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# Some Fixed Point Results on $M_b$-Cone Metric Space Over Banach Algebra

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ABSTRACT: In this paper, we introduce the notion of $M_b$-cone metric space over Banach algebra as a generalization of both $M$-cone metric space over Banach algebra and $b$-metric spaces. In this work, we prove Kannan's and Chatterjea's fixed point theorems in the framework of $M_b$-cone metric space over Banach algebra. Illustrative examples are presented to justify our main results.

Key Words: $M_b$-cone metric space over Banach algebra, Generalized Lipschitz mapping, fixed point.

## Contents



## 1. Introduction

A key and ever-evolving field of nonlinear analysis, fixed point theory has extensive applications in integral equations, optimization, and differential equations. It is a fundamental method used in applied sciences to demonstrate the existence and uniqueness of solutions to different mathematical models. Due to the growing necessity to study increasingly complicated structures over the years, especially in spaces with generalized or flexible distance ideas, this theory has expanded significantly.

Fréchet [8] presented the classical concept of a metric space, which served as the foundation for a large portion of contemporary topology and analysis. Over time, a variety of metric space generalizations have been developed in response to the need to handle more flexible and generalized frameworks. These include $b$-metric spaces [2,3], $G$-metric spaces, 2-metric spaces, and fuzzy metric spaces, all of which are intended to manage oddities in convergence, continuity, and mapping behaviors in larger contexts.

Specifically, Bakhtin [2] and Czerwik [3] proposed $b$-metric spaces, which loosened the triangle inequality to permit a multiplicative constant. This allowed for a wider range of fixed point solutions and expanded the class of contraction mappings. By introducing partial metric spaces, Matthews [13] expanded on this viewpoint and made room for non-zero self-distances, a notion that has proven particularly helpful in domain theory and computer science.

One important turning point was Huang and Zhang's invention of cone metric spaces [9], in which a cone-valued function defined in a Banach space is used in place of the real-valued metric. An extensive amount of follow-up research (e.g., [5–7]) was sparked by their work, which extended the Banach contraction principle to this context and investigated whether cone metric conclusions were in fact different

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from or reducible to results in classical metric space.

To address these issues, Liu and Xu [12] presented cone metric spaces over Banach algebras, substituting a Banach algebra for the underlying real Banach space, and showed via specific instances that this structure cannot be reduced to the classical situation. This was a significant advancement since the algebraic structure added additional nuances to operator theory and topology.

Through the generalization of pre existing frameworks, Shukla et al. [17,18] made a substantial contribution to fixed point theory. They used ideas from Banach algebra-valued spaces, metric, cone metric, and graphical metric to introduce graphical cone metric spaces over Banach algebras. Wider fixed point results with applications to systems of initial value difficulties were made possible as a result. They created a set-valued, relation-theoretic variant of the classical Prešić–Ćirić fixed point theorem for metric spaces with arbitrary binary relations in a different paper. By addressing differential and difference inclusion issues under less rigorous assumptions, their method broadened the theoretical and applied applicability of fixed point results in relational and nonlinear analysis.

As a further extension of partial metric spaces, $M$-metric spaces were established by Asadi et al. [1]. Inspired by these advancements, Fernandez et al. [4] presented $M$-cone metric spaces over Banach algebras, fusing the cone-valued structure and partial metric nature in an algebraic context. In addition to supporting generalized Lipschitz mappings, these spaces provide a fresh perspective on the analysis of nonlinear integral equations.

The unification of the structures of $b$-metric spaces with $M$-cone metric spaces over Banach algebras was still lacking, notwithstanding these developments. In order to overcome this gap between these frameworks, the current study introduces the innovative idea of $M_b$-cone metric space over Banach algebra. We prove analogues of Kannan's and Chatterjea's fixed point theorems by developing fixed point results for generalized Lipschitz mappings in this novel context. Crucially, we show from carefully built examples that the fixed point existence in our framework cannot be deduced from classical or previously known generalizations. Additionally, as an application, we design an operator on a complete $M_b$-cone metric space that satisfies the necessary contractive condition, proving the existence and uniqueness of solutions to a Fredholm-type integral problem.

## 2. Preliminaries

In this section, we give some definitions which are useful for our main results.

Let $A$ always be a real Banach algebra such that

1. $(\varsigma\varrho)\upsilon = \varsigma(\varrho\upsilon)$,
2. $\varsigma(\varrho + \upsilon) = \varsigma\varrho + \varsigma\upsilon$ and $(\varsigma + \varrho)\upsilon = \varsigma\upsilon + \varrho\upsilon$,
3. $\alpha(\varsigma\varrho) = (\alpha\varsigma)\varrho = \varsigma(\alpha\varrho)$,
4. $\| \varsigma\varrho \| \leq \| \varsigma \| \| \varrho \|$.

If $e\varsigma = \varsigma e = \varsigma$, for all $\varsigma \in A$ then $e \in A$ is called unit (i.e., a multiplicative identity). If there is an element $\varsigma \in A$ such that $\varsigma\varrho = \varrho\varsigma = e$, then $\varsigma \in A$ is said to be invertible. $\varsigma^{-1}$ is the inverse of $\varsigma$. For more details, we refer the readers to [16].

The following proposition is given in [16].

**Proposition 2.1.** Suppose that the spectral radius $\widehat{\rho}(\varsigma)$ of an element $\varsigma \in A$ is less than 1, i.e.

$$\widehat{\rho}(\varsigma) = \lim_{n\to+\infty} \|\varsigma^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|\varsigma^n\|^{\frac{1}{n}} < 1,$$

then $e - \varsigma$ is invertible, where $e \in A$ is unit. Moreover,

$$(e-\varsigma)^{-1} = \sum_{i=0}^{\infty} \varsigma^i.$$

**Remark 2.2.** From [16] we see that the spectral radius $\widehat{\rho}(\varsigma)$ of $\varsigma$ satisfies $\widehat{\rho}(\varsigma) \leq \| \varsigma \|$ for all $\varsigma \in \mathcal{A}$, where $\mathcal{A}$ is a Banach algebra with a unit $e$.

**Remark 2.3. (See [19]).** In Proposition 2.1, if the condition $'\widehat{\rho}(\varsigma) < 1'$ is replaced by $\| \varsigma \| \leq 1$, then the conclusion remains true.

**Remark 2.4. (See [19]).** If $\widehat{\rho}(\varsigma) < 1$ then $\| \varsigma^n \| \to 0 (n \to +\infty)$.
Now let us recall the concepts of cone over Banach algebra.
A subset $P$ of $\mathcal{A}$ is called a cone if

1. $P$ is non-empty closed and $\{\theta, e\} \subset P$;
2. $\alpha P + \beta P \subset P$ for all non-negative real numbers $\alpha, \beta$;
3. $P^2 = PP \subset P$;
4. $P \cap (-P) = \{\theta\}$,

where $\theta$ denotes the null of the Banach algebra $\mathcal{A}$. For a given cone $P \subset \mathcal{A}$, we can define a partial ordering $\preceq$ with respect to $P$ by $\varsigma \preceq \varrho$ if and only if $\varrho - \varsigma \in P$. $\varsigma < \varrho$ will stand for $\varsigma \leq \varrho$ and $\varsigma \neq \varrho$, while $\varsigma \ll \varrho$ will stand for $\varrho - \varsigma \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of $P$. If $\operatorname{int} P \neq \phi$, then $P$ is called a solid cone. The cone $P$ is called normal if there is a number $M \geq 1$ such that, for all $\varsigma, \varrho \in \mathcal{A}$,

$$\theta \preceq \varsigma \preceq \varrho \Rightarrow \| \varsigma \| \leq M \| \varrho \| .$$

The least positive number satisfying the above is called the normal constant of P [9].

We will require the following definitions and preliminary results to prove our results.

In 2013, Liu et al. [12] gave the definition of cone metric space over Banach algebra as follows:
**Definition 2.5.([12])** Let $\mathcal{M}$ be a nonempty set. Suppose the mapping $d : \mathcal{M} \times \mathcal{M} \to A$ satisfies

(1) $\theta \preceq d(\varsigma, \varrho)$ for all $\varsigma, \varrho \in \mathcal{M}$ and $d(\varsigma, \varrho) = \theta$ if and only if $\varsigma = \varrho$,
(2) $d(\varsigma, \varrho) = d(\varrho, \varsigma)$ for all $\varsigma, \varrho \in \mathcal{M}$;
(3) $d(\varsigma, \varrho) \preceq d(\varsigma, \upsilon) + d(\upsilon, \varrho)$ for all $\varsigma, \varrho, \upsilon \in \mathcal{M}$.

Then $d$ is called a cone metric on $\mathcal{M}$, and $(\mathcal{M}, d)$ is called a cone metric space over Banach algebra $\mathcal{A}$.

