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Ricci Yamabe Soliton on f-Kenmotsu Manifolds with Generalized Symmetric Metric Connection

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ABSTRACT: This research investigates Ricci Yamabe soliton on f-Kenmotsu manifolds whose potential vector field is torse-forming admits a generalized symmetric metric connection. Some results of such soliton on CR-submanifolds of f-Kenmotsu manifolds with generalized symmetric metric connection are obtained.

Key Words: Ricci Yamabe soliton, Torse-forming vector field, f-Kenmotsu manifold, CR-submanifold, generalized symmetric metric connections.

Contents

1. Introduction

In 1988, Hamilton [14] introduced the notions of Ricci flow and Yamabe flow concurrently. The solutions of these flows are known as Ricci solitons and Yamabe solitons, respectively [6,10]. The study of a new geometric flow which is a scalar combination of Ricci and Yamabe flow under the name Ricci-Yamabe map [15]. This is also known as Ricci Yamabe flow of the type (α, β). The Ricci Yamabe flow is an evolution for the metrics on the Riemannian or semi-Riemannian manifolds defined by [11,13,15]

$$\frac{\partial}{\partial t}g(t) = -2\alpha Ric(t) + \beta R(t)g(t), \qquad g_0 = g(0). \tag{1.1}$$

A soliton to the Ricci Yamabe flow is known as Ricci Yamabe soliton if it moves only by one parameter group of diffeomorphism and scaling. A Ricci Yamabe soliton on a Riemannian manifold $M^3(g, V, \lambda, \alpha, \beta)$ satisfies

$$(L_\kappa g)(U_1, U_2) + 2\alpha S(U_1, U_2) + (2\lambda - \beta r)g(U_1, U_2) = 0, \tag{1.2}$$

where r, S and L_κ is the scalar curvature, the Ricci tensor and the Lie-derivative along the vector field κ on M respectively and λ is a constant, is known as Ricci Yamabe soliton of (α, β)-type, which is a generalization of Ricci and Yamabe solitons. The Ricci Yamabe soliton is α-Ricci soliton if $\beta = 0$ and

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β-Yamabe soliton if $\alpha = 0$. The Ricci Yamabe soliton is said to be expanding if λ is negative or shrinking if λ is positive or steady if λ is zero.
A vector field κ on a Riemannian manifold (M, g) is called a torse-forming vector field [5,9] if it satisfies

$$\nabla_{U_1}\kappa = \Theta U_1 + \nu(U_1)\kappa \;\; \forall U_1 \in TM, \tag{1.3}$$

where $\Theta \in C^\infty(M)$ and ν is a linear form of M.
A torse-forming vector field κ is called
i. recurrent , if $\Theta = 0$,
ii. concircular, if the 1-form ν vanishes identically,
iii. parallel, if $\Theta = 0, \nu = 0$,
iv. concurrent, if $\Theta = 1, \nu = 0$.
In 2017, Chen [1] introduced a new type of vector field called torqued vector field if the vector field κ satisfying (1.2) with $\nu(\kappa) = 0$ where Θ is called the torqued function with the 1-form ν, called the torqued form of κ.
This paper deals with η-Ricci Soliton on f-Kenmotsu manifold[f-KM].

2. Preliminaries

A smooth manifold M of odd dimension is an almost contact metric manifold, if there exist ζ a vector field, a (1, 1) tensor field ϕ, η a 1-form and a Riemannian metric g on M so that

$$\phi^2 U_1 = -U_1 + \eta(U_1)\zeta, \quad \eta(\zeta) = 1, \quad \eta(U_1) = g(U_1, \zeta), \tag{2.1}$$

$$\phi\zeta = 0, \quad \eta \circ \phi = 0, \tag{2.2}$$

$$g(\phi U_1, \phi U_2) = g(U_1, U_2) - \eta(U_1)\eta(U_2). \tag{2.3}$$

A manifold of odd dimension is known as an [f-KM] [8] if the covariant differentiation of ϕ satisfies

$$(\nabla_{U_1}\phi)U_2 = f[g(\phi U_1, U_2)\zeta - \eta(U_2)\phi U_1], \tag{2.4}$$

where $f \in C^\infty(M)$ is such that $df \Lambda \eta = 0$. If $f = \beta \neq 0$, then the manifold is a β-Kenmotsu manifold. The 1-Kenmotsu manifold is a Kenmotsu manifold. The manifold is cosymplectic if $f = 0$ [2,3,8,17]. An [f-KM] is regular if $f^2 + f' \neq 0$, where $f' = \zeta f$.
For an [f-KM] from (2.1), we have

$$\nabla_{U_1}\zeta = f[U_1 - \eta(U_1)\zeta]. \tag{2.5}$$

In a three-dimensional Riemannian manifold, we have

$$\begin{aligned} R(U_1, U_2)U_3 &= [g(U_2, U_3)QU_1 - g(U_1, U_3)QU_2] + S(U_2, U_3)U_1 \\ &\quad -S(U_1, U_3)U_2 - \frac{r}{2}[g(U_2, U_3)U_1 - g(U_1, U_3)U_2]. \end{aligned} \tag{2.6}$$

