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Homogenization of a Stokes flow in porous media under a non-homogeneous slip boundary condition *

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ABSTRACT: We deal with the Stokes problem in a domain of $\mathbb{R}^{\mathbb{N}}, N > 2$, that is ε-periodically perforated by holes of sizes $r_1(\varepsilon\delta_1) = o(\varepsilon)$ and $r_2(\varepsilon\delta_2) = o(\varepsilon)$. A Robin-type condition depending on a parameter γ is prescribed on the boundary of some holes while a Dirichlet condition is imposed on the boundary of the remainding holes as well as on the external boundary of the domain. Our aim is to describe the asymptotic behavior of the fluid's velocity and pressure as ε tends to zero and derive the limit problem. To achieve this, we use the unfolding method introduced by Cioranescu et al. (C. R. Acad. Sci. Paris, Ser. I 335 (2002) 99-104.

Key Words: Periodic unfolding, Stokes system, Robin condition, strange term.

Contents

1. Introduction

In this work, we investigate the asymptotic behavior of the Stokes problem with a Robin type condition in a perforated domain $\Omega_{\varepsilon\delta_1\delta_2}$ with holes of size $\varepsilon\delta$, $\varepsilon-$periodically distributed. Our primary tool is the unfolding method.

We consider here the case where $\delta = \delta(\varepsilon)$ is such that $\delta \to 0$ as $\varepsilon \to 0$. These types of holes are referred to in literature as "small holes"(see e.g. [11]), or "tiny holes" (see e.g. ([1], [2], [3], [15]).
Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$ and $|\partial\Omega| = 0$. The reference (periodicity) cell Y is given by $Y = \left]-\frac{1}{2}, \frac{1}{2}\right[^N$.
We introduce two compact subsets T and B of Y such that $T \cap B = \emptyset$. We assume B and T have Lipschitz boundaries. There exists $\bar{\delta} > 0$ such that $\bar{\delta}\left(\overline{T} \cup \overline{B}\right) \subset Y$.

We denote by $\varepsilon, \delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ three small parameters satisfying $\left(\delta_1(\varepsilon), \delta_2(\varepsilon)\right) \in (0, \bar{\delta}]^2$.
The perforated domain $\Omega_{\varepsilon\delta_1\delta_2}$ is given by removing the following sets of holes:

$$B_{\varepsilon\delta_1} = \text{interior}\Big(\bigcup_{\xi\in\Xi_\varepsilon}(\varepsilon\delta_1\overline{B} + \varepsilon\xi)\Big), \qquad T_{\varepsilon\delta_2} = \text{interior}\Big(\bigcup_{\xi\in\Xi_\varepsilon}(\varepsilon\delta_2\overline{T} + \varepsilon\xi)\Big), \tag{1.1}$$

Thus, the perforated domain $\Omega_{\varepsilon\delta_1\delta_2}$ is defined as $\Omega_{\varepsilon\delta_1\delta_2} = \Omega \setminus (\overline{B}_{\varepsilon\delta_1} \cup \overline{T}_{\varepsilon\delta_2})$,

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where $\Xi_\varepsilon = \{\xi \in \mathbb{Z}^N \mid \varepsilon(\xi + Y) \subset \Omega\}$.
Let us consider the following problem:

$$(\mathcal{P}_{\varepsilon\delta_1\delta_2}) \begin{cases} -\nu \Delta u_{\varepsilon\delta_1\delta_2} + \nabla p_{\varepsilon\delta_1\delta_2} = f \text{ in } \Omega_{\varepsilon\delta_1\delta_2}, \\ \operatorname{div} u_{\varepsilon\delta_1\delta_2} = 0 \text{ in } \Omega_{\varepsilon\delta_1\delta_2}, \\ -p_{\varepsilon\delta_1\delta_2}\, n + \nu \dfrac{\partial u_{\varepsilon\delta_1\delta_2}}{\partial n} + \alpha\varepsilon^\gamma u_{\varepsilon\delta_1\delta_2} = G_{\varepsilon\delta_1} \text{ on } \partial B_{\varepsilon\delta_1}, \\ u_{\varepsilon\delta_1\delta_2} = 0 \text{ on } \partial\Omega \cup \partial T_{\varepsilon\delta_2}. \end{cases} \tag{1.2}$$

We denote f as the field of exterior body forces and $G_{\varepsilon\delta_1}$ is the field of exterior surface forces. The constants $\alpha \geq 0$ and γ are given, as well as $\nu > 0$ the viscosity of the fluid. The outward normal to $B_{\varepsilon\delta_1}$ is denoted by η. The Boundary condition on ∂S means that the stress vector gives rise to a breaking phenomenon due to the term $\alpha\varepsilon^\gamma u$ along with a proportionality effect with respect to the exterior surface forces due to the field G [7]. Our problem (1.2) modelizes the flow of an incompressible viscous fluid through a porous medium under the influence of an external electric field.For more details, see for instance [7,5].
There is an extensive body of literature on the homogenization of perforated domains in $\mathbb{R}^N$. For "small" holes of size ε^α, $\alpha > 0$, Cioranescu and Murat [11] studied the homogeneous Dirichlet problem for the Poisson equation. They established that the size $\varepsilon^{N/N-2}$ ($N > 2$) is "critical" in the sense that the limit problem not only involves the laplacian but also an additional zero- order term, referred by the authors as a "strange term", which depends on the capacity of the set of holes at the limit. The non homogeneous Neumann problem for the laplacian in the same geometrical framework, was analyzed by Conca and Donato [13] who identified a critical hole size of order $\varepsilon^{N/N-1}$. In this case, the contribution of the holes at the limit problem manifests as an additional right-hand side integral term.

Regarding the Stokes problem, a pioneering study by Ene and Sanchez Palencia [14] examined a flow in a periodic porous medium with ε-size holes under a Dirichlet boundary condition. They derived the Darcy law in the limite, by employing the multiple scale method (introduced in [4]), along with sharp error estimates. This provided the first mathematical justification of the experimental Darcy's law.
The Stokes problem with a non-homogeneous slip boundary condition, depending on a parameter $\gamma \in \mathbb{R}^N$ (with still ε-size holes), was studied by Cioranescu, Donato, and Ene [7] using energy methods. They derived, in the limit, for different values of γ, either a Darcy-type law, a Brinkman equation, or a Stokes-type system. However, in the context of small holes, it is the order of the hole size with respect to γ that determines the type of the homogenized problem. When working with perforated domains, the main difficulties arise from the fact that the equations and their solutions are defined on domains that strongly depend on γ. To study convergence as $\varepsilon \to 0$ (i.e., to "homogenize"), extension operators to a fixed domain must be introduced, and test functions specific to each situation must be constructed. Strong regularity of the geometry of the holes and the domain is required to tackle these challenges. These difficulties are addressed by the periodic unfolding method [8], [9], which avoids the need for extension operators and is particularly useful for complex geometries (see Cioranescu et al. [10]). The advantage of the unfolding method is that it separates the scales, allowing holes at different scales to be treated within the same period. Such problems, due to their complexity, cannot be solved using classical homogenization methods. For the Stokes problem, this method was applied by Capatina and Ene [6] and by Zaki [19] for ε-sized holes. This approach was later extended to domains with small holes in Cioranescu et al. [9] and applied by Ould-Hammouda [18,17,12] for the Poisson equation. It was also combined with energy methods for the Stokes problem in collaboration with Karek [16]. In this work, we extend this approach to investigate the asymptotic behavior of the Stokes problem in a perforated domain $\Omega_{\varepsilon\delta_1\delta_2}$ with two kind of holes, $B_{\varepsilon\delta_1}$on which a Robin-type condition depending on a parameter γ is prescribed and the others $T_{\varepsilon\delta_2}$ on which is imposed a Dirichlet conditions, both are being distributed $\varepsilon-$periodically which is a more general framework.

Let us now turn back to problem $(\mathcal{P}_{\varepsilon\delta_1\delta_2})$. We assume that ε, $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ are such that there exist two constants k_1 and k_2

$$k_1 = \lim_{\varepsilon \to 0} \frac{\delta_1^{N-1}}{\varepsilon} \in [0, +\infty[\quad \text{and } k_2 = \lim_{\varepsilon \to 0} \frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon} \in [0, +\infty[. \tag{1.3}$$

Under this assumption, and following the values of γ we show that the solution of (1.2) converges in an appropriate space to u which is characterized as the solution of a limit problem (see Theorems 4.2, 4.3 and 4.4). These new theorems show the presence of four types of contributions in the limit systems.
The term $k_2^2 \Theta$ from equations (4.32) and (4.33) represents the contribution of the Dirichlet holes $T_{\varepsilon\delta_2}$. It corresponds to the "strange term" introduced in Cioranescu and Murat [11] which depends on the capacity of the holes at the limit.

The paper is organized as follows. In Section 2, we list some notations and give the variational formulation of problem $(\mathcal{P}_{\varepsilon\delta_1\delta_2})$. Section 3, for the reader convenience, is devoted to some recalls of the unfolding method for "small holes". Finally, the homogenization results are proved in Section 4, we show that the solution of $(\mathcal{P}_{\varepsilon\delta_1\delta_2})$ converges in an appropriate space to the solution of a homogenized limit problem.

2. Setting of the problem and variational formulation

As in [8], for a.e. $z \in \mathbb{R}^N$, we denote by $[z] \in \mathbb{Z}^N$ the integer part of z and by $\{z\}$ the fractional part of z

$$\{z\} = z - [z] \in Y, \quad \text{for a.e. } z \in \mathbb{R}^N.$$

Hence

$$x = \varepsilon\Big(\Big[\frac{x}{\varepsilon}\Big] + \Big\{\frac{x}{\varepsilon}\Big\}\Big) \quad \text{for a.e. } x \in \mathbb{R}^N.$$

Introduce now the following sets:

$$\widehat{\Omega}_\varepsilon = \text{interior}\Big\{ \bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \overline{Y}) \Big\}, \quad \Lambda_\varepsilon = \Omega \setminus \overline{\widehat{\Omega}}_\varepsilon. \tag{2.1}$$

By definition, the set $\widehat{\Omega}_\varepsilon$ is the largest finite union of $\varepsilon(\xi + \overline{Y})$ cells, $\xi \in \mathbb{Z}^N$, contained in Ω while Λ_ε is the subset of Ω containing parts of $\varepsilon(\xi + \overline{Y})$ cells intersecting the boundary $\partial\Omega$.