Recently, Fernandez et al. [4] generalized the concepts of $M$-metric space and cone metric space over Banach algebra to define $M$-cone metric space over Banach algebra as follows:

**Definition 2.6 ([4])** Let $\mathcal{M}$ be a nonempty set. A function $m : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ is called a $m$-cone metric if the following conditions are satisfied:

$(m_1)$ $m(\varsigma, \varsigma) = m(\varrho, \varrho) = m(\varsigma, \varrho)$ if and only if $\varsigma = \varrho$,
$(m_2)$ $m_{\varsigma,\varrho} \preceq m(\varsigma, \varrho)$,
$(m_3)$ $m(\varsigma, \varrho) \preceq m(\varrho, \varsigma)$,
$(m_4)$ $\big(m(\varsigma, \varrho) - m_{\varsigma,\varrho}\big) \preceq \big(m(\varsigma, \upsilon) - m_{\varsigma,\upsilon}\big) + \big(m(\upsilon, \varrho) - m_{\upsilon,\varrho}\big)$.
where

$$m_{\varsigma,\varrho} = \min\big\{m(\varsigma, \varsigma), m(\varrho, \varrho)\big\}.$$

Then the pair $(\mathcal{M}, m)$ is called an $M$-cone metric space over Banach algebra.

The following definitions and Lemmas are needed in a sequel.

**Definition 2.7.([11])** Let $P$ be a solid cone in a Banach space $E$. A sequence $\{u_n\} \subset P$ is said to be a $c$-sequence if for each $c \gg \theta$ there exists a natural number $N$ such that $u_n \ll c$ for all $n > N$.

**Lemma 2.8.([19])** Let $P$ be a solid cone in a Banach algebra $\mathcal{A}$. Suppose that $k \in P$ be an arbitrary vector and $\{u_n\}$ is a $c$-sequence in $P$. Then $\{ku_n\}$ is a $c$-sequence.

**Lemma 2.9. ([16]).** Let $\mathcal{A}$ be a Banach algebra with a unit $e, k \in \mathcal{A}$, then $\lim_{n\to+\infty} \| k^n \|^{\frac{1}{n}}$ exists and the spectral radius $\widehat{\rho}(\varsigma)$ satisfies

$$\widehat{\rho}(\varsigma) = \lim_{n\to+\infty} \| k^n \|^{\frac{1}{n}} = \inf_{n\geq 1} \| k^n \|^{\frac{1}{n}} .$$

If $\widehat{\rho}(\varsigma) <| \lambda |$, then $(\lambda e - k)$ is invertible in $\mathcal{A}$. Moreover,

$$(\lambda e - k)^{-1} = \sum_{i=0}^{\infty} \frac{k^i}{\lambda^{i+1}}$$

where $\lambda$ is a complex constant.

**Lemma 2.10.([16])** Let $\mathcal{A}$ be a Banach algebra with a unit $e, a, b \in \mathcal{A}$. If a commutes with b, then

$$\widehat{\rho}(a+b) \leq \widehat{\rho}(a) + \widehat{\rho}(b), \quad \widehat{\rho}(ab) \leq \widehat{\rho}(a)\widehat{\rho}(b).$$

**Lemma 2.11.([10])** Let $\mathcal{A}$ be Banach algebra with a unit $e$ and $P$ be a solid cone in $\mathcal{A}$. Let $a, k, l \in P$ hold $l \preceq k$ and $a \preceq la$. If $\widehat{\rho}(k) < 1$, then $a = \theta$.

**Lemma 2.12.([10])** If $E$ is a real Banach space with a solid cone $P$ and $\{u_n\} \subset P$ be a sequence with $\| u_n \| \to 0 (n \to +\infty)$, then $\{u_n\}$ is a $c$-sequence.

**Lemma 2.13.([10])** If $E$ is a real Banach space with a solid cone $P$

(1) If $a, b, c \in E$ and $a \preceq b \ll c$, then $a \ll c$.
(2) If $a \in P$ and $a \ll c$ for each $c \gg \theta$, then $a = \theta$.

**Lemma 2.14.([10])** Let $\mathcal{A}$ be a Banach algebra with a unit $e$ and $k \in \mathcal{A}$. If $\lambda$ is a complex constant and $\widehat{\rho}(k) <| \lambda |$, then

$$\widehat{\rho}\Big((\lambda e - k)^{-1}\Big) \leq \frac{1}{| \lambda | - \widehat{\rho}(k)}.$$

## 3. $M_b$-cone metric space over Banach algebra

We begin with giving some notations that we shall need to state our results. In this section, we introduce the notion of $M_b$-cone metric space over Banach algebra.

**Notation 3.1.**

(1) $m_{b_{\varsigma,\varrho}} = \min\big\{m_b(\varsigma,\varsigma), m_b(\varrho,\varrho)\big\}$.
(2) $M_{b_{\varsigma,\varrho}} = \max\big\{m_b(\varsigma,\varsigma), m_b(\varrho,\varrho)\big\}$.

**Definition 3.2.** An $M_b$-cone metric space over Banach algebra on a nonempty set $\mathcal{M}$ is a function $m_b : \mathcal{M}^2 \to \mathcal{A}$ that satisfies the following conditions, for all $\varsigma, \varrho, \upsilon \in \mathcal{M}$ we have

$(m_b1)$ $m_b(\varsigma,\varsigma) = m_b(\varsigma,\varrho) = m_b(\varrho,\varrho)$ if and only if $\varsigma = \varrho$,
$(m_b2)$ $m_{b_{\varsigma,\varrho}} \preceq m_b(\varsigma,\varrho)$,
$(m_b3)$ $m_b(\varsigma,\varrho) = m_b(\varrho,\varsigma)$,
$(m_b4)$

$$\big(m_b(\varsigma,\varrho) - m_{b_{\varsigma,\varrho}}\big) \preceq s\Big[\big(m_b(\varsigma,\upsilon) - m_{b_{\varsigma,\upsilon}}\big) + \big(m_b(\upsilon,\varrho) - m_{b_{\upsilon,\varrho}}\big)\Big] - m_b(\upsilon,\upsilon).$$

for all $\varsigma, \varrho, \upsilon \in \mathcal{M}$. The number $s \geq 1$ is called the coefficient of the $M_b$-cone metric space over Banach algebra $(\mathcal{M}, m_b)$.

Now, we give an example of an $M_b$-cone metric space over Banach algebra which is not an $M$-cone metric space over Banach algebra.

**Example 3.3.** Let $\mathcal{A} = C_R^1[0,1]$ with norm defined by $\| \varsigma \| = \| \varsigma \|_\infty + \| \varsigma' \|_\infty$ under point-wise multiplication, $\mathcal{A}$ is a real unit Banach algebra with unit $e = 1$. Consider a cone $P = \{\varsigma \in \mathcal{A} : \varsigma \geq 0\}$ in $\mathcal{A}$. Moreover, $P$ is a non-normal cone (see[19]). Let $M = [0, +\infty)$. Define $m_b : \mathcal{M}^2 \to \mathcal{A}$ by

$$m_b(\varsigma, \varrho)(t) = \Big[\big(\max\{\varsigma, \varrho\}\big)^p + | \varsigma - \varrho |^p \Big] e^t, \text{ where } p > 1.$$

Note that $(\mathcal{M}, m_b)$ is an $M_b$-cone metric space over Banach algebra with coefficient $s = 2^p$, but as the triangle inequality is not satisfied, it is not a $M$-cone metric space over Banach algebra.

**Example 3.4.** Let $\mathcal{A} = C_R^1[0,1]$ with norm defined by $\| \varsigma \| = \| \varsigma \|_\infty + \| \varsigma' \|_\infty$. Under point-wise multiplication, $\mathcal{A}$ is a real unit Banach algebra with unit $e = 1$. Consider a cone $P = \{\varsigma \in \mathcal{A} : \varsigma \geq 0\}$ in $\mathcal{A}$. Moreover, $P$ is a non-normal cone (see[19]). Let $M = [0, +\infty)$. Define $m_b : \mathcal{M}^2 \to \mathcal{A}$ by

$$m_b(\varsigma, \upsilon)(t) = \big((max\{\varsigma, \upsilon\})^p + a\big) e^t, \text{ where } p > 1.$$

Note that $(\mathcal{M}, m_b)$ is an $M_b$-cone metric space over Banach algebra with coefficient $s = 2^p$, but as the triangle inequality is not satisfied, it is not a $M$-cone metric space over Banach algebra.