In a three-dimensional [f-KM], we get

$$\begin{aligned} R(U_1, U_2)U_3 &= (\frac{r}{2} + 2f^2 + 2f')(U_1 \Lambda U_2)U_3 \\ &\quad -(\frac{r}{2} + 3f^2 + 3f')[\eta(U_1)(\zeta \Lambda U_2)U_3 + \eta(U_2)(U_1 \Lambda \zeta)U_3], \end{aligned} \tag{2.7}$$

$$S(U_1, U_2) = -(f + \lambda)g(U_1, U_2) + (f - \mu)\eta(U_1)\eta(U_2), \tag{2.8}$$

$$S(U_1, \zeta) = -(\lambda + \mu)\eta(U_1), \tag{2.9}$$

$$QU_1 = -(\lambda + \mu)\zeta. \tag{2.10}$$

Let $\overline{M}$ be a submanifold of an [f-KM] $M^3(\phi, \zeta, \eta, g)$. The Gauss and Weingarten formulae are given by

$$\nabla_{U_1}U_2 = \overline{\nabla}_{U_1}U_2 + h(U_1, U_2), for \;\; all \;\; U_1, U_2 \in (T\overline{M}), \tag{2.11}$$

$$\nabla_{U_1}N = -A_N U_1 + \nabla^{\perp}_{U_1}N, for \;\; all \;\; U_1, U_2 \in (T^{\perp}\overline{M}), \tag{2.12}$$

where $\nabla_{U_1}U_2 \in (T\overline{M})$ and $[h(U_1, U_2), \nabla^{\perp}_{U_1}N] \in (T^{\perp}\overline{M})$.

3. Generalized symmetric metric connection

In an [f-KM] $M^3(\phi,\zeta,\eta,g)$, the generalized symmetric metric connection $\overline{\nabla}$ is defined as [4,7,12]

$$\overline{\nabla}_{U_1}U_2 = \nabla_{U_1}U_2 + p[\eta(U_2)U_1 - g(U_1,U_2)\zeta] - q\eta(U_1)\phi U_2, \tag{3.1}$$

for any U_1, U_2 on M. The generalized symmetric metric connection reduces to a semi-symmetric metric connection if $(p,q)=(1,0)$ and quarter-symmetric metric connection if $(p,q)=(0,1)$.
Let $M^3(\phi,\zeta,\eta,g)$ be [f-KM] with a generalized symmetric metric connection $\overline{\nabla}$. Then we have the following results [7]

$$\begin{aligned}
(\overline{\nabla}_{U_1}\phi)(U_2) &= (f+p)[g(\phi U_1,U_2)\zeta - \eta(U_2)\phi U_1], && (3.2)\\
\overline{\nabla}_{U_1}\zeta &= (f+p)[U_1 - \eta(U_1)\zeta], && (3.3)\\
\overline{R}(U_1,U_2)U_3 &= R(U_1,U_2)U_3 + (f+p)(p[2g(U_1,U_3)U_2 - 2g(U_2,U_3)U_1 - \eta(U_1)\eta(U_3)U_2\\
&+ \eta(U_2)\eta(U_3)U_1 - \eta(U_2)g(U_1,U_3)\zeta + \eta(U_1)g(U_2,U_3)\zeta)]\\
&- q[g(\phi U_1,U_3)\eta(U_2)\zeta - \eta(U_2)\eta(U_3)\phi U_1 - g(\phi U_2,U_3)\eta(U_1)\zeta\\
&+ \eta(U_1)\eta(U_3)\phi U_2] + p^2[g(U_2,U_3)U_1 - g(U_1,U_3)U_2], && (3.4)\\
\overline{R}(U_1,U_2)\zeta &= [f(f+p)+f'][\eta(U_1)U_2 - \eta(U_2)U_1]\\
&- (f+p)q[\eta(U_1)\phi U_2 - \eta(U_2)\phi U_1], && (3.5)\\
\overline{S}(U_1,U_2) &= S(U_1,U_2) + (f+p)(p[(1-4n)g(U_1,U_2) + (2n-1)\eta(U_1)\eta(U_2)]\\
&+ qg(\phi U_1,U_2)) + p^2 2ng(U_1,U_2), && (3.6)
\end{aligned}$$

for all U_1, U_2 in (TM).