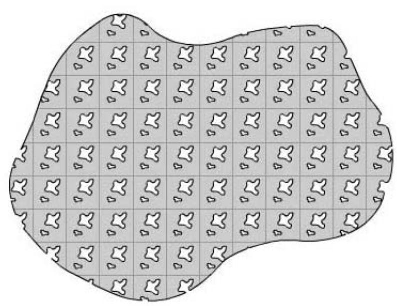

Figure 1: The domain $\Omega_{\varepsilon\delta_1\delta_2}$

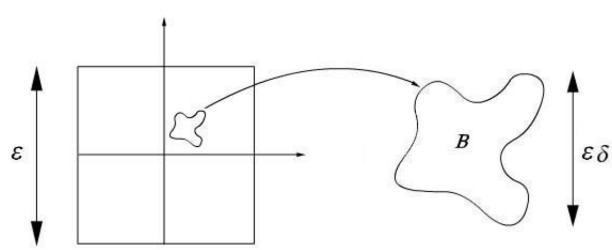

Figure 2: The cell εY_δ

For every bounded domain K included in $\mathbb{R}^N$ with Lipschitz boundary and for every $\varphi \in L^1(\partial K)$, we denote

$$\mathcal{M}_{\partial K}(\varphi) = \frac{1}{|\partial K|} \int_{\partial K} \varphi(z)\, d\sigma(z),$$

the average of φ over ∂K.

Consider the system (1.2), where, $u_{\varepsilon\delta_1\delta_2}$ is the velocity field, $p_{\varepsilon\delta_1\delta_2}$ is the pressure, f is the field of exterior body forces, $G_{\varepsilon\delta_1}$ is the field of exterior surface forces. Recall that $\alpha \geq 0$ and γ are constant, and assume that the data f and $G_{\varepsilon\delta_1}$ satisfy

(i) $f \in L^2(\Omega)^N$.

(ii) $G_{\varepsilon\delta_1} = g_0 + g_{\varepsilon\delta_1}$, where $g_0 \in L^2(\Omega)^N$ and $g_{\varepsilon\delta_1} = g\Big(\dfrac{1}{\delta_1}\Big\{\dfrac{\cdot}{\varepsilon}\Big\}\Big)$, $g \in L^2(\partial B)^N$ satisfying $\mathcal{M}_{\partial B}(g) = 0$.

Let us introduce the following Hilbert spaces:

$$\begin{aligned}\mathbf{H}_{\varepsilon\delta_1\delta_2} &= \left\{ v \mid v \in H^1(\Omega_{\varepsilon\delta_1\delta_2})^N,\ \ v = 0 \text{ on } \partial\Omega \cup \partial T_{\varepsilon\delta_2} \right\},\\ \mathbf{V}_{\varepsilon\delta_1\delta_2} &= \{ v \mid v \in \mathbf{H}_{\varepsilon\delta_1\delta_2},\ \ \operatorname{div} v = 0 \text{ in } \Omega_{\varepsilon\delta_1\delta_2} \},\end{aligned}$$

endowed with the scalar product

$$\langle \varphi, \psi \rangle = \int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla\varphi : \nabla\psi\, dx = \sum_{i,j=1}^{N} \int_{\Omega_{\varepsilon\delta_1\delta_2}} \frac{\partial \varphi_i}{\partial x_j} \frac{\partial \psi_i}{\partial x_j} dx.$$

The variational formulation of system (1.2) is

$$\begin{cases} \text{Find } u_{\varepsilon\delta_1\delta_2} \in \mathbf{V}_{\varepsilon\delta_1\delta_2},\ p_{\varepsilon\delta_1\delta_2} \in L^2(\Omega_{\varepsilon\delta_1\delta_2}) \text{ satisfying} \\ \displaystyle\int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_{\varepsilon\delta_1\delta_2} : \nabla\varphi\, dx + \alpha\varepsilon^\gamma \int_{\partial B_{\varepsilon\delta_1}} u_{\varepsilon\delta_1\delta_2} \cdot \varphi\, d\sigma_{\varepsilon\delta_1}(x) - \int_{\Omega_{\varepsilon\delta_1\delta_2}} p_{\varepsilon\delta_1\delta_2} \operatorname{div}\varphi\, dx \\ \displaystyle = \int_{\Omega_{\varepsilon\delta_1\delta_2}} f \cdot \varphi\, dx + \int_{\partial B_{\varepsilon\delta_1}} G_{\varepsilon\delta_1} \cdot \varphi\, d\sigma_{\varepsilon\delta_1}(x), \\ \forall \varphi \in \mathbf{H}_{\varepsilon\delta_1\delta_2}. \end{cases} \tag{2.2}$$

Classical results (see for details [?]) give the existence of a unique solution to problem (2.2). Our aim is to give the asymptotic behavior of $(u_{\varepsilon\delta_1\delta_2}, p_{\varepsilon\delta_1\delta_2})$ as $\varepsilon \to 0$. To do so, we apply the tools of the periodic unfolding method.

For simplicity, from now on we denote $(u_\varepsilon, p_\varepsilon)$ the solution of the problem (2.2).

3. The unfolding operators $\mathcal{T}_{\varepsilon\delta}$ and $\mathcal{T}^b_{\varepsilon\delta}$

In this section, we recall the definitions and the main properties of the periodic unfolding operators with two small parameters introduced in [9].

Let $\mathbf{C}$ be a bounded domain with Lipschitz boundary, for δ small enough $\delta\overline{\mathbf{C}} \subset Y$. Denote

$$\Omega_{\varepsilon\delta} = \Omega \setminus \overline{\mathbf{C}}_{\varepsilon\delta}, \qquad \mathbf{C}_{\varepsilon\delta} = \text{interior}\Big(\bigcup_{\xi \in \Xi_\varepsilon} (\varepsilon\delta\overline{\mathbf{C}} + \varepsilon\xi) \Big).$$

3.1. The unfolding operator $\mathcal{T}_{\varepsilon\delta}$

According to [9], the geometry of the domains with small holes requires a specific unfolding operator depending on both parameters ε and δ. Below, we will consider functions in $W^{1,p}(\Omega_{\varepsilon\delta})$, $p \in [1, +\infty]$, which vanish on the boundary of the open set $\mathbf{C}_{\varepsilon\delta}$. These functions are naturally extended by zero to the whole domain Ω. Consequently, from now on, we will not distinguish these functions and their extension.

Definition 3.1 *For $\phi \in L^p(\Omega)$, $p \in [1, +\infty]$, the linear and continuous unfolding operator $\mathcal{T}_{\varepsilon\delta} : L^p(\Omega) \to L^p(\Omega \times \mathbb{R}^N)$ is defined by*

$$\mathcal{T}_{\varepsilon\delta}(\phi)(x, z) = \begin{cases} \phi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big] + \varepsilon\delta z\Big) & \textit{for a.e.} \quad (x, z) \in \widehat{\Omega}_\varepsilon \times \dfrac{1}{\delta} Y, \\ 0 & \textit{otherwise.} \end{cases}$$

Observe that for any ϕ in $L^p(\Omega)$, $p\in[1,+\infty]$, we define the local average operator $M^\varepsilon_Y : L^p(\Omega)\to L^p(\Omega)$ by:

$$M^\varepsilon_Y(\phi)=\delta^N\int_{\frac{1}{\delta}Y}\mathcal{T}_{\varepsilon\delta}(\phi)(\cdot,z)dz,$$

and if (v_ε) is a sequence such that $v_\varepsilon\to v$ strongly in $L^p(\Omega)$, then, $M^\varepsilon_Y(v_\varepsilon)\to v$ strongly in $L^p(\Omega)$. Also, we recal the following propositions are proved in [9]:

Proposition 3.1 *For every u in $L^2(\Omega)$ one has*

$$\|\mathcal{T}_{\varepsilon\delta}(u)\|^2_{L^2(\Omega\times\mathbb{R}^N)}\le\frac{1}{\delta^N}\|u\|^2_{L^2(\Omega)}. \tag{3.1}$$

If $\{w_\varepsilon\}_\varepsilon$ is a sequence in $L^1(\Omega)$ satisfying $\displaystyle\int_{\Lambda_\varepsilon}|w_\varepsilon|\,dx\to0$, then

$$\int_\Omega w_\varepsilon\,dx-\delta^N\int_{\Omega\times\mathbb{R}^N}\mathcal{T}_{\varepsilon\delta}(w_\varepsilon)\,dxdz\to0.$$

Proposition 3.2 *Suppose $N\ge3$ and denote by 2^* the Sobolev exponent $\dfrac{2N}{N-2}$ associated to 2. For every $u\in H^1(\Omega)$ the following estimates hold:*

$$\begin{aligned}&\|\nabla_z\big(\mathcal{T}_{\varepsilon\delta}(u)\big)\|^2_{L^2(\Omega\times\frac{1}{\delta}Y)}\le\frac{\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)},\\&\|\mathcal{T}_{\varepsilon\delta}\big(u-M^\varepsilon_Y(u)\big)\|^2_{L^2(\Omega;L^{2^*}(\mathbb{R}^N))}\le\frac{C\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)},\\&\|\mathcal{T}_{\varepsilon\delta}\big(u\big)\|^2_{L^2(\Omega\times\mathbf{C})}\le\frac{C\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)}+C\|u\|^2_{L^2(\Omega)},\end{aligned} \tag{3.2}$$

where C does not depend on ε and δ.