**Definition 3.5.** Let $(\mathcal{M}, m_b)$ be a $M_b$-cone metric space over Banach algebra. Then:

(a) A sequence $\{\varsigma_n\}$ in $\mathcal{M}$ converges to a point $\varsigma \in \mathcal{M}$ whenever for every $c \gg \theta$ there is a natural number $N$ such that $m_b(\varsigma_n, \varsigma) \ll M_{b_{\varsigma_n,\varsigma}} + c$, for all $n \geq N$. It is denoted by

$$\lim_{n \to +\infty} \Big( m_b(\varsigma_n, \varsigma) - m_{b_{\varsigma_n,\varsigma}} \Big) = \theta.$$

(b) A sequence $\{\varsigma_n\}$ in $\mathcal{M}$ is said to be $m_b$-Cauchy sequence, if for every $c \gg \theta$ $\exists N$ such that, for all $n, m \geq N$, we have

$$m_b(\varsigma_n, \varsigma_m) \ll m_{b_{\varsigma_n,\varsigma_m}} + c.$$

(c) An $M_b$-cone metric space over Banach algebra is said to be complete if every $m_b$-Cauchy sequence $\{\varsigma_n\}$ converges to a point $\mathcal{M}$ such that

$$\lim_{n \to +\infty} \Big( m_b(\varsigma_n, \varsigma) - m_{b_{\varsigma_n,\varsigma}} \Big) = \theta \quad \text{and}$$

$$\lim_{n \to +\infty} \Big( M_b(\varsigma_n, \varsigma_m) - m_{b_{\varsigma_n,\varsigma_m}} \Big) = \theta.$$

**Lemma 3.6.** Let $\{\varsigma_n\}$ be a sequence is an $m_b$-cone metric space over Banach algebra $(\mathcal{M}, m_b)$, such that $\exists\, r \in [0,1)$ such that

$$m_b(\varsigma_{n+1}, \varsigma_n) \preceq r\, m_b(\varsigma_n, \varsigma_{n-1}) \text{ for all } n \in N. \tag{2.3}$$

Then

(A) $\lim_{n \to +\infty} m_b(\varsigma_n, \varsigma_{n-1}) = \theta$,
(B) $\lim_{n \to +\infty} m_b(\varsigma_n, \varsigma_n) = \theta$,
(C) $\lim_{m,n \to +\infty} m_{b_{\varsigma_m,\varsigma_n}} = \theta$.

First, we introduce the notion of generalized Lipschitz mapping on $M_b$-cone metric space over Banach algebra.

**Definition 3.7.** Let $(\mathcal{M}, m_b)$ be a $M_b$-cone metric space over Banach algebra $\mathcal{A}$ and let $P$ be a cone in $\mathcal{A}$. A mapping $T : \mathcal{M} \to \mathcal{M}$ is said to be a generalized Lipschitz mapping if there exist a vector $k \in P$ with $\widehat{\rho}(k) < 1$ such that

$$m_b(T\varsigma, T\varrho) \preceq k m_b(\varsigma, \varrho) \text{ for all } \varsigma, \varrho \in \mathcal{M}.$$

**Example 3.8.** Consider a Banach algebra $\mathcal{A}$ and cone $P$ be as in Example 3.3 and let $\mathcal{M} = \mathbb{R}^+$. Define a mapping $M_b : \mathcal{M}^2 \to \mathcal{A}$ by

$$m_b(\varsigma, \varrho)(t) = \Big[\Big(\max\{\varsigma, \varrho\}\Big)^2 + \mid \varsigma - \varrho \mid^2 \Big] e^t,$$

for all $\varsigma, \varrho \in \mathcal{M}$. Then $(\mathcal{M}, m_b)$ be a $M_b$-cone metric space over Banach algebra $\mathcal{A}$. Define a self-map $T$ by $T\varsigma = \frac{\varsigma}{\varsigma+1}$ for all $\varsigma \in \mathcal{M}$. Since $0 \leq \varsigma \leq 1$ and $0 \leq \varrho \leq 1$, $\frac{\varsigma}{\varsigma+1} \leq \frac{\varsigma}{2}$ and $\frac{\varrho}{\varrho+1} \leq \frac{\varrho}{2}$. Then

$$\begin{aligned} m_b(T\varsigma, T\varrho)(t) &= \Bigg[\Big(\max\Big\{\frac{\varsigma}{\varsigma+1}, \frac{\varrho}{\varrho+1}\Big\}\Big)^2 + \Big|\frac{\varsigma}{\varsigma+1} - \frac{\varrho}{\varrho+1}\Big|^2\Bigg] e^t \\ &\preceq \Bigg[\Big(\max\Big\{\frac{\varsigma}{2}, \frac{\varrho}{2}\Big\}\Big)^2 + \Big|\frac{\varsigma}{2} - \frac{\varrho}{2}\Big|^2\Bigg] e^t \\ &= \frac{1}{4}\Bigg[\Big(\max\{\varsigma, \varrho\}\Big)^2 + \mid \varsigma - \varrho \mid^2\Bigg] e^t \\ &= \frac{1}{4} m_b(\varsigma, \varrho)(t). \end{aligned}$$

for all $\varsigma, \varrho \in \mathcal{M}$. Hence $T$ is a generalized Lipschitz mapping on $\mathcal{M}$, where $k = \frac{1}{4}$.

## 4. Fixed Point Theorems

In this section, as an application, we prove fixed point results for generalized Lipschitz mappings with an Example which demonstrates the strength and applicability of our main results.

**Theorem 4.1.** Let $(\mathcal{M}, m_b)$ be a complete $M_b$-cone metric space over Banach algebra $\mathcal{A}$ with the coefficient $s \geq 1$ and $P$ be a solid cone in $\mathcal{A}$. Let $k \in P$ be generalized Lipschitz constant with $\widehat{\rho}(k) < \frac{1}{s}$. Suppose $T : \mathcal{M} \to \mathcal{M}$ satisfying the following condition:

$$m_b(T\varsigma, T\varrho) \preceq k\big[m_b(\varsigma, T\varsigma) + m_b(\varrho, T\varrho)\big], \tag{4.1}$$

for all $\varsigma, \varrho \in \mathcal{M}$. Then $T$ has a unique fixed point $u$ such that $m_b(u, u) = \theta$.

**Proof:** Let $\varsigma_0 \in \mathcal{M}$ be arbitrary. Consider the sequence $\{\varsigma_n\}$ defined by $\varsigma_n = T^n\varsigma_0$ and $m_{b_n} = m_b(\varsigma_n, \varsigma_{n+1})$. Note that if there exist a natural $n$ such that $m_{b_n} = \theta$, then $\varsigma_n$ is a fixed point of $T$. So, we may assume that $m_{b_n} > \theta$, for $n \geq 0$. By (4.1) we obtain

$$\begin{aligned} m_{b_n} &= m_b(\varsigma_n, \varsigma_{n+1}) = m_b(T\varsigma_{n-1}, T\varsigma_n) \\ &\preceq k\Big[m_b(\varsigma_{n-1}, T\varsigma_{n-1}) + m_b(\varsigma_n, T\varsigma_n)\Big] \\ &= k\big[m_b(\varsigma_{n-1}, \varsigma_n) + m_b(\varsigma_n, \varsigma_{n+1})\big] \\ &= k\big[m_{b_{n-1}} + m_{b_n}\big], \end{aligned}$$

for any $n \geq 0$, $m_{b_n} \preceq k m_{b_{n-1}} + k m_{b_n}$, which implies $(e - k) m_{b_n} \preceq k m_{b_{n-1}}$. Since $(s+1)\widehat{\rho}(k) < 1$ leads to $\widehat{\rho}(k) < 1$, by Lemma 2.9, $(e - k)$ is invertible. So,

$$m_{b_n} \preceq k(e - k)^{-1} m_{b_{n-1}}.$$

Put $h = k(e-k)^{-1}$. Hence

$$m_{b_n} \preceq h m_{b_{n-1}} \preceq \cdots \preceq h^n m_{b_0}.$$

By Lemma 2.10 and Lemma 2.14, we have

$$\begin{aligned}\widehat{\rho}(h) &= \widehat{\rho}\big(k(e-k)^{-1}\big)\\ &\le \widehat{\rho}(k).\widehat{\rho}\big((e-k)^{-1}\big)\\ &\le \frac{\widehat{\rho}(k)}{1-\widehat{\rho}(k)} < \frac{1}{s}.\end{aligned}$$

Thus, $(e - sh)$ is invertible, so $\| sh^n \| \to 0 (n \to +\infty)$. Thus, $\lim_{n\to+\infty} m_{b_n} = \theta$. By (4.1) for all natural numbers $n, m$ we have

$$\begin{aligned}m_b(\varsigma_n, \varsigma_m) &= m_b(T^n\varsigma_0, T^m\varsigma_0)\\ &= m_b(T\varsigma_{n-1}, T\varsigma_{m-1})\\ &\preceq k\big[m_b(\varsigma_{n-1}, T\varsigma_{n-1}) + m_b(\varsigma_{m-1}, T\varsigma_{m-1})\big]\\ &= k\big[m_b(\varsigma_{n-1}, \varsigma_n) + m_b(\varsigma_{m-1}, \varsigma_m)\big]\\ &= k\big[m_{b_{n-1}} + m_{b_{m-1}}\big]\end{aligned}$$