4. CR-submanifolds of an [f-KM] with generalized symmetric metric connection

Definition 4.1 *A three-dimensional Riemannian manifold (M,g) of an [f-KM] $M^3(\phi,\zeta,\eta,g)$ is known as a CR-submanifold [16] if ζ is tangent to M and there exists on M a differentiable distribution $G : x \to G_x \subset T_xM$ such that*
i. G is invariant under ϕ,
ii. The orthogonal complement distribution $G^\perp : x \to G_x^\perp \subset T_xM$ of the distribution G on M is totally real.

Definition 4.2 *If the distribution G (resp., $G^\perp$) is horizontal (resp., vertical), then the pair $(G,G^\perp)$ is known as $\zeta-horizontal$ $(resp.,\zeta-vertical)$ if $\zeta\in\Gamma(G)$ $(resp.,\zeta\in\Gamma(G^\perp))$ [16]. The CR-submanifold is also known as $\zeta-horizontal$ $(resp.,\zeta-vertical)$ if $\zeta\in\Gamma(G)$ $(resp.,\zeta\in\Gamma(G^\perp))$.*

The orthogonal complement $\phi G^\perp \in T^\perp M$ is defined by

$$TM = G\oplus G^\perp, \;\; T^\perp M = \phi G^\perp \oplus \mu, \tag{4.1}$$

where $\phi\mu=\mu$. Let $\overline{M}$ be a CR-submanifold of $[f-KM]$ $M^3(g,\phi,\zeta,\eta)$ with a generalized symmetric metric connection $\overline{\nabla}$. For every $U_1\in\Gamma(TM)$ and $N\in\Gamma(T^\perp M)$, we can write as

$$\begin{aligned}
U_1 &= DU_1 + EU_1, \;\; DU_1\in\Gamma G, \;\; EU_1\in\Gamma G^\perp, && (4.2)\\
\phi N &= BN + CN, \;\; BN\in\Gamma G^\perp, CN\in\Gamma\mu. && (4.3)
\end{aligned}$$

The Gauss and Weingarten formulae with respect to $\overline{\nabla}$ are given by

$$\begin{aligned}
\overline{\nabla}_{U_1}U_2 &= \overline{\overline{\nabla}}_{U_1}U_2 + \overline{h}(U_1,U_2), && (4.4)\\
\overline{\nabla}_{U_1}N &= -\overline{A}_N U_1 + \overline{\nabla}^\perp_{U_1}N && (4.5)
\end{aligned}$$

respectively, where $\overline{\nabla}_{U_1}U_2, \;\overline{\nabla}_{U_1}N \in \Gamma(TM)$.
Here $\overline{\overline{\nabla}}$ is the induced connection on $\overline{M}$, $\overline{h}$ is the second fundamental form and $\overline{A}_N$ is the Weingarten

mapping with respect to $\overline{\nabla}$.
By virtue of (2.11), (3.1) and (4.4), we yield

$$\overline{\overline{\nabla}}_{U_1}U_2+\overline{h}(U_1,U_2)=\overline{\nabla}_{U_1}U_2+h(U_1,U_2)+p[\eta(U_2)U_1-g(U_1,U_2)\zeta]-q\eta(U_1)\phi U_2. \tag{4.6}$$

In view of (4.2, 4.3, 4.6) and comparing the normal and the tangential components, we have

$$D\overline{\overline{\nabla}}_{U_1}U_2 = D\overline{\nabla}_{U_1}U_2+p[\eta(U_2)DU_1-g(U_1,U_2)D\zeta]-Dq\eta(U_1)\phi U_2, \tag{4.7}$$
$$\overline{h}(U_1,U_2) = h(U_1,U_2)-qE\eta(U_1)\phi U_2, \tag{4.8}$$
$$E\overline{\overline{\nabla}}_{U_1}U_2 = E\overline{\nabla}_{U_1}U_2+p[\eta(U_2)EU_1-g(U_1,U_2)E\zeta]-qE\eta(U_1)\phi U_2, \tag{4.9}$$

for any $U_1,U_2\in(TM)$.