As a consequence of the above proposition, one obtains

Lemma 3.1 *For every $u\in H^1(\Omega)$ one has*

$$\|\mathcal{T}_{\varepsilon\delta}\big(u-M^\varepsilon_Y(u)\big)\|^2_{L^2(\Omega\times\partial\mathbf{C})}\le\frac{C\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)}. \tag{3.3}$$

The constant does not depend on ε and δ.

Proof: Estimates $(3.2)_{1,2}$ lead to

$$\|\mathcal{T}_{\varepsilon\delta}\big(u-M^\varepsilon_Y(u)\big)\|^2_{L^2(\Omega;H^1(\mathbf{C}))}\le\frac{C\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)}.$$

Then (3.3) follows using the trace theorem for the functions in $H^1(\mathbf{C})$. □

When $\delta=1$ we have the following proposition and definition.

Proposition 3.3 *Suppose that $p\in]1,\inf[$. Let (w_ε) be a sequence converging to some w in $W^{1,p}(\Omega)$. Up to a subsequence, there exists some $\hat w$ in $L^p(\Omega;W^{1,p}_{per}(Y))$ such that:*

$$\begin{cases}\mathcal{T}_{\varepsilon1}(w_\varepsilon)\rightharpoonup w & \textit{weakly in } L^p(\Omega;W^{1,p}(Y)),\\ \mathcal{T}_{\varepsilon1}(\nabla w_\varepsilon)\rightharpoonup\nabla w+\nabla\hat w & \textit{weakly in } L^p(\Omega\times Y).\end{cases} \tag{3.4}$$

3.2. The boundary unfolding operator $\mathcal{T}^b_{\varepsilon\delta}$

Definition 3.2 For ϕ in $L^p(\partial\mathbf{C}_{\varepsilon\delta})$, $p\in[1,+\infty]$, the boundary unfolding operator $\mathcal{T}^b_{\varepsilon\delta}: L^p(\partial\mathbf{C}_{\varepsilon\delta})\mapsto L^p(\Omega\times\partial\mathbf{C})$ is defined by

$$\mathcal{T}^b_{\varepsilon\delta}(\phi)(x,z)=\begin{cases}\phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]+\varepsilon\delta z\Big) & \text{for a.e. } (x,z)\in\widehat{\Omega}_\varepsilon\times\partial C,\\ 0 & \text{for a.e. } (x,z)\in\Lambda_\varepsilon\times\partial C.\end{cases}\tag{3.5}$$

The boundary unfolding operator $\mathcal{T}^b_{\varepsilon\delta}$ is linear and continuous, it satisfies

$$\int_{\partial\mathbf{C}_{\varepsilon\delta}}\phi(x)\,d\sigma_{\varepsilon\delta}(x)=\frac{\delta^{N-1}}{\varepsilon}\int_{\Omega\times\partial\mathbf{C}}\mathcal{T}^b_{\varepsilon\delta}(\phi)(x,z)\,dxd\sigma(z),\qquad\forall\phi\in L^1(\partial\mathbf{C}_{\varepsilon\delta}).\tag{3.6}$$

Proposition 3.4 *Let g be in $L^2(\partial\mathbf{C})$. Set*

$$g_{\varepsilon\delta}(x)=g\Big(\frac{1}{\delta}\Big\{\frac{x}{\varepsilon}\Big\}\Big)\quad\textit{for a.e. } x\in\partial\mathbf{C}_{\varepsilon\delta}.\tag{3.7}$$

For every v in $H^1(\Omega)$, one has

$$\Big|\int_{\partial\mathbf{C}_{\varepsilon\delta}}g_{\varepsilon\delta}(x)v(x)d\sigma_{\varepsilon\delta}(x)\Big|\le C\delta^{\frac{N}{2}}\Big(\|g\|_{L^2(\partial\mathbf{C})}\|\nabla v\|_{L^2(\Omega)}+\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}|\mathcal{M}_{\partial\mathbf{C}}(g)|\|v\|_{L^2(\Omega)}\Big).\tag{3.8}$$

The constant does not depend on ε and δ.

Proof: We transform the integral over $\partial\mathbf{C}_{\varepsilon\delta}$ by unfolding. That gives

$$\begin{aligned}\Big|\int_{\partial\mathbf{C}_{\varepsilon\delta}}g_\varepsilon(x)v(x)d\sigma_{\varepsilon\delta}(x)\Big|&=\Big|\frac{\delta^{N-1}}{\varepsilon}\int_{\Omega\times\partial\mathbf{C}}g(z)\mathcal{T}^b_{\varepsilon\delta}(v)(x,z)dxd\sigma(z)\Big|\\&\le\frac{\delta^{N-1}}{\varepsilon}\Big|\int_{\Omega\times\partial\mathbf{C}}g(z)\big(\mathcal{T}^b_{\varepsilon\delta}(v)-M^\varepsilon_Y(v)\big)(x,z)dxd\sigma(z)\Big|+\frac{\delta^{N-1}}{\varepsilon}\Big|\int_{\Omega\times\partial\mathbf{C}}g(z)M^\varepsilon_Y(v)(x)dxd\sigma(z)\Big|.\end{aligned}$$

Due to (3.3), the first integral in the right hand side is estimated by

$$\frac{\delta^{N-1}}{\varepsilon}\Big|\int_{\Omega\times\partial\mathbf{C}}g(z)\big(\mathcal{T}^b_{\varepsilon\delta}(v)-M^\varepsilon_Y(v)\big)(x,z)dxd\sigma(z)\Big|\le C\delta^{\frac{N}{2}}|\Omega|^{1/2}\|g\|_{L^2(\partial\mathbf{C})}\|\nabla v\|_{L^2(\Omega)}.$$

On the other hand one has

$$\int_{\Omega\times\partial\mathbf{C}}g(z)M^\varepsilon_Y(v)(x)dxd\sigma(z)=\mathcal{M}_{\partial\mathbf{C}}(g)\int_\Omega M^\varepsilon_Y(v)(x)dx.$$

Besides

$$\int_\Omega M^\varepsilon_Y(v)(x)dx=\int_{\widehat{\Omega}_\varepsilon}v(x)dx.$$

Hence, the Cauchy Schwartz inequality yields

$$\Big|\int_\Omega M^\varepsilon_Y(v)(x)dx\Big|\le|\Omega|^{1/2}\|v\|_{L^2(\Omega)}.$$

Finally, summarizing the above inequalities and equalities give (3.8). □

4. Main results

We denote K_T the space

$$K_T = \{\Phi \in L^{2^*}(\mathbb{R}^N) \mid \nabla\Phi \in L^2(\mathbb{R}^N),\ \Phi \text{ constant on } T\}. \tag{4.1}$$

Theorem 4.1 *Let $(u_\varepsilon, p_\varepsilon)$ be the solution of (2.2). For $\gamma < -2$ the sequence $\{\varepsilon^{-2}\widetilde{u_\varepsilon}, \widetilde{p_\varepsilon}\}_\varepsilon$ is uniformly bounded in $L^2(\Omega)^N \times L^2(\Omega)$.*
So, up to a subsequence, there exist $(u,p) \in L^2(\Omega)^N \times L^2(\Omega)$ such that:

$$\varepsilon^{-2}\widetilde{u_\varepsilon} \rightharpoonup u \quad \text{weakly in } L^2(\Omega)^N,$$
$$\widetilde{p_\varepsilon} \rightharpoonup p \quad \text{weakly in } L^2(\Omega),$$

then

$$u = 0.$$

Theorem 4.2 (Darcy type law) *Let $(u_\varepsilon, p_\varepsilon)$ be the solution of (2.2). For $-2 \le \gamma < 0$ the sequence $\{\varepsilon^{\gamma}\widetilde{u_\varepsilon}, \varepsilon^{-\gamma}\widetilde{p_\varepsilon}\}_\varepsilon$ is uniformly bounded in $L^2(\Omega)^N \times L^2(\Omega)$.*
So, up to a subsequence, there exist $(u,p) \in L^2(\Omega)^N \times L^2(\Omega)$ such that:

$$\varepsilon^{\gamma}\widetilde{u_\varepsilon} \rightharpoonup u \quad \text{weakly in } L^2(\Omega)^N,$$
$$\varepsilon^{-\gamma}\widetilde{p_\varepsilon} \rightharpoonup p \quad \text{weakly in } L^2(\Omega).$$

Note $U = \mathcal{M}_{\partial B}(u)$ then

$$U = \frac{-1}{\alpha k_1 |\partial B|}\nabla p.$$

Theorem 4.3 $(0 \le \gamma < 2)$ *The sequence $\{\varepsilon^{1+\gamma}\widetilde{u_\varepsilon}, \varepsilon\widetilde{p_\varepsilon}\}_\varepsilon$ is bounded in $L^2(\Omega)^N \times L^2(\Omega)$. So, up to a subsequence, there exist $(u,p) \in L^2(\Omega)^N \times L^2(\Omega)$, such that*

$$\varepsilon^{1+\gamma}\widetilde{u_\varepsilon} \rightharpoonup u \quad \text{weakly in } L^2(\Omega)^N,$$
$$\varepsilon p_\varepsilon \rightharpoonup p \quad \text{weakly in } L^2(\Omega).$$

It satisfies the following equation:

$$u = \frac{1}{\alpha k_1 |\partial B|}\Big(-\nabla p + |\partial B| g_0 + |\partial B|\mathcal{M}_{\partial B} g\Big).$$