As $\lim_{n\to+\infty} m_{b_n} = \theta$. From Lemma 2.12, it follows that, for any $c \in A$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that, for any $m > n > n_0$ we have $m_b(\varsigma_n, \varsigma_m) \ll c$ and $m_b(\varsigma_n, \varsigma_m) \to \theta$ as $n, m \to +\infty$

$$m_b(\varsigma_n, \varsigma_m) - m_{b_{\varsigma_n,\varsigma_m}} \ll c \text{ for all } n, m > n_0.$$

Now, for all natural numbers $n, m$ we have

$$\begin{aligned}m_{b_{\varsigma_n,\varsigma_m}} &= m_b(T\varsigma_{n-1}, T\varsigma_{n-1})\\ &\preceq k\big[m_b(\varsigma_{n-1}, T\varsigma_{n-1}) + m_b(\varsigma_{n-1}, T\varsigma_{n-1})\big]\\ &= k\big[m_b(\varsigma_{n-1}, \varsigma_n) + m_b(\varsigma_{n-1}, \varsigma_n)\big]\\ &= 2k m_b(\varsigma_{n-1}, \varsigma_n)\\ &= 2k m_{b_{n-1}} \to \theta \text{ as } n \to \infty.\end{aligned}$$

So, $M_{b_{\varsigma_n,\varsigma_m}} - m_{b_{\varsigma_n,\varsigma_m}} \ll c$ for all $n, m > n_0$. Thus, $\{\varsigma_n\}$ is an $m_b$-Cauchy sequence in $\mathcal{M}$. For some $u \in \mathcal{M}$,

$$\lim_{n\to+\infty} m_b(\varsigma_n, u) - m_{b_{\varsigma_n,u}} = \theta.$$

Now, we show that $u$ is a fixed point of $T$ in $\mathcal{M}$. For any natural number $n$ we have,

$$\begin{aligned}\lim_{n\to+\infty} m_b(\varsigma_n, u) - m_{b_{\varsigma_n,u}} &= \theta\\ &= \lim_{n\to+\infty} m_b(\varsigma_{n+1}, u) - m_{b_{\varsigma_{n+1},u}}\\ &= \lim_{n\to+\infty} m_b(T\varsigma_n, u) - m_{b_{T\varsigma_n,u}}\\ &= m_b(Tu, u) - m_{b_{Tu,u}},\end{aligned}$$

which implies that $m_b(Tu, u) - m_{b_{u,Tu}} = \theta$, hence $m_b(Tu, u) = m_{b_{u,Tu}}$, therefore $Tu = u$. Thus, $u$ is a fixed point of $T$. Now, we show that $u$ is a fixed point, then $m_b(u, u) = \theta$, assume that $u$ is a fixed point of $T$, hence

$$\begin{aligned}m_b(u, u) &= m_b(Tu, Tu)\\ &\preceq k\big[m_b(u, Tu) + m_b(u, Tu)\big]\\ &= 2k m_b(u, Tu)\\ &= 2k m_b(u, u).\end{aligned}$$

That is,

$$(e-2k)m_b(u,u) \preceq \theta.$$

The multiplication by

$$(e-2k)^{-1} = \sum_{i=0}^{+\infty}(2k)^i \geq 0,$$

yields that $m_b(u,u) \preceq \theta$. Thus, $m_b(u,u) = \theta$. To prove uniqueness, assume that $T$ has two fixed points say $u, v \in \mathcal{M}$, hence,

$$\begin{aligned} m_b(u,v) &= m_b(Tu,Tv) \\ &\preceq k\big[m_b(u,Tu) + m_b(v,Tv)\big] \\ &= k\big[m_b(u,u) + m_b(v,v)\big] \\ &= \theta, \end{aligned}$$

which implies that $m_b(u,v) = \theta$, and hence $u = v$ as required.

**Theorem 4.2:** Let $(\mathcal{M}, m_b)$ be a complete $M_b$-cone metric space over Banach algebra $\mathcal{A}$ with the coefficient $s \geq 1$ and $P$ be a solid cone in $\mathcal{A}$. Let $k \in P$ be generalized Lipschitz constant with $\widehat{\rho}(k) < \frac{1}{s}$. Suppose $T : \mathcal{M} \to \mathcal{M}$ satisfying the following condition:

$$m_b(T\varsigma, T\varrho) \preceq k\big[m_b(\varsigma, T\varrho) + m_b(T\varsigma, \varrho)\big], \text{ for all } \varsigma, \varrho \in \mathcal{M}. \tag{4.2}$$

Then $T$ has a unique fixed point $u$ such that $m_b(u,u) = \theta$.

**Proof:** Let $\varsigma_0 \in \mathcal{M}$ be arbitrary. Consider the sequence $\{\varsigma_n\}$ defined by $\varsigma_n = T^n\varsigma_0$ and $m_{b_n} = m_b(\varsigma_n, \varsigma_{n+1})$. Note that if $\exists$ a natural number $n$ such that $m_{b_n} = \theta$, then $\varsigma_n$ is a fixed point of $T$. So, we may assume that $m_{b_n} > \theta$, for $n \geq 0$. By (4.2) we obtain

$$\begin{aligned} m_{b_n} &= m_b(\varsigma_n, \varsigma_{n+1}) = m_b(T\varsigma_{n-1}, T\varsigma_n) \\ &\preceq k\big[m_b(\varsigma_{n-1}, T\varsigma_n) + m_b(\varsigma_n, T\varsigma_{n-1})\big] \\ &= k\big[m_b(\varsigma_{n-1}, \varsigma_{n+1}) + m_b(\varsigma_n, \varsigma_n)\big] \\ &\preceq k\big[sm_b(\varsigma_{n-1}, \varsigma_n) - sm_{b_{\varsigma_{n-1},\varsigma_n}} + sm_b(\varsigma_n, \varsigma_{n+1}) \\ &\quad - sm_{b_{\varsigma_n,\varsigma_{n+1}}} + m_{b_{\varsigma_{n-1},\varsigma_{n+1}}} - m_{b_{\varsigma_n,\varsigma_n}} + m_{b_{\varsigma_n,\varsigma_n}}\big]. \end{aligned}$$

By $(m_b2)$, we have

$$\begin{aligned} m_{b_n} &\preceq k\big[sm_b(\varsigma_{n-1}, \varsigma_n) - sm_b(\varsigma_{n-1}, \varsigma_n) + sm_b(\varsigma_n, \varsigma_{n+1}) \\ &\quad - sm_{b_{\varsigma_n,\varsigma_{n+1}}} + m_{b_{\varsigma_{n-1},\varsigma_{n+1}}} - m_b(\varsigma_n, \varsigma_n) + m_b(\varsigma_n, \varsigma_n)\big] \\ &= sk\big[m_b(\varsigma_{n-1}, \varsigma_n) + m_b(\varsigma_n, \varsigma_{n+1}) - m_{b_{\varsigma_{n-1},\varsigma_n}} \\ &\quad - m_{b_{\varsigma_n,\varsigma_{n+1}}} + m_{b_{\varsigma_{n-1},\varsigma_{n+1}}}\big] \end{aligned} \tag{4.3}$$

Put $A_n = m_{b_{\varsigma_{n-1},\varsigma_{n+1}}} - m_{b_{\varsigma_{n-1},\varsigma_n}} - m_{b_{\varsigma_n,\varsigma_{n+1}}}$.
Hence we will have six cases:

**Case 1:** Let $m_b(\varsigma_{n-1}, \varsigma_{n-1}) \preceq m_b(\varsigma_n, \varsigma_n) \preceq m_b(\varsigma_{n+1}, \varsigma_{n+1})$, then $m_{b_{\varsigma_n,\varsigma_{n+1}}} = m_b(\varsigma_n, \varsigma_n)$, $m_{b_{\varsigma_{n-1},\varsigma_n}} = m_b(\varsigma_{n-1}, \varsigma_{n-1})$ and $m_{b_{\varsigma_{n-1},\varsigma_{n+1}}} = m_b(\varsigma_{n-1}, \varsigma_{n-1})$.

Thus,

$$\begin{aligned} A_n &= m_b(\varsigma_{n-1}, \varsigma_{n-1}) - m_b(\varsigma_{n-1}, \varsigma_{n-1}) - m_b(\varsigma_n, \varsigma_n). \\ A_n &= -m_b(\varsigma_n, \varsigma_n). \end{aligned}$$

At first by (4.2), for all $n \in N$, we have

$$\begin{aligned} m_b(\varsigma_n, \varsigma_n) &= m_b(T\varsigma_{n-1}, T\varsigma_{n-1}) \\ &\preceq k\big[m_b(\varsigma_{n-1}, \varsigma_n) + m_b(\varsigma_{n-1}, \varsigma_n)\big] \\ &= 2km_b(\varsigma_{n-1}, \varsigma_n). \end{aligned} \tag{4.4}$$

Therefore, $A_n < -2km_b(\varsigma_{n-1}, \varsigma_n)$.