5. Ricci Yamabe soliton on Einstein-like [f-KM]

Definition 5.1 *An [f-KM] is called an Einstein-like if S Ricci tensor satisfies*

$$S(U_1,U_2)=a_1g(U_1,U_2)+a_2g(\phi U_1,U_2)+a_3\eta(U_1)\eta(U_2), \tag{5.1}$$

where a_1,a_2 and a_3 are some real constants.

Taking $V=\zeta$ in (1.2) and then from (5.1), we obtain

$$g(\nabla_{U_1}\zeta,U_2)+g(\nabla_{U_2}\zeta,U_1)+2\alpha S(U_1,U_2)+(2\lambda-\beta r)g(U_1,U_2)=0. \tag{5.2}$$

In view of (2.5) and (5.2), we have

$$[2f+2\alpha a_1+2\lambda-\beta r]g(U_1,U_2)+[2\alpha a_3-2f]\eta(U_1)\eta(U_2)+2\alpha a_2g(\phi U_1,U_2)=0. \tag{5.3}$$

From the above equation we yield
$a_1=\frac{1}{\alpha}[\frac{\beta r}{2}-(f+\lambda)]$, $a_2=0$ and $a_3=\frac{f}{\alpha}$.
Now

$$\nabla_{U_1}\zeta=[\frac{\beta r}{2}-(\alpha a_1+\lambda)][U_1-\eta(U_1)], \tag{5.4}$$

and

$$S(U_1,U_2) = \frac{1}{\alpha}[\frac{\beta r}{2}-(f+\lambda)]g(U_1,U_2)+\frac{f}{\alpha}\eta(U_1)\eta(U_2), \tag{5.5}$$
$$S(U_1,\zeta) = \frac{1}{\alpha}[\frac{\beta r}{2}-\lambda]\eta(U_1), \tag{5.6}$$
$$QU_1 = \frac{1}{\alpha}[\frac{\beta r}{2}-\lambda]\zeta. \tag{5.7}$$

Thus, we state the following:

Theorem 5.1 *If a non-cosymplectic [f-KM] $M^3(g,\phi,\zeta,\alpha,\beta)$ admits a Ricci Yamabe soliton with an Einstein-like Ricci tensor, then the Ricci Yamabe soliton will expand if $\beta r<2(\alpha a_1+f)$ or shrink if $\beta r>2(\alpha a_1+f)$ or steady if $f=\frac{\beta r}{2}-\alpha a_1$.*

If $\beta=0$, then we have

$$\lambda=-(f+\alpha a_1).$$

Corollary 5.1 *If a non-cosymplectic [f-KM] $M^3(g,\phi,\zeta)$ admits a Ricci Yamabe soliton with an Einstein-like Ricci tensor, then the α-Ricci soliton will expand if $f+\alpha a_1<0$ or shrink if $f+\alpha a_1>0$ or steady if $f=-\alpha a_1$.*

6. Ricci Yamabe soliton whose potential vector field is torse-forming

Let (g,λ,κ) be a Ricci Yamabe soliton $M^3(g,\phi,\eta,\zeta)$ with respect to a generalized symmetric metric connection. From (1.2), we yield

$$(\overline{L}_\kappa g)(U_1,U_2)+2\alpha\overline{S}(U_1,U_2)+(2\lambda-\beta\overline{r})g(U_1,U_2)=0. \tag{6.1}$$

By Lie derivative's definition,(1.3) and (3.1), we have

$$\begin{aligned}(\overline{L}_\kappa g)(U_1,U_2) &= 2\Theta g(U_1,U_2)+\nu(U_1)g(\kappa,U_2)+\nu(U_2)g(U_1,\kappa)\\ &\quad +p[2\eta(\kappa)g(U_1,U_2)-\eta(U_2)g(U_1,\kappa)-\eta(U_1)g(U_2,\kappa)]\\ &\quad -q[\eta(U_2)g(\phi U_1,\kappa)+\eta(U_1)g(\phi U_2,\kappa)].\end{aligned} \tag{6.2}$$

In view of (6.1) and (6.2), we have

$$\begin{aligned}[2\Theta+2p\eta(\kappa) &+ 2\lambda(1-\alpha)-\beta\overline{r}-22\alpha p(f+p)-2\alpha f+6p^2]g(U_1,U_2)\\ &= p[\eta(U_1)g(U_2,\kappa)+\eta(U_2)g(U_1,\kappa)]\\ &\quad -q[\eta(U_1)g(\phi U_2,\kappa)+\eta(U_2)g(\phi U_1,\kappa)+2\alpha g(\phi U_1,U_2)]\\ &\quad -[\nu(U_1)g(U_2,\kappa)+\nu(U_2)g(U_1,\kappa)]\\ &\quad -[10\alpha p(f+p)+2\alpha(f-\mu)]\eta(U_1)\eta(U_2).\end{aligned} \tag{6.3}$$

