{t \in I} |\xi(t) - \zeta(t)|$ is a complete metric space. If $d_{\mathfrak{m}}(\xi, \zeta) = e^{d(\xi,\zeta)}$, then by Theorem 1.10, the pair (Δ, d_m) is a complete multiplicative metric space.

Let $\phi : [0, +\infty) \to [0, +\infty)$ be a function satisfying the following conditions:

1. ϕ is increasing;
2. $\phi(t) = tg(t)$, where $g(t) \in \mathcal{G}$.

As examples of such functions, we can list the following $\phi(t) = \frac{t}{1+t}$ and $\phi(t) = \ln(1 + t)$.

Theorem 4.1 *Besides the above conditions, for all $t \in I, a, b \in \mathbb{R}$, if there exists $\delta \in \mathbb{R} - \{0\}$, we have $\frac{1}{8}|f(t, a) - f(t, b)| \le \phi(|a - b|)$ such that $\varphi(e^{\phi(t)}) \le \frac{\phi(e^t)}{e^t}\varphi(e^t)$ where $t \ge 0$. Then equation (4.1) has a unique solution in $\xi^* \in C^2(I)$.*

Proof: Define the operator $\digamma : C(I) \to C(I)$ by

$$\digamma\xi(t) = \int_0^1 G(t, s) f(s, \xi(s))\, ds, \quad \text{for all } t \in I.$$

Then the problem (4.1) is equivalent to finding $\xi^* \in C(I)$ that is a fixed point of $\digamma$. Let $\xi, \zeta \in C(I)$. We have

$$\begin{aligned} |\digamma\xi(t) - \digamma\zeta(t)| &= \left| \int_0^1 G(t, s)[f(s, \xi(s)) - f(s, \zeta(s))]\, ds \right| \\ &\le \int_0^1 G(t, s)|f(s, \xi(s)) - f(s, \zeta(s))|\, ds \\ &\le 8 \int_0^1 G(t, s)\phi(|\xi(s) - \zeta(s)|) ds \\ &\le 8 \left(\sup_{t \in I} \int_0^1 G(t, s)\, ds \right) \phi(\|\xi - \zeta\|_{+\infty}) \\ &\le \phi(d(\xi, \zeta)). \end{aligned} \tag{4.3}$$

Note that for all $t \in I$,

$$\int_0^1 G(t,s)\,ds = -\frac{t^2}{2} + \frac{t}{2},$$

which implies that $\sup_{t\in I} \int_0^1 G(t,s)\,ds = \frac{1}{8}$. Taking $\max_{t\in I}$ and exponential on both sides of (4.3), we get

$$d_{\mathfrak{m}}(\xi,\zeta) \le e^{\phi(d(\xi,\zeta))}$$

Since φ is increasing, we have

$$\begin{aligned}\varphi(d_{\mathfrak{m}}(\xi,\zeta)) &\le \varphi(e^{\phi(d(\xi,\zeta))})\\ &\le \frac{\phi(e^{d(\xi,\zeta)})}{e^{d(\xi,\zeta)}}\varphi(e^{d(\xi,\zeta)})\\ &= \frac{\phi(d_{\mathfrak{m}}(\xi,\zeta))}{d_{\mathfrak{m}}(\xi,\zeta)}\varphi(d_{\mathfrak{m}}(\xi,\zeta))\\ &= g(d_{\mathfrak{m}}(\xi,\zeta))\varphi(d_{\mathfrak{m}}(\xi,\zeta)).\end{aligned}$$

Therefore, all hypotheses of Theorem 2.2 are satisfied, and so $\digamma$ has a unique fixed point; that is, the problem (4.1) has a unique solution $\xi^* \in C^2(I)$. □

Example 4.1 *Consider the boundary value problem*

$$\xi'' = t + \xi, \xi(0) = \xi(1) = 0 \tag{4.4}$$

which is equivalent to the integral equation

$$\xi(t) = \int_0^t t(1-s)(\xi(s)+s)ds + \int_t^1 s(1-t)(\xi(s)+s)ds. \tag{4.5}$$

Now, for all $t \in I = [0,1], a, b \in \mathbb{R}$

$$\frac{1}{8}|f(t,a) - f(t,b)| = \frac{1}{8}|b+t-a-t| \le |b-a| \le \phi(|b-a|) \quad taking \quad \phi(t) = t.$$

Also, it is obvious that $\varphi(e^{\phi(t)}) \le \frac{\phi(e^t)}{e^t}\varphi(e^t)$ *for all* $t \ge 0$.

So, all the conditions of Theorem 4.1 are satisfied and hence BVP (4.4) has a unique solution which is $\xi^*(t) = \frac{e^2t - t + e^{1-t} - e^{t+1}}{1-e^2} \in C^2[0,1]$ *when* $t \in I$.

5. Conclusion

In this work, we establish the Geraghty and Ćirić-Reich-Rus type fixed point theorem in the framework of multiplicative metric spaces, which are not equivalent to those of metric spaces, using the function $\varphi : [1,+\infty) \to [0,+\infty)$ that satisfies the condition: $\varphi(s) \ge b(\ln s)^c, s > 1$, where b and c are positive constants. Our results generalize the results of Stojan Radenović and Bessem Samet [9]. We give an application of our main theorem (Theorem 2.2) to the boundary value problem for the second-order differential equation. We also give several examples to validate our results. For future research, one can establish fixed point theorems for Chatterjea, Meir-Keeler, Ćirić, etc., type of contractions.

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References

1. R. P. Agarwal, E. Karapinar and B. Samet, *An essential remark on fixed point results on multiplicative metric spaces*, *Fixed Point Theory and Applications*, **2016** (2016), 21.
2. S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, *Fundamenta Mathematicae*, **3** (1922), 133–181.
3. A. E. Bashirov, E. M. Kurpınar and A. Özyapıcı, *Multiplicative calculus and its applications*, *Journal of Mathematical Analysis and Applications*, **337** (2008), 36–48.
4. T. Došenović, M. Postolache and S. Radenović, *On multiplicative metric spaces: survey*, *Fixed Point Theory and Applications*, **2016** (2016), 1–17.
5. M. Geraghty, *On contractive mappings*, *Proceedings of the American Mathematical Society*, **40** (1973), 604–608.
6. M. Grossman and R. Katz, *Non-Newtonian Calculus*, Lee Press, Pigeon Cove, 1972.
7. R. Kannan, *Some results on fixed points*, *Bulletin of the Calcutta Mathematical Society*, **10** (1968), 71–76.
8. M. Özavşar and A. C. Çevikel, *Fixed point of multiplicative contraction mappings on multiplicative metric spaces*, *Journal of Engineering Technology and Applied Sciences*, **2** (2017), 65–79.
9. S. Radenović and B. Samet, *New directions in fixed point theory in multiplicative metric spaces*, *Filomat*, **39**(9) (2025), 3083–3097.
10. S. Reich, *Some remarks concerning contraction mappings*, *Canadian Mathematical Bulletin*, **14** (1971), 121–124.
11. S. Shukla, *Some critical remarks on the multiplicative metric spaces and fixed point results*, *Journal of Advanced Mathematical Studies*, **9**(3) (2016), 454–458.
12. I. Shashikanta, Y. Mahendra, G. A. Hirankumar and A. Moussaoui *A New Interpolative (φ, ψ)-Type $\mathfrak{Z}$-Contraction with Application to Nonlinear Matrix Equation Systems*, *Boletim da Sociedade Paranaense de Matemática*, **43** (2025).

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