**Case 2:** Let $m_b(\varsigma_{n+1}, \varsigma_{n+1}) \preceq m_b(\varsigma_n, \varsigma_n) \preceq m_b(\varsigma_{n-1}, \varsigma_{n-1})$, we have $m_{b_{\varsigma_n,\varsigma_{n+1}}} = m_b(\varsigma_{n+1}, \varsigma_{n+1})$, $m_{b_{\varsigma_{n-1},\varsigma_n}} = m_b(\varsigma_n, \varsigma_n)$ and $m_{b_{\varsigma_{n-1},\varsigma_{n+1}}} = m_b(\varsigma_{n+1}, \varsigma_{n+1})$. Thus,

$$\begin{aligned} A_n &= m_b(\varsigma_{n+1}, \varsigma_{n+1}) - m_b(\varsigma_n, \varsigma_n) - m_b(\varsigma_{n+1}, \varsigma_{n+1}) \\ A_n &= -m_b(\varsigma_n, \varsigma_n) \end{aligned}$$

By (4.4)

$$A_n < -2km_b(\varsigma_{n-1}, \varsigma_n).$$

**Case 3:** Let $m_b(\varsigma_{n-1}, \varsigma_{n-1}) \preceq m_b(\varsigma_{n+1}, \varsigma_{n+1}) \preceq m_b(\varsigma_n, \varsigma_n)$, and $m_{b_{\varsigma_n,\varsigma_{n+1}}} = m_b(\varsigma_{n+1}, \varsigma_{n+1}), m_{b_{\varsigma_{n-1},\varsigma_n}} = m_b(\varsigma_{n-1}, \varsigma_{n-1})$ and $m_{b_{\varsigma_{n-1},\varsigma_{n+1}}} = m_b(\varsigma_{n-1}, \varsigma_{n-1})$. Hence we have,

$$\begin{aligned} A_n &= m_b(\varsigma_{n-1}, \varsigma_{n-1}) - m_b(\varsigma_{n-1}, \varsigma_{n-1}) - m_b(\varsigma_{n+1}, \varsigma_{n+1}) \\ A_n &= -m_b(\varsigma_{n+1}, \varsigma_{n+1}). \end{aligned}$$

By (4.2), we have

$$\begin{aligned} A_n &= -m_b(T\varsigma_n, T\varsigma_n) \\ &\preceq -k\big[m_b(\varsigma_n, \varsigma_{n+1}) + m_b(\varsigma_n, \varsigma_{n+1})\big] \\ &\preceq -2km_b(\varsigma_n, \varsigma_{n+1}). \end{aligned} \tag{4.5}$$

**Case 4:** Let $m_b(\varsigma_n, \varsigma_n) \preceq m_b(\varsigma_{n+1}, \varsigma_{n+1}) \preceq m_b(\varsigma_{n-1}, \varsigma_{n-1})$ and so, $m_{b_{\varsigma_n,\varsigma_{n+1}}} = m_b(\varsigma_n, \varsigma_n), m_{b_{\varsigma_{n-1},\varsigma_n}} = m_b(\varsigma_n, \varsigma_n)$ and $m_{b_{\varsigma_{n-1},\varsigma_{n+1}}} = m_b(\varsigma_{n+1}, \varsigma_{n+1})$. So, we get

$$\begin{aligned} A_n &= m_b(\varsigma_{n+1}, \varsigma_{n+1}) - m_b(\varsigma_n, \varsigma_n) - m_b(\varsigma_n, \varsigma_n) \\ &\preceq m_b(\varsigma_{n+1}, \varsigma_{n+1}) \\ &\preceq 2km_b(\varsigma_n, \varsigma_{n+1}). \end{aligned}$$

**Case 5:** Let $m_b(\varsigma_n, \varsigma_n) \preceq m_b(\varsigma_{n-1}, \varsigma_{n-1}) \preceq m_b(\varsigma_{n+1}, \varsigma_{n+1})$ and so, $m_{b_{\varsigma_n,\varsigma_{n+1}}} = m_b(\varsigma_n, \varsigma_n), m_{b_{\varsigma_{n-1},\varsigma_n}} = m_b(\varsigma_n, \varsigma_n)$ and $m_{b_{\varsigma_{n-1},\varsigma_{n+1}}} = m_b(\varsigma_{n-1}, \varsigma_{n-1})$.

Thus, we get

$$\begin{aligned} A_n &= m_b(\varsigma_{n-1}, \varsigma_{n-1}) - m_b(\varsigma_n, \varsigma_n) - m_b(\varsigma_n, \varsigma_n) \\ &\preceq m_b(\varsigma_{n-1}, \varsigma_{n-1}) \\ &\preceq m_b(\varsigma_{n+1}, \varsigma_{n+1}) \\ &\preceq 2km_b(\varsigma_n, \varsigma_{n+1}). \qquad \text{[by (4.5)]} \end{aligned}$$

**Case 6:** Let $m_b(\varsigma_{n+1}, \varsigma_{n+1}) \preceq m_b(\varsigma_{n-1}, \varsigma_{n-1}) \preceq m_b(\varsigma_n, \varsigma_n)$ and $m_{b_{\varsigma_n,\varsigma_{n+1}}} = m_b(\varsigma_{n+1}, \varsigma_{n+1}), m_{b_{\varsigma_{n-1},\varsigma_n}} = m_b(\varsigma_{n-1}, \varsigma_{n-1})$ and $m_{b_{\varsigma_{n-1},\varsigma_{n+1}}} = m_b(\varsigma_{n-1}, \varsigma_{n-1})$.

Thus,

$$\begin{aligned} A_n &= m_b(\varsigma_{n-1}, \varsigma_{n-1}) - m_b(\varsigma_{n-1}, \varsigma_{n-1}) - m_b(\varsigma_{n+1}, \varsigma_{n+1}) \\ &= -m_b(\varsigma_{n+1}, \varsigma_{n+1}) \\ &\preceq 2km_b(\varsigma_n, \varsigma_{n+1}). \end{aligned}$$

Now if $A_n < -m_b(\varsigma_n, \varsigma_n)$ and by (4.4) we have $A_n < -2km_b(\varsigma_{n-1}, \varsigma_n)$, and hence

$$\begin{aligned} m_b(\varsigma_n, \varsigma_{n+1}) &\preceq sk\big[m_b(\varsigma_{n-1}, \varsigma_n) + m_b(\varsigma_n, \varsigma_{n+1}) - 2km_b(\varsigma_{n-1}, \varsigma_n)\big] \\ (e - sk)m_b(\varsigma_n, \varsigma_{n+1}) &\preceq ksm_b(\varsigma_{n-1}, \varsigma_n) - 2sk^2 m_b(\varsigma_{n-1}, \varsigma_n) \\ (e - sk)m_b(\varsigma_n, \varsigma_{n+1}) &\preceq sk(e - 2k)m_b(\varsigma_n, \varsigma_{n-1}). \end{aligned}$$

Since $s\widehat{\rho}\Big[k(s - 2sk + e)\Big] < 1$ leads to $\widehat{\rho}(sk) < 1$, by Lemma 2.9, $(e - sk)$ is invertible. So,

$$m_b(\varsigma_n, \varsigma_{n+1}) \preceq sk(e - 2k)(e - sk)^{-1} m_b(\varsigma_n, \varsigma_{n-1}).$$

Put $h = sk(e - 2k)(e - sk)^{-1}$. Hence

$$m_b(\varsigma_n, \varsigma_{n+1}) \preceq hm_b(\varsigma_n, \varsigma_{n-1}) \preceq ... \preceq h^n m_b(\varsigma_0, \varsigma_1).$$

By Lemma 2.10 and 2.14, we have

$$\begin{aligned} \widehat{\rho}(h) &= \widehat{\rho}\big[sk(e - 2k)(e - sk)^{-1}\big] \\ &\leq \widehat{\rho}(sk)\widehat{\rho}(e - 2k)\widehat{\rho}(e - sk)^{-1} \\ &\leq \frac{\widehat{\rho}(sk)[1 - \widehat{\rho}(2k)]}{1 - \widehat{\rho}(sk)} < \frac{1}{s}. \end{aligned}$$

Thus, $(e - sh)$ is invertible, so $\| sh^n \| \to 0 (n \to +\infty)$. Thus, $m_b(\varsigma_n, \varsigma_{n+1}) \to \theta$ as $n \to +\infty$. If $A_n < m_b(\varsigma_{n+1}, \varsigma_{n+1})$, by (4.4), we have $A_n < 2km_b(\varsigma_n, \varsigma_{n+1})$, and have

$$\begin{aligned} m_b(\varsigma_n, \varsigma_{n+1}) &\preceq sk\big[m_b(\varsigma_{n-1}, \varsigma_n) + m_b(\varsigma_n, \varsigma_{n+1}) + 2km_b(\varsigma_n, \varsigma_{n+1})\big] \\ \big[e - (sk + 2sk^2)\big]m_b(\varsigma_n, \varsigma_{n+1}) &\preceq skm_b(\varsigma_{n-1}, \varsigma_n). \end{aligned}$$