Theorem 4.4 (Brinkman type law) *Assume (1.3). Let $(u_\varepsilon, p_\varepsilon)$ be the solution of problem (2.2).*
If $\gamma = 2$, the sequence $\{(\varepsilon^2\widetilde{u_\varepsilon}, p_\varepsilon)\}_\varepsilon$ is uniformly bounded in $L^2(\Omega)^N \times L^2(\Omega)$. So, up to a subsequence, there exists $(u,p) \in L^2(\Omega)^N \times L^2(\Omega)$ such that

$$\varepsilon^2\widetilde{u_\varepsilon} \rightharpoonup u \quad \text{weakly in } L^2(\Omega)^N,$$
$$\widetilde{p_\varepsilon} \rightharpoonup p \quad \text{weakly in } L^2(\Omega).$$

Moreover, there exist $\widehat{u} \in L^2(\Omega; H^1_{per}(Y))^N$, $U \in L^2(\Omega; L^2_{loc}(\mathbb{R}^N))$ with $U - k_2 u \in L^2(\Omega; K_T)$, such that $(u, \widehat{u}, U, p)$ solves the equations:

$$\nu\int_Y (\nabla u(x) + \nabla_y\widehat{u}(x,y))\nabla_y\phi(y)dy = 0 \text{ for a.e. } x \in \Omega,\ \ \forall\phi \in H^1_{per}(Y)^N. \tag{4.2}$$

$$\int_{\mathbb{R}\setminus T} \nabla_z U(x,z)\nabla_z v(z)dz = 0 \text{ for a.e. } x \in \Omega,\ \ \forall v \in K_T,\ \ v(T) = 0. \tag{4.3}$$

$$\begin{aligned}\nu\int_{\Omega\times Y}(\nabla u + \nabla_y\widehat{u})\nabla\psi dxdy - \nu k_2\int_{\Omega\times\partial T}\nabla_z U n_T\psi dxd\sigma + \alpha k_1|\partial B|\int_\Omega u\psi dx + \int_\Omega \nabla p\psi dx \\ = \int_\Omega f\psi dx + k_1|\partial B|\int_\Omega g_0\psi dx + k_1|\partial B|\ M_{\partial B}\int_\Omega \psi dx.\end{aligned} \tag{4.4}$$

If $\gamma > 2$, the sequence $\left\{\left(\widetilde{u_\varepsilon}, \widetilde{p_\varepsilon}\right)\right\}_\varepsilon$ is uniformly bounded in $L^2(\Omega)^N \times L^2(\Omega)$. Then, up to a subsequence, there exist $(u,p) \in H_0^1(\Omega)^N \times L^2(\Omega)$ such that

$$\begin{aligned}
&\widetilde{u_\varepsilon} \rightharpoonup u \quad \text{weakly in } \; L^2(\Omega)^N,\\
&\|\widetilde{u}_\varepsilon - u\|_{L^2(\Omega)} \longrightarrow 0,\\
&\widetilde{p_\varepsilon} \rightharpoonup p \quad \text{weakly in } \; L^2(\Omega).
\end{aligned}$$

Moreover, there exist $\widehat{u} \in L^2(\Omega; H^1_{per}(Y))^N$, $U \in L^2(\Omega; L^2_{loc}(\mathbb{R}^N))^N$ with $U - k_2 u \in L^2(\Omega; K_T)$, such that $(u, \widehat{u}, U, p)$ solves the equations (4.2), (4.3) and

$$\begin{aligned}
\nu \int_{\Omega\times Y} (\nabla u + \nabla_y \widehat{u}) : \nabla\psi dxdy - \nu k_2 \int_{\Omega\times\partial T} \nabla_z U n_T \psi dx d\sigma + \int_\Omega \nabla p \psi dx = \\
\int_\Omega f\cdot\psi dx + k_1|\partial B| \int_\Omega g_0 \cdot \psi \, dx + k_1 |\partial B| \; M_{\partial B} \int_\Omega \psi dx \;\; \forall \psi \in H_0^1(\Omega).
\end{aligned} \tag{4.5}$$

Lemma 4.1 *There exists a positive constant c independent of ε and δ such that*

$$\|v\|_{L^2(\Omega_{\varepsilon\delta})} \le c\Big(\varepsilon\delta \|\nabla v\|_{L^2(\Omega_{\varepsilon\delta})} + (\varepsilon\delta)^{\frac{1}{2}} \|v\|_{L^2(\partial B_{\varepsilon\delta})}\Big) \qquad \forall v \in H_{\varepsilon\delta_1\delta_2}.$$

Proof: We use the following inequality (see [?, Lemma 6.1])

$$\|v\|^2_{L^2(\Omega_{\varepsilon\delta})} \le c\Big(\|\nabla_y v\|^2_{L^2(\Omega_{\varepsilon\delta})} + \|v\|^2_{L^2(\partial B)}\Big), \qquad \forall v \in H_{\varepsilon\delta_1\delta_2}.$$

and by the change $z = \dfrac{y}{\delta}$, we get the result. □

Lemma 4.2 *(see [?, Lemma 5.1]) For every $\phi \in L^2(\Omega_{\varepsilon\delta_1\delta_2})$, there exists $\varphi \in V_{\varepsilon\delta_1\delta_2}$ such that*

$$\begin{cases} div\,\varphi = \phi, \\ \|\varphi\|_{H^1(\Omega_{\varepsilon\delta_1\delta_2})} \le C \|\phi\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}. \end{cases}$$

Apriori estimates for u_ε and p_ε

Proposition 4.1 *Let $(u_\varepsilon, p_\varepsilon)$ be the solution of problem (2.2). Then the following a priori estimates hold true:*
(i) For $\gamma < -2$

$$\begin{aligned}
&\|\varepsilon^{-2} u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le C,\\
&\|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le C \; .
\end{aligned} \tag{4.6}$$

(ii) For $-2 \le \gamma < 0$

$$\begin{aligned}
&\|\varepsilon^{\gamma} u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le C,\\
&\|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le C\\
&\|\varepsilon^{-\gamma} p_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le C.
\end{aligned} \tag{4.7}$$

(iii) For $0 \le \gamma < 2$

$$\begin{aligned}
&\|\varepsilon^{1+\gamma} u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le C,\\
&\|\varepsilon^{\frac{\gamma}{2}} \nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le C,\\
&\|\varepsilon p_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le C.
\end{aligned} \tag{4.8}$$

(iv) For $\gamma = 2$

$$\begin{aligned}
&\|\varepsilon^{2} u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le C,\\
&\|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le C,
\end{aligned} \tag{4.9}$$

$$\|p_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le C.$$

(iiv) For $\gamma > 2$

$$\begin{aligned} \|u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} &\le C, \\ \|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} &\le C, \\ \|p_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} &\le C. \end{aligned} \tag{4.10}$$

Proof: To prove this proposition, we combine the estimates given in Proposition 3.4. Using u_ε as test fonction in (2.2) we get

$$\nu \int_{\Omega_{\varepsilon\delta_1\delta_2}} |\nabla u_\varepsilon|^2 dx + \alpha\varepsilon^\gamma \int_{\partial B_{\varepsilon\delta_1}} |u_\varepsilon|^2 d\sigma(x)$$
$$= \int_{\Omega_{\varepsilon\delta_1\delta_2}} f \cdot u_\varepsilon dx + \int_{\partial B_{\varepsilon\delta_1}} g_0 \cdot u_\varepsilon d\sigma(x) + \langle g_\varepsilon, u_\varepsilon \rangle_{(H^{-1/2}(\partial B_{\varepsilon\delta_1}))^N,(H^{1/2}(\partial B_{\varepsilon\delta_1}))^N}.$$

Then

$$\nu\|\nabla u_\varepsilon\|^2_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} + \alpha\varepsilon^\gamma\|u_\varepsilon\|^2_{L^2(\partial B_{\varepsilon\delta_1})} \le \Big|\int_{\Omega_{\varepsilon\delta_1\delta_2}} f\cdot u_\varepsilon dx\Big| + \Big|\int_{\partial B_{\varepsilon\delta_1}} g_0 \cdot u_\varepsilon d\sigma(x)\Big|$$
$$+ \Big|\langle g_\varepsilon, u_\varepsilon \rangle_{H^{-1/2}(\partial B_{\varepsilon\delta_1})^N,H^{1/2}(\partial B_{\varepsilon\delta_1})^N}\Big|.$$

By the Cauchy Schwartz and Poincaré inequalities, we have

$$\Big|\int_{\Omega_{\varepsilon\delta_1\delta_2}} f\cdot u_\varepsilon dx\Big| \le \|f\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}\|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}, \tag{4.11}$$

and we have successively, by using Proposition 3.4

$$\Big|\int_{\partial B_{\varepsilon\delta_1}} g_0 \cdot u_\varepsilon d\sigma(x)\Big| \le C\delta_1^{\frac{N}{2}}\|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} + C\frac{\delta_1^{N-1}}{\varepsilon}\|u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})},$$

and finally

$$\Big|\langle g_\varepsilon, u_\varepsilon\rangle\Big|_{H^{-1/2}(\partial B_{\varepsilon\delta_1})^N,H^{1/2}(\partial B_{\varepsilon\delta_1})^N} \le C\delta_1^{\frac{N}{2}}\|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} + C\frac{\delta_1^{N-1}}{\varepsilon}\|u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}.$$

Hence

$$\nu\|\nabla u_\varepsilon\|^2_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} + \alpha\varepsilon^\gamma\|u_\varepsilon\|^2_{L^2(\partial B_{\varepsilon\delta_1})} \le C(1+\delta_1^{\frac{N}{2}})\|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} + C\frac{\delta_1^{N-1}}{\varepsilon}\|u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}.$$

From which, using the Poincaré inequality and due to assumption on k_1 in $(1.3)_1$, that gives

$$\|\nabla u_\varepsilon\|^2_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le C\Big(1+\frac{\delta_1^{N-1}}{\varepsilon}\Big)\|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}.$$

Thus

$$\|u_\varepsilon\|_{H^1(\Omega_{\varepsilon\delta_1\delta_2})} \le C. \tag{4.12}$$