Since $\widehat{\rho}\big[s^2k + sk + 2sk^2\big] < 1$ leads to $\widehat{\rho}\big(sk + 2sk^2\big) < 1$, by Lemma 2.9, $\big[e - (sk + 2sk^2)\big]$ is invertible. So,

$$m_b(\varsigma_n, \varsigma_{n+1}) \preceq sk\big[e - (sk + 2sk^2)\big]^{-1} m_b(\varsigma_{n-1}, \varsigma_n).$$

Put $\lambda = sk\big[e - (sk + 2sk^2)\big]^{-1}$. Hence

$$m_b(\varsigma_n, \varsigma_{n+1}) \preceq \lambda m_b(\varsigma_{n-1}, \varsigma_n) \preceq ... \preceq \lambda^n m_b(\varsigma_0, \varsigma_1).$$

By Lemma 2.10 and Lemma 2.14, we have

$$\begin{aligned} \widehat{\rho}(\lambda) &= \widehat{\rho}\big(sk[e - (sk + 2sk^2)]^{-1}\big) \\ &\leq \widehat{\rho}(sk)\widehat{\rho}[e - (sk + 2sk^2)]^{-1} \\ &\leq \frac{\widehat{\rho}(sk)}{1 - \widehat{\rho}(sk + 2sk^2)} < \frac{1}{s}. \end{aligned}$$

Thus, $(e - s\lambda)$ is invertible, so $\| s\lambda^n \| \to 0 (n \to +\infty)$. Thus, $m_b(\varsigma_n, \varsigma_{n+1}) \to \theta$ as $n \to +\infty$. If $A_n < -m_b(\varsigma_{n+1}, \varsigma_{n+1})$, by (4.4), we have $A_n < -2km_b(\varsigma_n, \varsigma_{n+1})$, and have

$$\begin{aligned} m_b(\varsigma_n, \varsigma_{n+1}) &\preceq sk\big[m_b(\varsigma_{n-1}, \varsigma_n) + m_b(\varsigma_n, \varsigma_{n+1}) - 2km_b(\varsigma_n, \varsigma_{n+1})\big] \\ \big[e - (sk - 2sk^2)\big]m_b(\varsigma_n, \varsigma_{n+1}) &\preceq skm_b(\varsigma_{n-1}, \varsigma_n). \end{aligned}$$

Since $\widehat{\rho}\big[s^2k + sk - 2sk^2\big] < 1$ leads to $\widehat{\rho}\big(sk - 2sk^2\big) < 1$, by Lemma 2.9, $\big[e - (sk - 2sk^2)\big]$ is invertible. So,

$$m_b(\varsigma_n, \varsigma_{n+1}) \preceq sk\big[e - (sk - 2sk^2)\big]^{-1} m_b(\varsigma_{n-1}, \varsigma_n).$$

Put $\lambda = sk\big[e - (sk - 2sk^2)\big]^{-1}$. Hence

$$m_b(\varsigma_n, \varsigma_{n+1}) \preceq \lambda m_b(\varsigma_{n-1}, \varsigma_n) \preceq ... \preceq \lambda^n m_b(\varsigma_0, \varsigma_1).$$

By Lemma 2.10 and Lemma 2.14, we have

$$\begin{aligned}\widehat{\rho}(\lambda) &= \widehat{\rho}\big(sk[e-(sk-2sk^2)]^{-1}\big)\\ &\leq \widehat{\rho}(sk)\widehat{\rho}[e-(sk-2sk^2)]^{-1}\\ &\leq \frac{\widehat{\rho}(sk)}{1-\widehat{\rho}(sk-2sk^2)} < \frac{1}{s}.\end{aligned}$$

Thus, $(e-s\lambda)$ is invertible, so $\| s\lambda^n \| \to 0 (n \to +\infty)$. Thus, $m_b(\varsigma_n, \varsigma_{n+1}) \to \theta$ as $n \to +\infty$. By completeness of $\mathcal{M}$, we get $\varsigma_n \to \varsigma$ for some $\varsigma \in \mathcal{M}$. So, $m_b(\varsigma_n, \varsigma) - m_{b_{\varsigma_n,\varsigma}} \to \theta$ as $n \to +\infty$ and $M_b(\varsigma_n, \varsigma) - m_{b_{\varsigma_n,\varsigma}} \to \theta$ as $n \to +\infty$. By Lemma 2.8, $m_{b_{\varsigma_n,\varsigma}} \to \theta$ as $n \to +\infty$, so we have $m_b(\varsigma_n, \varsigma) \to \theta$ as $n \to +\infty$. By Remark 3.3, we have

$$m_b(\varsigma, \varsigma) = \theta = m_{b_{\varsigma,T\varsigma}}.$$

Now, we prove $m_b(\varsigma, T\varsigma) = \theta$. By $(m_b4)$ and (4.2), we get

$$\begin{aligned}m_b(\varsigma, T\varsigma) &\preceq s\big[m_b(\varsigma, \varsigma_n) - m_{b_{\varsigma,\varsigma_n}} + m_b(\varsigma_n, T\varsigma)\\ &\quad - m_{b_{\varsigma_n,T\varsigma}}\big] - m_{b_{\varsigma_n,\varsigma_n}} + m_{b_{\varsigma,T\varsigma}}\\ &= sm_b(\varsigma_n, T\varsigma)\\ &= sm_b(T\varsigma_{n-1}, T\varsigma)\\ &\preceq ks\big[m_b(\varsigma_{n-1}, T\varsigma) + m_b(T\varsigma_{n-1}, \varsigma)\big]\\ &\preceq ks\big[m_b(\varsigma_{n-1}, T\varsigma) + m_b(\varsigma_n, \varsigma)\big]\\ \\ &\preceq ks^2\big[m_b(\varsigma_{n-1}, T\varsigma) - m_{b_{\varsigma_{n-1},\varsigma_n}} + m_b(\varsigma_n, T\varsigma)\\ &\quad - m_{b_{\varsigma_n,T\varsigma}}\big] - ksm_{b_{\varsigma_n,\varsigma_n}} + ksm_{b_{\varsigma_{n-1},T\varsigma}}\\ &\preceq ks^2 m_b(\varsigma, T\varsigma)\\ (e-ks^2)m_b(\varsigma, T\varsigma) &\preceq \theta.\end{aligned}$$

The multiplication by

$$(e-ks^2)^{-1} = \sum_{i=0}^{+\infty}(ks^2)^i \geq 0,$$

yields that $m_b(\varsigma, T\varsigma) \preceq \theta$. Thus, $m_b(\varsigma, T\varsigma) = \theta$. By (4.2), we have

$$\begin{aligned}\theta \preceq m_b(T\varsigma, T\varsigma) &\preceq k\big[m_b(\varsigma, T\varsigma) + m_b(T\varsigma, \varsigma)\big]\\ &= 2km_b(\varsigma, T\varsigma)\\ &= \theta.\end{aligned}$$

So, we have $m_b(T\varsigma, T\varsigma) = m_b(\varsigma, T\varsigma) = m_b(\varsigma, \varsigma)$. By $(m_b1)$, we get $T\varsigma = \varsigma$. Now, let $\varsigma, \varrho \in \mathcal{M}$ and $\varsigma \neq \varrho$ are two fixed points of $T$, so we have

$$\begin{aligned}m_b(\varsigma, \varrho) = m_b(T\varsigma, T\varrho) &\preceq k\big[m_b(\varsigma, T\varrho) + m_b(\varrho, T\varsigma)\big]\\ &= k\big[m_b(\varsigma, \varrho) + m_b(\varsigma, \varrho)\big]\\ &= 2km_b(\varsigma, \varrho)\\ (e-2k)m_b(\varsigma, \varrho) &\preceq \theta.\end{aligned}$$

The multiplication by

$$(e-2k)^{-1} = \sum_{i=0}^{+\infty}(2k)^i \geq 0,$$

yields that $m_b(\varsigma,\varrho)\preceq\theta$. Thus, $m_b(\varsigma,\varrho)=\theta$. Thus $\varsigma=\varrho$ and $T$ has a unique fixed point.

**Corollary 4.3** Let $(\mathcal{M},m_b)$ be a complete $M_b$-cone metric space over Banach algebra $\mathcal{A}$ with the coefficient $s\geq 1$ and $P$ be a solid cone in $\mathcal{A}$. Let $k\in P$ be generalized Lipschitz constant with $\widehat{\rho}(k)<\frac{1}{s}$. Suppose $T:\mathcal{M}\to\mathcal{M}$ satisfying the following condition:

$$m_b(T\varsigma,T\varrho)\preceq km_b(\varsigma,\varrho),\ \text{ for all }\ \varsigma,\varrho\in\mathcal{M}. \tag{4.2}$$

Then $T$ has a unique fixed point $u$ such that $m_b(u,u)=\theta$.

In this work, given below are few examples that support our main results.

**Example 4.4** Let $A=C^1_R[0,1]$ with norm defined by $\parallel\varsigma\parallel=\parallel\varsigma\parallel_\infty+\parallel\varsigma'\parallel_\infty$. Under point-wise multiplication, $\mathcal{A}$ is a real unit Banach algebra with unit $e=1$. Consider a cone $P=\{\varsigma\in\mathcal{A}:\varsigma\geq 0\}$ in $\mathcal{A}$. Moreover, $P$ is a non-normal cone (see[19]). Let $\mathcal{M}=[0,+\infty)$. Define $m_b:\mathcal{M}^2\to A$ by $m_b(\varsigma,\varrho)(t)=\left(\frac{\varsigma+\varrho}{2}\right)^2e^t$ for all $\varsigma,\varrho\in\mathcal{M}$. Then $(\mathcal{M},m_b)$ is complete $M_b$-cone metric space over Banach algebra. Define $T:\mathcal{M}\to\mathcal{M}$ by

$$T\varsigma=\begin{cases}0, & 0<\varsigma<3\\ \frac{\varsigma}{1+\varsigma}, & \varsigma\geq 3.\end{cases}$$

We will show that the condition (4.1) is satisfied with $k=\frac{1}{8}$. Suppose that $\varsigma,\varrho\in\mathcal{M}$. Then there are 4 possible cases:

**Case 1:** If $\varsigma,\varrho\in[0,3)$, the claim is obvious.

**Case 2:** If $\varsigma,\varrho\in[3,+\infty)$, we get

$$\begin{aligned}m_b(T\varsigma,T\varrho)(t)&=\frac{1}{4}\Big[\frac{\varsigma}{1+\varsigma}+\frac{\varrho}{1+\varrho}\Big]^2e^t\\ &\preceq\frac{1}{4}\Big[\frac{\varsigma}{4}+\frac{\varrho}{4}\Big]^2e^t\\ &=\frac{1}{16}\Big[\frac{\varsigma}{2}+\frac{\varrho}{2}\Big]^2e^t\\ &\preceq\frac{1}{16}\Big[\frac{\varsigma+\frac{\varsigma}{1+\varsigma}}{2}+\frac{\varrho+\frac{\varrho}{1+\varrho}}{2}\Big]^2e^t.\end{aligned}$$

Since

$$\begin{aligned}(\varsigma+\varrho)^p&\preceq 2^{p-1}(\varsigma^p+\varrho^p)\\ &=\frac{1}{8}\Big[\big(\frac{\varsigma+\frac{\varsigma}{1+\varsigma}}{2}\big)+\big(\frac{\varrho+\frac{\varrho}{1+\varrho}}{2}\big)\Big]^2e^t\\ &=\frac{1}{8}\Big[m_b(\varsigma,T\varsigma)+m_b(\varrho,T\varrho)\Big](t).\end{aligned}$$

**Case 3:** Assume that $\varsigma,\varrho\in[3,+\infty)\times[0,3)\cup[0,3)\times[3,+\infty)$ without loss of generality, we may assume that $\varsigma\in[0,3)$ and $\varrho\in[3,+\infty)$. Then we obtain

$$\begin{aligned}m_b(T\varsigma,T\varrho)(t)&=\frac{1}{4}\Big(\frac{\varrho}{1+\varrho}\Big)^2e^t\\ &\preceq\frac{1}{4}\Big(\frac{\varrho}{4}\Big)^2e^t\\ &\preceq\frac{1}{16}\Big(\frac{\varrho+\frac{\varrho}{1+\varrho}}{2}\Big)^2e^t\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{8}\Big[\big(\frac{\varrho}{2}\big)^2 + \big(\frac{\frac{\varrho}{1+\varrho}}{2}\big)\Big]^2 e^t \\
&= \frac{1}{8}\Big[m_b(\varsigma, T\varsigma) + m_b(\varrho, T\varrho)\Big](t).
\end{aligned}
$$

**Case 4:** If $\varsigma \in [3, +\infty)$ and $\varrho \in [0, 3)$, then we have

$$
\begin{aligned}
m_b(T\varsigma, T\varrho)(t) &= \frac{1}{4}\Big(\frac{\varsigma}{1+\varsigma}\Big)^2 e^t \\
&\preceq \frac{1}{4}\Big(\frac{\varsigma}{4}\Big)^2 e^t \\
&= \frac{1}{16}\Big(\frac{\varsigma}{2}\Big)^2 e^t \\
&= \frac{1}{16}\Big(\frac{\varsigma + \frac{\varsigma}{1+\varsigma}}{2}\Big)^2 e^t \\
&= \frac{1}{8}\Big[m_b(\varsigma, T\varsigma) + m_b(\varrho, T\varrho)\Big](t).
\end{aligned}
$$

Since

$$
\begin{aligned}
(\varsigma + \varrho)^p &\preceq 2^{p-1}(\varsigma^p + \varrho^p) \\
&= \frac{1}{8}\Big[\big(\frac{\varsigma}{2}\big)^2 + \Big(\frac{\frac{\varsigma}{1+\varsigma}}{2}\Big)^2\Big] e^t \\
&= \frac{1}{8}\Big[m_b(\varsigma, T\varsigma) + m_b(\varrho, T\varrho)\Big](t).
\end{aligned}
$$

Therefore, all the condition in Theorem (4.1) are satisfied so $T$ has a unique fixed point. In this case $\varsigma = \theta$ is the unique fixed point of $T$.

**Example 4.5** Consider a Banach algebra $\mathcal{A}$ and cone $P$ be as in Example 4.4 and let $\mathcal{M} = R^+$. Define a mapping $m_b : \mathcal{M}^2 \to A$ by $m_b(\varsigma, \varrho)(t) = \big(\frac{\varsigma+\varrho}{2}\big)^2 e^t$, for all $\varsigma, \varrho \in \mathcal{M}$. Then $(\mathcal{M}, m_b)$ is a $M_b$-cone metric space over Banach algebra $\mathcal{A}$. Define $T : \mathcal{M} \to \mathcal{M}$ by

$$
T\varsigma = \begin{cases} \varsigma^2, & \text{if } 0 \leq \varsigma < \frac{1}{2} \\ \frac{1}{4}, & \text{if } \varsigma \geq \frac{1}{2}. \end{cases}
$$

We will show that the condition (4.2) is satisfied with $k = \frac{1}{3}$. Suppose that $\varsigma, \varrho \in \mathcal{M}$. Then there are three possible cases:

**Case 1:** If $\varsigma, \varrho \in [0, \frac{1}{2})$, then we have

$$
\begin{aligned}
m_b(T\varsigma, T\varrho)(t) &= \Big(\frac{\varsigma^2 + \varrho^2}{2}\Big)^2 e^t \\
&= \frac{1}{4}\big(\varsigma^2 + \varrho^2\big)^2 e^t \\
&\preceq \frac{1}{2}\Big[\big(\varsigma^2\big)^2 + \big(\varrho^2\big)^2\Big] e^t \\
&\preceq \Big[\frac{\varsigma^2}{6} + \frac{\big(\varrho^2\big)^2}{6} + \frac{\varrho}{6} + \frac{\big(\varsigma^2\big)^2}{6}\Big] e^t
\end{aligned}
$$

$$\begin{aligned}&= \frac{1}{3}\Big[\frac{\varsigma^2}{2} + \frac{\big(\varrho^2\big)^2}{2} + \frac{\varrho}{2} + \frac{\big(\varsigma^2\big)^2}{2}\Big]e^t\\ &= \frac{1}{3}\Big[m_b(\varsigma, T\varrho) + m_b(\varrho, T\varsigma)\Big](t).\end{aligned}$$

**Case 2:** If $\varsigma, \varrho \in [\frac{1}{2}, +\infty)$, then we have

$$\begin{aligned}m_b(T\varsigma, T\varrho)(t) &= \frac{1}{4}\Big[\frac{1}{4} + \frac{1}{4}\Big]^2 e^t\\ &\preceq \frac{1}{2}\Big[\big(\frac{1}{4}\big)^2 + \big(\frac{1}{4}\big)^2\Big]e^t\\ &\preceq \Big[\frac{\varsigma^2}{6} + \frac{1}{96} + \frac{\varrho^2}{6} + \frac{1}{96}\Big]e^t\\ &= \frac{1}{3}\Big[\frac{\varsigma^2}{2} + \frac{1}{32} + \frac{\varrho^2}{2} + \frac{1}{32}\Big]e^t\\ &= \frac{1}{3}\Big[m_b(\varsigma, T\varrho) + m_b(\varrho, T\varsigma)\Big](t).\end{aligned}$$