This estimate can be refined following the different values of γ. To do so, observe that according to Lemma 4.1

$$\|u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le C\Big(\varepsilon\delta_1\|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} + (\varepsilon\delta_1)^{\frac{1}{2}}\|u_\varepsilon\|_{L^2(\partial B_{\varepsilon\delta_1})}\Big),$$

then

$$\frac{\delta_1^{N-1}}{\varepsilon}\|u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le c\Big(\delta_1^N\|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} + \Big(\frac{\delta_1^{N-1}}{\varepsilon}\Big)^{\frac{1}{2}}\delta_1^{\frac{N}{2}}\|u_\varepsilon\|_{L^2(\partial B_{\varepsilon\delta_1})}\Big).$$

Using Young's inequality, we get

$$\frac{\delta_1^{N-1}}{\varepsilon}\|u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le C\Big(\delta_1^N\|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} + \frac{1}{\eta}\frac{\delta_1^{N-1}}{\varepsilon}\varepsilon^{-\gamma} + \eta\varepsilon^\gamma\delta_1^N\|u_\varepsilon\|^2_{L^2(\partial B_1)}\Big).$$

Consequently

$$\nu\|\nabla u_\varepsilon\|^2_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} + (\alpha - C\eta\delta_1^N)\varepsilon^\gamma\|u_\varepsilon\|^2_{L^2(\partial B_{\varepsilon\delta_1})} \le C(1+\delta_1^N)\|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} + \frac{C}{\eta}\frac{\delta_1^{N-1}}{\varepsilon}\varepsilon^{-\gamma}.$$

Then for suitable η we finally obtain the following a priori estimate:

$$\nu\|\nabla u_\varepsilon\|^2_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} + \alpha\varepsilon^\gamma\|u_\varepsilon\|^2_{L^2(\partial B_{\varepsilon\delta_1})} \le C\|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} + C\frac{\delta_1^{N-1}}{\varepsilon}\varepsilon^{-\gamma}. \tag{4.13}$$

(i) Case $\gamma < -2$. From (4.13), one has

$$\|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le C.$$

Now, by using estimate (4.12) and the fact that for $H^1(\Omega_\varepsilon)$ (see for example, [13]),

$$\|v\|_{(L^2(\Omega_\varepsilon))^N} \le C\varepsilon\|\nabla v\|_{(L^2(\Omega_\varepsilon))^{N\times N}},$$

and by the Young inequality we get the estimate (4.6).
(ii) Case $-2 \le \gamma < 0$. From (4.13), one has

$$\|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le C,$$

and then

$$\|u_\varepsilon\|_{L^2(\partial B_{\varepsilon\delta_1})} \le C\varepsilon^{-\frac{\gamma}{2}}.$$

By lemma 4.1 and using the Young inequality, we get

$$\|u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le C(\varepsilon\delta_1 + \frac{\varepsilon\delta_1}{\eta} + \eta\varepsilon^{-\gamma}).$$

Hence

$$\|\varepsilon^\gamma u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le C,$$

which is the first estimate in (4.7)
(ii) Case $0 \le \gamma < 2$. On one hand by (4.13), one has

$$\|\varepsilon^{\frac{\gamma}{2}}\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le C.$$

On the other hand we have

$$\|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le C\Big(\frac{\delta_1^{N-1}}{\varepsilon}\Big)^{\frac{1}{2}}\varepsilon^{-\frac{\gamma}{2}} \implies \|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le C\delta_1^{\frac{N-1}{2}}\varepsilon^{\frac{-1-\gamma}{2}}.$$

Then by the Poincaré inequality and again using the Young inequality, we get

$$\|u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le \frac{C}{\eta}\delta_1^{N-1} + \eta\varepsilon^{-1-\gamma}.$$

Consequently,

$$\|\varepsilon^{1+\gamma}u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le C.$$

(iii) Case $\gamma = 2$. From (4.13), one has

$$\nu\|\nabla u_\varepsilon\|^2_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le C\frac{\delta_1^{N-1}}{\varepsilon}\varepsilon^{-2},$$

then, again successively by Poincaré and Young inequalities, we get

$$\|\varepsilon^2 u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \leq C.$$

The estimate of ∇u_ε follows easily from (4.12).

(iv) Case $\gamma > 2$. The estimate of ∇u_ε and u_ε in (4.10) follow easily from (4.12).

Now, we prove the upper estimates of the pressure.

In the end, we shall establish the a priori estimates of the pressure p_ε. Indeed, we choose $\phi \in L^2(\Omega_{\varepsilon\delta_1\delta_2})$ as a test function in the variational formulation (2.2) and by Proposition 3.4 and Lemma 4.2, we get

$$\begin{aligned}\Big|\int_{\Omega_{\varepsilon\delta_1\delta_2}} p_\varepsilon\,\phi\,dx\Big| \leq C\Big[&(1+\varepsilon^\gamma\delta_1^{\frac{N}{2}})\|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}\\ &+\varepsilon^\gamma\frac{\delta_1^{N-1}}{\varepsilon}\|u_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}+\frac{\delta_1^{N-1}}{\varepsilon}\Big]\|\phi\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}.\end{aligned} \tag{4.14}$$

The a priori estimates for the pressure follow now from (4.14) and estimates the u_ε and ∇u_ε for the different values of γ obtained above. □

Proof of theorem 4.1

Proof: Case $\gamma < -2$. The corresponding estimates (4.6) from proposition 4.1 as wel as the following ones:

$$\varepsilon^{-2}\widetilde{u_\varepsilon} \rightharpoonup u \quad \text{weakly in} \quad L^2(\Omega)^N.$$

Let $\varphi \in D(\Omega)^N$ be a test fonction in (2.2). Multiplying (2.2) by $\varepsilon^{-\gamma-2}$, then unfolding. That gives

$$\begin{aligned}&\nu\varepsilon^{-\gamma-2}\int_{\Omega\times Y}\mathcal{T}_\varepsilon(\nabla u_\varepsilon):\mathcal{T}_\varepsilon(\nabla\varphi)dxdy+\alpha\frac{\delta_1^{N-1}}{\varepsilon}\int_{\Omega\times\partial B}\mathcal{T}^b_{\varepsilon\delta_1}(\varepsilon^{-2}u_\varepsilon)\cdot\mathcal{T}^b_{\varepsilon\delta_1}(\varphi)dxd\sigma(z)\\ &-\int_{\Omega_{\varepsilon\delta_1\delta_2}}\varepsilon^{-\gamma-2}p_\varepsilon\operatorname{div}\varphi\,dx=\varepsilon^{-\gamma-2}\int_{\Omega_{\varepsilon\delta_1\delta_2}}f\cdot\varphi dx+\varepsilon^{-\gamma-2}\int_{\partial B_{\varepsilon\delta_1}}g_0\cdot\varphi d\sigma_{\varepsilon\delta_1}(x)\\ &+\varepsilon^{-\gamma}\int_{\partial B_{\varepsilon\delta_1}}g_\varepsilon\cdot\varphi d\sigma_{\varepsilon\delta_1}(x).\end{aligned}$$

Passing to the limit we observe that all the integrals vanish, except for the second integral we get

$$\alpha k_1\int_{\Omega\times\partial B}u\varphi dxd\sigma(z)=0, \tag{4.15}$$

this imply

$$u=0.$$

□

Proof of theorem 4.2

Proof: Case $-2 \leq \gamma < 0$. The corresponding estimates (4.7) from proposition 4.1 as wel as the following ones:

$$\varepsilon^{\gamma}\widetilde{u_\varepsilon} \rightharpoonup u \quad \text{weakly in} \quad L^2(\Omega)^N, \qquad \varepsilon^{-\gamma}\widetilde{p_\varepsilon} \rightharpoonup p \quad \text{weakly in} \quad L^2(\Omega).$$

Let $\varphi \in D(\Omega)^N$ be a test fonction in (2.2). Multiplying (2.2) by $\varepsilon^{-\gamma}$, then unfolding. That gives

$$\begin{aligned}&\nu\varepsilon^{-\gamma}\int_{\Omega\times Y}\mathcal{T}_\varepsilon(\nabla u_\varepsilon):\mathcal{T}_\varepsilon(\nabla\varphi)dxdy+\alpha\frac{\delta_1^{N-1}}{\varepsilon}\int_{\Omega\times\partial B}\mathcal{T}^b_{\varepsilon\delta_1}(u_\varepsilon)\cdot\mathcal{T}^b_{\varepsilon\delta_1}(\varphi)dxd\sigma(z)\\ &-\int_{\Omega_{\varepsilon\delta_1\delta_2}}\varepsilon^{-\gamma}p_\varepsilon\operatorname{div}\varphi\,dx=\varepsilon^{-\gamma}\int_{\Omega_{\varepsilon\delta_1\delta_2}}f\cdot\varphi dx+\varepsilon^{-\gamma}\int_{\partial B_{\varepsilon\delta_1}}g_0\cdot\varphi d\sigma_{\varepsilon\delta_1}(x)\\ &+\varepsilon^{-\gamma}\int_{\partial B_{\varepsilon\delta_1}}g_\varepsilon\cdot\varphi d\sigma_{\varepsilon\delta_1}(x).\end{aligned}$$

Now, we can pass to limit in all terms. Using again the fact that ∇u_ε is uniformly bounded in $L^2(\Omega_{\varepsilon\delta_1\delta_2})^{N\times N}$ and as $-\gamma > 0$, we get at the limit the following identity:

$$\alpha k_1 \int_{\Omega\times\partial B} u\varphi dx d\sigma(z) - \int_\Omega p \operatorname{div} \varphi dx = 0, \tag{4.16}$$

one has Darcy law

$$u = \frac{-1}{\alpha k_1 |\partial B|} \nabla p.$$

□

Proof of theorem 4.3

Proof: $0 \leq \gamma < 2$. From corresponding estimates (4.8) in Proposition 4.1,it follow that

$$\varepsilon^{1+\gamma}\widetilde{u_\varepsilon} \rightharpoonup u \quad \text{weakly in } \ L^2(\Omega), \qquad \varepsilon p_\varepsilon \rightharpoonup p \quad \text{weakly in } \ L^2(\Omega).$$