**Case 3:** Let $\varsigma, \varrho \in [0, \frac{1}{2}) \times [\frac{1}{2}, +\infty) \cup [\frac{1}{2}, +\infty) \times [0, \frac{1}{2})$ without loss of generality, we may assume that $\varsigma \in [0, \frac{1}{2})$ and $\varrho \in [\frac{1}{2}, +\infty)$. Then we get,

$$\begin{aligned}m_b(T\varsigma, T\varrho)(t) &= \left(\frac{\varsigma^2 + \frac{1}{4}}{2}\right)^2 e^t\\ &\preceq \frac{1}{2}\Big[(\varsigma^2)^2 + \frac{1}{4}2\Big]e^t\\ &\preceq \Big[\frac{\varsigma^2}{6} + \frac{1}{96} + \frac{\varrho^2}{6} + \frac{1}{6}(\varsigma^2)^2\Big]e^t\\ &= \frac{1}{3}\Big[\frac{\varsigma^2}{2} + \frac{1}{32} + \frac{\varrho^2}{2} + \frac{(\varsigma^2)^2}{2}\Big]e^t\\ &= \frac{1}{3}\Big[m_b(\varsigma, T\varrho) + m_b(\varrho, T\varsigma)\Big](t).\end{aligned}$$

Then $T$ satisfies the condition (4.2) of Theorem 4.2 for all $\varsigma, \varrho \in \mathcal{M}$ with $k = \frac{1}{3}$ and so there exist a unique fixed point of $T$. In this case $\varsigma = \theta$ is a unique fixed point of $T$.

## 5. Application to the Existence of a Solution of Integral Equation

In this section, we will provide an application of the theorem proved in the previous section.

Consider $C\big([0,1], \mathbb{R}\big)$, the class of continuous functions on [0,1]. Let $A = C[0,1]$ be equipped with the norm $\| \varsigma \| = \| \varsigma \|_\infty + \| \varsigma' \|_\infty$. Take the usual multiplication, then $A$ is a Banach algebra with the unit $e = 1$. Let $m_b$ be the $M_b$-cone metric given as

$$m_b(\varsigma, \varrho)(t) = \sup_{t\in[0,1]} \left(\frac{\varsigma + \varrho}{2}\right)^2 e^t, \tag{5.1}$$

for all $\varsigma, \varrho \in C\big([0,1], \mathbb{R}\big)$. Note that $\Big(C\big([0,1], \mathbb{R}\big), m_b\Big)$ is a $\theta$-complete $M_b$-cone metric space over Banach algebra $\Big(C[0,1], \mathbb{R}\Big)$.

**Theorem 5.1.** Assume that for all $\varsigma, \varrho \in C\big([0,1], \mathbb{R}\big)$

$$\left|K\big(t,s,\varsigma(t)\big) + K\big(t,s,\varrho(t)\big)\right| \leq \frac{1}{2}\left|\varsigma(t) + \varrho(t)\right|, \tag{5.2}$$

for all $t, s \in [0,1]$. Then the integral equation

$$\varsigma(t) = \int_0^T K\big(t,s,\varsigma(t)\big)ds \tag{5.3}$$

where $t \in [0,1]$, admits a unique solution in $C\big([0,1], \mathbb{R}\big)$.

**Proof.** Define $T : \mathcal{M} \to \mathcal{M}$ by

$$T\varsigma(t) = \int_0^1 K\big(t,s,\varsigma(t)\big)ds \tag{5.4}$$

for all $t, s \in [0,1]$.
We have

$$\begin{aligned}
m_b(T\varsigma, T\varrho)(t) &= \left|\frac{T\varsigma(t) + T\varrho(t)}{2}\right|^2 \\
&= \left|\int_0^1 \left(\frac{K\big(t,s,\varsigma(t)\big) + K\big(t,s,\varrho(t)\big)}{2}\right)ds\right|^2 e^t \\
&\preceq \left(\int_0^1 \left|\frac{K\big(t,s,\varsigma(t)\big) + K\big(t,s,\varrho(t)\big)}{2}\right| ds\right)^2 e^t \\
&\preceq \left(\frac{1}{2}\int_0^1 \left|\frac{\varsigma(t) + \varrho(t)}{2}\right| ds\right)^2 e^t \\
&\preceq \left(\frac{1}{2}\int_0^1 \left(\frac{|\varsigma(t)| + |\varrho(t)|}{2}\right) ds\right)^2 e^t \\
&\preceq \left(\frac{1}{2}\sup_{t\in[0,1]} \left(\frac{|\varsigma(t)| + |\varrho(t)|}{2}\right) \int_0^1 ds\right)^2 e^t \\
&\preceq \frac{1}{4}\, m_b(\varsigma, \varrho)(t).
\end{aligned}$$

Thus, condition (4.1) is satisfied. Therefore, all conditions of Theorem 4.1 are satisfied. Hence $T$ has a unique fixed point, which means that the Fredholm integral equation (5.3) has a unique solution. This completes the proof.

**Open Problems :** Prove analogue of Reich contraction, Ćirić contraction and Hardy-Rogers contraction in $M_b$-cone metric space over Banach algebra.

## Conclusion

Fixed point theory plays an essential role in various fields of sciences and engineering. In the present article, we introduce the concept of $M_b$-cone metric space over Banach algebra and investigate fixed points for using generalized Lipschitz maps. An application of the theorem proved as main result is given in Section 5. At last some open problems are given for the readers.

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## References

1. M. Asadi, E. Karapinar and P. Salimi, *New extension of p-metric spaces with some fixed-point results on M-metric spaces*, J. of Inequalities and Appl., 2014, 18 (2014). https://doi.org/10.1186/1029-242X-2014-18
2. I. A. Bakhtin, *The contraction mapping principle in almost metric spaces*, Funct. Anal. Gos. Ped. Inst. Unianowsk 30 (1989), 26-37.
3. S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math Inf Univ Ostraviensis 1(1): 5–11, (1993).
4. J. Fernandez, N. Malviya and Ersin Gilić, *Fixed point results in M-cone metric space over Banach algebra with an application*, Filomat 36:16 (2022), 5547–5562 https://doi.org/10.2298/FIL2216547.
5. J. Fernandez, N. Malviya, Z. D. Mitrović, A. Hussain, V. Parvaneh, *Some fixed point results on $N_b$-cone metric spaces over Banach algebra*, Adv. Diff. Equ., 2020:529 https://doi.org/10.1186/s13662-020-02991-5, (2020).
6. J. Fernandez, K. Saxena and N. Malviya, *On cone $b_2$-metric spaces over Banach algebra*, Sao Paulo J. Math. Sci., 11, 221-239, (2017).
7. J. Fernandez, N. Malviya, D. Djekic-Dolićanin, D. Pučić, *The $p_b$-cone metric spaces over Banach algebra with applications*, Filomat, 34(3), 983-998, (2020).
8. M. Fréchet, *Sur quelques points du calcul fonctionnel*, Palermo Rend., 22, 1-74, (1906).
9. L. G. Huang, and X. Zhang, *Cone metric spaces and fixed point theorems for contractive mappings*, J. Math. Anal. Appl., 332(2), 1468-1476, (2007).
10. H. Huang and S. Radenović, *Common fixed point theorems of Generalized Lipschitz mappings in cone metric spaces over Banach algebras*, Appl. Math. Inf. Sci. 9, No. 6, 2983-2990 (2015).
11. Z. Kadelburg and S. Radenović, *A note on various types of cones and fixed point results in cone metric spaces*, Asian J. Math. Appl., (2013).
12. H. Liu, and S. Xu, *Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings*, Fixed Point Theory Appl., 320 (2013).
13. S. Matthews, *Partial metric topology*, 8th Summer Conference on General topology and Appl., 183-197, (1994).
14. S. Radenović, and B.E. Rhoades, *Fixed point theorem for two non-self mappings in cone metric spaces*, Comput. Math. Appl. 57, 1701-1707 (2009).
15. Sh. Rezapour, and R. Hamlbarani, *Some notes on the paper, Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., 345, 719-724, (2008).
16. W. Rudin, *Functional Analysis*, 2nd Edn. McGraw-Hill, New York (1991).
17. S. Shukla, S. Rai and R. Shukla, *A relation-theoretic set-valued version of Prešić-Ćirić theorem and applications*, Bound Value Probl 2023, 59 (2023).
18. S. Shukla, N. Dubey and R. Shukla, *Fixed point theorems in graphical cone metric spaces and application to a system of initial value problems*, J Inequal Appl 2023, 91 (2023).
19. S. Xu and S. Radenović, *Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality*, Fixed Point Theory and Appl., (2014).

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