Let $\varphi \in (D(\Omega))^N$ be a test function in (2.2). Multiply equation (2.2) by ε and we use the unfolding operator $\mathcal{T}_\varepsilon$ we get

$$\nu\varepsilon^{1-\frac{\gamma}{2}} \int_{\Omega\times Y^*} \mathcal{T}^*_\varepsilon(\varepsilon^{\frac{\gamma}{2}}\nabla u_\varepsilon) : \mathcal{T}^*_\varepsilon(\nabla\varphi) dx dy + \alpha\frac{\delta_1^{N-1}}{\varepsilon} \int_{\Omega\times\partial B} \mathcal{T}^b_{\varepsilon\delta_1}(\varepsilon^{1+\gamma}u_\varepsilon)\cdot\mathcal{T}^b_{\varepsilon\delta_1}(\varphi) dx d\sigma(z)$$
$$- \int_{\Omega_{\varepsilon\delta_1\delta_2}} \varepsilon p_\varepsilon \operatorname{div}\varphi\, dx = \int_{\Omega_{\varepsilon\delta_1\delta_2}} \varepsilon f\cdot\varphi dx + \int_{\Omega\times\partial B} \mathcal{T}^b_\varepsilon(g_0)\cdot\varphi dx d\sigma(z) + \int_{\Omega\times\partial B} \mathcal{T}^b_\varepsilon(g_\varepsilon)\varphi dx d\sigma(z).$$

We can now pass to limit in all the expression. By using the estimate (4.8) and the assumption for f, we get

$$\alpha k_1 \int_{\Omega\times\partial B} u\varphi dx d\sigma(z) - \int_\Omega p\operatorname{div}\varphi dx = \int_{\Omega\times\partial B} g_0\cdot\varphi dx d\sigma(y) + \int_{\Omega\times\partial B} g\varphi dx d\sigma(y).$$

Then

$$\int_\Omega \Big(\alpha k_1|\partial B|u + \nabla p - |\partial B|g_0 - |\partial B|\mathcal{M}_{\partial B}(g)\Big)\varphi dx = 0.$$

So that

$$u = \frac{1}{\alpha k_1|\partial B|}\Big(-\nabla p + |\partial B|g_0 + |\partial B|\mathcal{M}_{\partial B}(g)\Big).$$

□

Proof of theorem 4.4 The proof of theorem 4.4, makes use the next two elementary results.

Lemma 4.3 (see [9]) *Suppose $N \geq 3$. Then, there exists $\delta_0 > 0$ such that*

$$\bigcup_{0<\delta<\delta_0} \{\phi \in H^1_{per}(Y) \ \mid \ \phi = 0 \ \text{on } \delta T\},$$

is dense in $H^1_{per}(Y)$.

Lemma 4.4 *(see [9, Lemma 3.3]) Let v be in $\mathcal{D}(\mathbb{R}^N)\cap K_T$ (i.e $v = v(T)$ on T). Set*

$$w_{\varepsilon\delta_2}(x) = v(T) - v\Big(\frac{1}{\delta_2}\Big\{\frac{x}{\varepsilon}\Big\}\Big) \ \text{for a.e. } x\in\mathbb{R}^N.$$

Then

$$w_{\varepsilon\delta_2} \rightharpoonup v(T) \ \text{weakly in } H^1(\Omega). \tag{4.17}$$

Proof: [Proof of Theorem 4.4] Due to the estimates (4.9) in Proposition 4.1, the case $\gamma = 2$ is the easiest.
From (4.12) we have $\|u_\varepsilon\|_{(H^1(\Omega_{\varepsilon\delta_1\delta_2}))^N} \leq C$, and by (3.4) one has the following convergences

$$\begin{aligned} \mathcal{T}_\varepsilon(\nabla u_\varepsilon) \rightharpoonup \nabla u + \nabla_y \widehat{u} \quad &\text{weakly in } L^2(\Omega \times Y)^{N\times N},\\ \mathcal{T}_\varepsilon(u_\varepsilon) \rightharpoonup u \quad &\text{weakly in } \; L^2(\Omega; H^1(Y))^N, \end{aligned}$$

where $u \in H_0^1(\Omega)^N$ and $\widehat{u} \in L^2(\Omega; H^1_{per}(Y))^N$.
By Proposition 3.2, there exists U in $L^2(\Omega; L^2_{loc}(\mathbb{R}^N))$ such that, up to a subsequence,

$$\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}\mathcal{T}_{\varepsilon\delta_2}(u_\varepsilon) \rightharpoonup U \text{ weakly in } L^2(\Omega; L^2_{loc}(\mathbb{R}^N)). \tag{4.18}$$

On one hand, by definition 3.1, one has

$$\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}\mathcal{M}^\varepsilon_Y(u_\varepsilon) \longrightarrow k_2 u \text{ in } L^2(\Omega; L^2_{loc}(\mathbb{R}^N)). \tag{4.19}$$

On the other hand, by Proposition 3.2 there exists a W in $L^2(\Omega; L^{2^*}(\mathbb{R}^N))$ with $\nabla_z W$ in $L^2(\Omega\times\mathbb{R}^N)$ such that

$$\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}\Big(\mathcal{T}_{\varepsilon\delta_2}(u_\varepsilon) - \mathcal{M}^\varepsilon_Y(u_\varepsilon)\Big) \rightharpoonup W \text{ weakly in } L^2(\Omega; L^{2^*}(\mathbb{R}^N)). \tag{4.20}$$

From (4.18),(4.19) and (4.20), it yields

$$U = W + k_2 u \;\text{ and }\; \nabla_z U = \nabla_z W,$$

and again by Proposition 3.2

$$\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}\nabla_z\mathcal{T}_{\varepsilon\delta_2}(u_\varepsilon) = \delta_2^{\frac{N}{2}}\mathcal{T}_{\varepsilon\delta_2}(\nabla u_\varepsilon) \rightharpoonup \nabla_z U \text{ weakly in } L^2(\Omega\times\mathbb{R}^N). \tag{4.21}$$

Now, in (2.2) choose the test function $v_\varepsilon = \varepsilon\Phi\psi\Big(\frac{\cdot}{\varepsilon}\Big)$ with $\Phi \in \mathcal{D}(\Omega)^N$ and ψ in $H^1_{per}(Y)^N$. Since $\nabla v_\varepsilon = \varepsilon\psi\Big(\frac{x}{\varepsilon}\Big)\nabla\Phi + \Phi\nabla_y\psi\Big(\frac{x}{\varepsilon}\Big)$ we get

$$\begin{aligned} \nu\int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_\varepsilon\Big(\varepsilon\psi\Big(\frac{x}{\varepsilon}\Big)\nabla\Phi + \Phi\nabla_y\psi\Big(\frac{x}{\varepsilon}\Big)\Big)dx + \alpha\varepsilon^2\int_{\partial B_{\varepsilon\delta_1}} u_\varepsilon\varepsilon\Phi\psi\Big(\frac{x}{\varepsilon}\Big)d\sigma(x)\\ -\varepsilon\int_{\Omega_{\varepsilon\delta_1\delta_2}} p_\varepsilon \operatorname{div}\Phi\psi\Big(\frac{x}{\varepsilon}\Big)dx = \varepsilon\int_{\Omega_{\varepsilon\delta_1\delta_2}} f\cdot\Phi\psi\Big(\frac{x}{\varepsilon}\Big)dx + \varepsilon\int_{\partial B_{\varepsilon\delta_1}} g_0\cdot\Phi\psi\Big(\frac{x}{\varepsilon}\Big)d\sigma(x)\\ +\varepsilon\int_{\partial B_{\varepsilon\delta_1}} g_\varepsilon\Phi\psi\Big(\frac{x}{\varepsilon}\Big)d\sigma(x). \end{aligned}$$

When ε goes to 0 we obtain

$$\nu\int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_\varepsilon\Phi(x)\nabla\psi\Big(\frac{x}{\varepsilon}\Big)dxdy = 0.$$

By unfolding with $\mathcal{T}_\varepsilon$ one has

$$\nu\int_{\Omega\times Y}\mathcal{T}_\varepsilon\Big(\nabla u_\varepsilon\Big)\mathcal{T}_\varepsilon(\Phi(x))\mathcal{T}_\varepsilon\Big(\nabla\psi\Big(\frac{x}{\varepsilon}\Big)\Big)dxdy = 0.$$

By passing to limit we obtain (4.2)

$$\nu\int_{\Omega\times Y}\Big(\nabla u(x) + \nabla_y\widehat{u}(x,y)\Big)\nabla_y\Psi(x,y)dxdy = 0.$$

In order to describe the contribution of the perforations, we use the function $w_{\varepsilon\delta_2}$ introduced in lemma 4.4. For ψ in $D(\Omega)$, choose $w_{\varepsilon\delta_2}\psi$ as a test function in (2.2). We get

$$\begin{aligned} \nu \int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_\varepsilon \nabla w_{\varepsilon\delta_2}\psi dx + \nu \int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_\varepsilon w_{\varepsilon\delta_2}\nabla\psi dx + \alpha\varepsilon^2 \int_{\partial B_{\varepsilon\delta_1}} u_\varepsilon w_{\varepsilon\delta_2} d\sigma(x) - \\ \int_{\Omega_{\varepsilon\delta_1\delta_2}} p_\varepsilon \mathrm{div}\,(w_{\varepsilon\delta_2}\psi)dx = \int_{\Omega_{\varepsilon\delta_1\delta_2}} f w_{\varepsilon\delta_2}\psi dx + \int_{\partial B_{\varepsilon\delta_1}} g_0 w_{\varepsilon\delta_2}\psi d\sigma_{\varepsilon\delta_1}(x) + \int_{\partial B_{\varepsilon\delta_1}} g_\varepsilon w_{\varepsilon\delta_2} d\sigma_{\varepsilon\delta_1}(x). \end{aligned} \tag{4.22}$$

By unfolding the first term in (4.22) with $\mathcal{T}_{\varepsilon\delta_2}$, we get

$$\int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_\varepsilon \nabla w_{\varepsilon\delta_2}\psi dx \overset{\mathcal{T}_{\varepsilon\delta_2}}{\simeq} \delta_2^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon\delta_2}(\nabla u_\varepsilon)\mathcal{T}_{\varepsilon\delta_2}(\nabla w_{\varepsilon\delta_2})\mathcal{T}_{\varepsilon\delta_2}(\psi), \tag{4.23}$$

since by lemma 4.4, one has:

$$\mathcal{T}_{\varepsilon\delta_2}(\nabla w_{\varepsilon\delta_2}) = -\frac{1}{\varepsilon\delta_2}\nabla_z v \quad ,$$

then

$$\int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_\varepsilon \nabla w_{\varepsilon\delta_2}\psi dx \overset{\mathcal{T}_{\varepsilon\delta_2}}{\simeq} \frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon} \int_{\Omega\times\mathbb{R}^N} \delta_2^{\frac{N}{2}} \mathcal{T}_{\varepsilon\delta_2}(\nabla u_\varepsilon)(-\nabla_z v)\mathcal{T}_{\varepsilon\delta_2}(\psi).$$

Convergence (4.21) as well as hypothesis (1.3), allows us to pass to the limit in (4.23) to obtain:

$$\lim_{\varepsilon\longrightarrow 0} \int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_\varepsilon \nabla w_{\varepsilon\delta_2}\psi dx = -k_2 \int_{\Omega\times(\mathbb{R}^N\setminus B)} \nabla_z U(x,z)\nabla_z v(z)\psi(x) dx dz. \tag{4.24}$$

The second term in (4.22) is unfolded with $\mathcal{T}_\varepsilon$ and we have,

$$\int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_\varepsilon w_{\varepsilon\delta_2}\nabla\psi dx \overset{\mathcal{T}_\varepsilon}{\simeq} \int_{\Omega\times Y} \mathcal{T}_\varepsilon(\nabla u_\varepsilon)\mathcal{T}_\varepsilon(w_{\varepsilon\delta_2})\mathcal{T}_\varepsilon(\nabla\psi) dx dy.$$

Using proposition 3.3 and convergence (4.17), we can pass to the limit with respect to ε in tne above equality to get :

$$\lim_{\varepsilon\longrightarrow 0} \int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_\varepsilon w_{\varepsilon\delta_2}\nabla\psi dx = v(T)\int_{\Omega\times Y} (\nabla u + \nabla_y \widehat{u})\nabla_x\psi(x) dx dy. \tag{4.25}$$

For the third term in (4.22) by using (3.6) we get

$$\lim_{\varepsilon\longrightarrow 0} \alpha \int_{\partial B_{\varepsilon\delta_1}} \varepsilon^2 u_\varepsilon w_{\varepsilon\delta_2}\psi d\sigma(x) = \lim_{\varepsilon\longrightarrow 0} \frac{\delta_1^{N-1}}{\varepsilon}\alpha \int_{\mathbb{R}^N\times\partial B} \mathcal{T}^b_{\varepsilon\delta_1\delta_2}(\varepsilon^2 u_\varepsilon)\mathcal{T}^b_{\varepsilon\delta_1\delta_2}(w_{\varepsilon\delta_2})\mathcal{T}^b_{\varepsilon\delta_1\delta_2}(\psi) dx d\sigma(z).$$

Passing to the limit yields

$$\lim_{\varepsilon\longrightarrow 0} \alpha \int_{\partial B_{\varepsilon\delta_1}} \varepsilon^2 u_\varepsilon w_{\varepsilon\delta_2}\psi d\sigma(x) = \alpha k_1 v(T)\int_{\mathbb{R}^N\times\partial B} u\psi dx d\sigma(z). \tag{4.26}$$

For the fourth term in (4.22) we have

$$\int_{\Omega_{\varepsilon\delta_1\delta_2}} p_\varepsilon \mathrm{div}\,(w_{\varepsilon\delta_2}\psi)dx = \int_{\Omega_{\varepsilon\delta_1\delta_2}} p_\varepsilon w_{\varepsilon\delta_2}\mathrm{div}\,(\psi)dx + \int_{\Omega_{\varepsilon\delta_1\delta_2}} p_\varepsilon \nabla w_{\varepsilon\delta_2}\psi dx. \tag{4.27}$$

For the second term of right-hand sid of this equation we apply the operator $\mathcal{T}_{\varepsilon\delta_2}$ and we use the lemma 4.4 we get

$$\begin{aligned} \int_{\Omega_{\varepsilon\delta_1\delta_2}} p_\varepsilon \nabla w_{\varepsilon\delta_2}\psi dx &= \delta_2^N \int_{\Omega\times\mathbb{R}} \mathcal{T}_{\varepsilon\delta_2}(p_\varepsilon)\mathcal{T}_{\varepsilon\delta_2}(\nabla w_{\varepsilon\delta_2})\mathcal{T}_{\varepsilon\delta_2}(\psi) dx dz \\ &= \frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}\delta_2^{\frac{N}{2}} \int_{\Omega\times\mathbb{R}} \mathcal{T}_{\varepsilon\delta_2}(p_\varepsilon)(-\nabla_z v_z)\mathcal{T}_{\varepsilon\delta_2}(\psi) dx dz. \end{aligned}$$

For $\varepsilon \to 0$ this integral go to zero Passing to the limit in (4.27) we get

$$\lim_{\varepsilon \longrightarrow 0} \int_{\Omega_{\varepsilon\delta_1\delta_2}} p_\varepsilon \mathrm{div}\,(w_{\varepsilon\delta_2}\psi)dx = v(T)\int_\Omega p\ \mathrm{div}\,(\psi)dx. \tag{4.28}$$

Similarly, for the fifth, sixth and seventh terms, in view again (3.6), we get

$$\lim_{\varepsilon \longrightarrow 0} \int_{\Omega_{\varepsilon\delta_1\delta_2}} f_\varepsilon w_{\varepsilon\delta_2}\psi dx = v(T)\int_\Omega f\psi dx, \tag{4.29}$$

$$\lim_{\varepsilon \longrightarrow 0} \int_{\partial B_{\varepsilon\delta_1}} g_0 w_{\varepsilon\delta_2}\psi d\sigma(x) = k_1 v(T)\int_{\mathbb{R}^N\times\partial B} g_0(z)\psi dx d\sigma(z), \tag{4.30}$$

$$\lim_{\varepsilon \longrightarrow 0} \int_{\partial B_{\varepsilon\delta_1}} g_\varepsilon w_{\varepsilon\delta_2}\psi d\sigma(x) = k_1 v(T)\int_{\mathbb{R}^N\times\partial B} g(z)\psi dx d\sigma(z). \tag{4.31}$$

Passing to limit in (4.22) and using (4.23), (4.25), (4.26), (4.28), (4.29), (4.30), and (4.31) we obtain

$$\begin{aligned} v(T)\int_{\Omega\times Y}(\nabla u+\nabla_y\widehat{u})\nabla_x\psi(x)dxdy - k_2\int_{\Omega\times(\mathbb{R}^N\setminus B)}\nabla_z U(x,z)\nabla_z v(z)\psi(x)dxdz \\ +\alpha k_1 v(T)\int_{\mathbb{R}^N\times\partial B} u\psi dx d\sigma(z) + v(T)\int_\Omega \nabla p\psi dx = v(T)\int_\Omega f\psi dx \\ +k_1 v(T)\int_{\mathbb{R}^N\times\partial B} g_0(z)\psi dx d\sigma(z) + k_1 v(T)\int_{\mathbb{R}^N\times\partial B} g(z)\psi dx d\sigma(z). \end{aligned}$$

Which by density, holds true for all $\psi \in H_0^1(\Omega)$ and $v \in K_T$. With $v(T) = 0$ above we obtain the limit equation (4.3). Equations (4.4) follow simply by integrating by parting the above equation.
If $\gamma > 2$ by similar argument as those used in all termes of exeprition (4.22) except the third term the bihaviour at the limit for $\gamma > 2$, this term goes to zero, in view of (4.23), (4.25), (4.28), (4.29), (4.30), and (4.31) we obtain

$$\begin{aligned} v(T)\int_{\Omega\times Y}(\nabla u+\nabla_y\widehat{u})\nabla_x\psi(x)dxdy - k_2\int_{\Omega\times(\mathbb{R}^N\setminus B)}\nabla_z U(x,z)\nabla_z v(z)\psi(x)dxdz \\ + v(T)\int_\Omega \nabla p\psi dx = v(T)\int_\Omega f\psi dx + k_1 v(T)\int_{\mathbb{R}^N\times\partial B} g_0(z)\psi dx d\sigma(z) \\ + k_1 v(T)\int_{\mathbb{R}^N\times\partial B} g(z)\psi dx d\sigma(z). \end{aligned}$$

Equation (4.5) follow simply by integrating by parting the above equation. □

To finish let us give the classical "strong" formulation of the homogenized problem obtained in theorem (4.4). We skip the proof since the strong formulation from the unfolded problem is standard, we refer the reader for instance to [12] or [17].

Remark 4.1 *For the case $\gamma \geqslant 2$, a tensor $B = b_{ijkh}$ is introduced. Its form is found following standard argument, as shown in [7].*

The next theorem gives the classical (standard) from of homogenized system (4.2)-(4.5). To state it, we follow the procedure from [9] , where more details can be found.

Theorem 4.5 *If $\gamma = 2$, (u, p) is the unique solution of the homogenized problem*

$$\begin{cases} -b_{ijkh}\frac{\partial^2 u_k}{\partial x_j \partial x_h} + \alpha k_1|\partial B|u_i + \nabla p + k_2^2\Theta u = k_1|\partial B|g_{0i} + k_1|\partial B|\mathcal{M}_{g_i} + f_i \ \textit{in}\ \Omega, 1 \le i \le N, \\ u = 0 \ \textit{on}\ \partial\Omega, \end{cases} \tag{4.32}$$

if $\gamma > 2$

$$\begin{cases} -b_{ijkh}\frac{\partial^2 u_k}{\partial x_j \partial x_h} + \nabla p + k_2^2 \Theta u = k_1|\partial B| g_{0i} + k_1|\partial B|\mathcal{M}_{g_i} + f_i \ \ in\ \Omega, 1 \leq i \leq N, \\ u = 0 \ \ on\ \partial\Omega. \end{cases} \tag{4.33}$$

In both systems the function Θ *giving rise to a "strange term" in the second sustem, is given by*

$$\Theta(x) = \int_{\mathbb{R}^N \setminus B} \nabla_z \chi(x,z) dz. \tag{4.34}$$

Introduce first the classical correctors $\widehat{\chi}_j, j = 1, ..., n$, for the homogenization in fixed domains (see for instance [4]). They are defined by the cell problems

$$\begin{cases} \widehat{\chi}_j \in L^\infty(\Omega; H^1_{per}(Y)), \\ \int_Y \nabla(\widehat{\chi}_j - y_j)\nabla\phi \ = \ 0 \quad a.e.x \in \Omega, \\ \forall \phi \in H^1_{per}(Y). \end{cases} \tag{4.35}$$

Here χ is the solution of the cell problem corresponding

$$\begin{cases} \chi \in L^\infty(\Omega; K_T) \quad (\chi, T) \equiv 1, \\ \int_{\mathbb{R}\setminus T} \nabla_z \chi(x,z) \nabla_z \Psi(z) dz \ = \ 0 \quad a.e.x \in \Omega, \forall \Psi \in K_T, \Psi(T) = 0. \end{cases} \tag{4.36}$$

To do so, observe that (4.2) gives $\widehat{u}$ in terms of ∇u and a tensor $B = (b_{ijkh})$ expressed as integrals of function defined on cell problems. The procedure is standard, for the Stokes problem. The details can be found in [7]. We will just recall here the definition of B.
For $k, h = 1, ..., N$ let $\Pi^{kh} = (\Pi_i^{kh})_i$ with $\Pi_i^{kh} = \delta_{ki} y_h$ (δ_{ki} being the Kronecker symbols) and introduce the solution (χ^{kh}, q^{kh}) of the Stokes cell system

$$\begin{cases} -\Delta\chi^{kh} + \nabla q^{kh} = 0 \text{ in } Y^*, \\ \operatorname{div}(\chi^{kh} - \Pi^{kh}) = 0 \text{ in } Y^*, \\ -\frac{\partial(\chi^{kh} - \Pi)}{\partial n} + q^{kh}.n = 0 \text{ on } \partial B, \\ M_{Y^*}(\chi^{KH}) = 0 \ \chi^{kh} \text{ Y-periodic}, \end{cases}$$

the tensor $B = (b_{ijkh})$ is defined as follows:

$$b_{ijkh} = \int_{Y^*} \frac{\partial(\chi^{kh} - \Pi^{kh})_l}{\partial y_m} \frac{\partial(\chi^{ij} - \Pi^{ij})_l}{\partial y_m} dy. \tag{4.37}$$

Proof: The proof follows the reasoning from [9], we just emphasize the main points. The correctors defined by (4.35) enable us to express $\widehat{u}(x,y)$ in equation (4.2) in terms of u as

$$\widehat{u} = \sum_{ijkh} b_{ijkh} \frac{\partial u_k}{\partial x_l} \widehat{\chi}_j^{hl}(x,y).$$

replacing this expression (4.4) it is easily seen that the limit function u is solution of

$$\begin{aligned} \int_\Omega b_{ijkh} \nabla u \nabla \psi dx dy - \nu k_2 \int_{\Omega\times\partial T} \nabla_z U n_T \psi dx d\sigma + \alpha k_1 |\partial B| \int_\Omega u\psi dx + \int_\Omega \nabla p \psi dx = \\ \int_\Omega f\psi dx + k_1|\partial B| \int_\Omega g_0 \psi dx + k_1 |\partial B| \ M_{\partial B} \int_\Omega \psi dx, \end{aligned} \tag{4.38}$$

with b_{ijkh} given by (4.37). Now, by integrating by parts in (4.3), one easily gets

$$\int_{\partial T} \nabla_z U v_T d\sigma = \int_{\partial T} \nabla_z (U - k_2 u) v_T d\sigma = -k_2 u \Big(\int_{\partial T} \nabla_z \chi v_T d\sigma \Big).$$

Wich, replaced into (4.38) gives (4.32) with Θ defined by (4.34)
By similar method we get the equation (4.33) □

Remark 4.2 For $k_1 = 0$ and $\gamma \geq 2$, arises the following equation

$$\begin{cases} -b_{ijkh}\frac{\partial^2 u_k}{\partial x_j \partial x_h} + \nabla p + k_2^2 \Theta u = f_i \text{ in } \Omega, 1 \leq i \leq N, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Remark 4.3 For $k_2 = 0$, two following cases arise:
If $\gamma = 2$ we get

$$\begin{cases} -b_{ijkh}\frac{\partial^2 u_k}{\partial x_j \partial x_h} + \alpha k_1 |\partial B| u_i + \nabla p = k_1 |\partial B| g_{0i} + k_1 |\partial B| \mathcal{M}_{g_i} + f_i \text{ in } \Omega, 1 \leq i \leq N, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

If $\gamma > 0$ we get

$$\begin{cases} -b_{ijkh}\frac{\partial^2 u_k}{\partial x_j \partial x_h} + \nabla p = k_1 |\partial B| g_{0i} + k_1 |\partial B| \mathcal{M}_{g_i} + f_i \text{ in } \Omega, 1 \leq i \leq N, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Remark 4.4 If $k_1 = 1$ and $k_2 = 0$ we become in the case of classical homogenization and we find all the results established in [7,19].

References

1. G. Allaire, *Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes,* I. Abstract framework, a volume distribution of holes, Arch. Rational Mech. Anal. 113, 209-259, (1984).
2. G. Allaire, *Homogenization of the Stokes flow in a connected porous medium,* Asymptotic Analysis 3, 203-222, (1989).
3. G.Allaire, *Homogenization of Navier-Stokes equations with a slip boundary condition,* Comm. Pure Appl. XLIV, 605-642 (1991).
4. A. Bensoussan, J.-L. Lions and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures,* North Holland, Amsterdam, 1978.
5. A. Brillard, *Asymptotic analysis of incompressible and viscous fluid flow through porous media,* Brinkman's law via epi-convergence methods, Ann. Fac. Sc. Toulouse, 8, 225-252(1996).
6. A. Capatina and H. Ene, *Homogenization of the Stokes problem with a pure non-homogeneous slip boundary condition by the periodic unfolding method,* European J. of Math. 22 , 333-345,(2011).
7. D.Cioranescu, P.Donato and H.Ene, *Homogenisation of Stokes problem with non homogeneouse slip boundary conditions,* Math.Meth. Appl. Sci. 19,857-881,(1996).
8. D. Cioranescu, A. Damlamian and G. Griso, *The periodic unfolding method in homogenization,* SIAM Journal on Mathematical Analysis 40 (4), 1585-1620,(2008).
9. D. Cioranescu, A. Damlamian, G. Griso and D. Onofrei, *The periodic unfolding method for perforated domains and Neumann sieve models,* J. Math. Pures Appl., 89 , 248-277,(2008).
10. D. Cioranescu, A. Damlamian, P. Donato, G.Griso and R. Zaki, *The periodic unfolding method in domains with holes,* SIAM Journal on Mathematical Analysis, 44 (2), 718-760,(2012).
11. D. Cioranescu and F. Murat, *Un terme étrange venu d'ailleurs, in Nonlinear partial differential equations and their applications,* College de France Seminar & II, ed. H.Brezis and J.L.Lions, Research Notes in Math. 60-70, Pitman, Boston , 98-138, 154-178,(1982).
12. D. Cioranescu and A. Ould Hammouda., *Homogenization of elliptic problems in perforated domains with mixed boundary conditions,* Rev. Roumaine Math. Pures Appl., 53 , 5-6, 389-406, (2008).
13. C. Conca and P. Donato, *Non-homogeneous Neumann problems in domains with small holes,* RAIRO Modél. Math. Anal. Numér., 4, 22 , 561–608.(1988).
14. H. I, Ene and Sanchez-Palencia E., *Equation et phénomènes de surface pour l'écoulement dans un modèle de milieu poreux,* J. Mécanique, 14, 73-108(1975).
15. S. Kaizu, *The Poisson equation with semilinear boundary conditions in domains with many tiny holes,* J. Faculty of Science, Univ. Tokyo. Sect. 1A 36(1) , 43-64,(1989).
16. C. Karek, A. Ould Hammouda, *Asymptotic behavior of the Stokes problem with Robin condition in a domain with small holes,* Asymptotic Analysis, Ios Press, vol. 115, no. 3-4, pp. 147-167, (2019).
17. A. Ould Hammouda, *Periodic unfolding and nonhomogeneous Neumann problems in domains with holes,* C. R. Acad. Sci. Paris, Ser. I, 346 963-968, (2008).

18. A. Ould Hammouda, *Homogenization of a class of Neumann problems in perforated domains,* Asymptotic Analysis, Vol. 71 No. 1-2, 33-57 (2011).

19. R. Zaki,*Homogenization of a Stokes problem in a porous medium by the periodic unfolding method,* Asymptotic Analysis 79 (2012) 229-250.

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