Taking $U_1=U_2=e_i$ in the above equation, we obtain

$$\lambda=\frac{1}{\alpha-1}\left[\Theta+\frac{2p}{3}\eta(\kappa)+\frac{\nu(\kappa)}{3}-\alpha f(1+11p)+p^2(3-11\alpha)-\frac{\beta\overline{r}}{2}\right]. \tag{6.4}$$

Thus, we can state the following theorem:

Theorem 6.1 *Let (g,λ,κ) be a Ricci Yamabe soliton on 3-dimensional [f-KM] with respect to a generalized symmetric metric connection. If κ is a torse-forming vector field, then (g,λ,κ) is shrinking or steady or expanding accordingly as*
$\frac{1}{\alpha-1}\left[\Theta+\frac{2p}{3}\eta(\kappa)+\frac{\nu(\kappa)}{3}-\alpha f(1+11p)+p^2(3-11\alpha)-\frac{\beta\overline{r}}{2}\right]\gtreqless 0.$

In this continuation, we state the following corollaries:

Corollary 6.1 *Let (g,λ,κ) be a Ricci Yamabe soliton on 3-dimensional [f-KM] with respect to a generalized symmetric metric connection. Then, the following relations hold:*

κ	***Existence condition***	***Nature of solitons (expanding or steady or shrinking)***
torse-forming	$\frac{1}{\alpha-1}[\Theta+\frac{2p}{3}\eta(\kappa)+\frac{\nu(\kappa)}{3}-\frac{\beta\overline{r}}{2}$ $-\alpha f(1+11p)+p^2(3-11\alpha)]=constant$	$\frac{1}{\alpha-1}[\Theta+\frac{2p}{3}\eta(\kappa)+\frac{\nu(\kappa)}{3}-\frac{\beta\overline{r}}{2}$ $-\alpha f(1+11p)+p^2(3-11\alpha)]\lesseqgtr 0$
recurrent	$\frac{1}{\alpha-1}[\frac{2p}{3}\eta(\kappa)+\frac{\nu(\kappa)}{3}-\frac{\beta\overline{r}}{2}$ $-\alpha f(1+11p)+p^2(3-11\alpha)]=constant$	$\frac{1}{\alpha-1}[\frac{2p}{3}\eta(\kappa)+\frac{\nu(\kappa)}{3}-\frac{\beta\overline{r}}{2}$ $-\alpha f(1+11p)+p^2(3-11\alpha)]\lesseqgtr 0$
concircular	$\frac{1}{\alpha-1}[\Theta+\frac{2p}{3}\eta(\kappa)-\frac{\beta\overline{r}}{2}$ $-\alpha f(1+11p)+p^2(3-11\alpha)]=constant$	$\frac{1}{\alpha-1}[\Theta+\frac{2p}{3}\eta(\kappa)-\frac{\beta\overline{r}}{2}$ $-\alpha f(1+11p)+p^2(3-11\alpha)]\lesseqgtr 0$
concurrent	$\frac{1}{\alpha-1}[1+\frac{2p}{3}\eta(\kappa)-\frac{\beta\overline{r}}{2}$ $-\alpha f(1+11p)+p^2(3-11\alpha)]=constant$	$\frac{1}{\alpha-1}[1+\frac{2p}{3}\eta(\kappa)-\frac{\beta\overline{r}}{2}$ $-\alpha f(1+11p)+p^2(3-11\alpha)]\lesseqgtr 0$
parallel	$\frac{1}{\alpha-1}[\frac{2p}{3}\eta(\kappa)-\frac{\beta\overline{r}}{2}$ $-\alpha f(1+11p)+p^2(3-11\alpha)]=constant$	$\frac{1}{\alpha-1}[\frac{2p}{3}\eta(\kappa)-\frac{\beta\overline{r}}{2}$ $-\alpha f(1+11p)+p^2(3-11\alpha)]\lesseqgtr 0$
torqued	$\frac{1}{\alpha-1}[\Theta+\frac{2p}{3}\eta(\kappa)-\frac{\beta\overline{r}}{2}$ $-\alpha f(1+11p)+p^2(3-11\alpha)]=constant$	$\frac{1}{\alpha-1}[\Theta+\frac{2p}{3}\eta(\kappa)-\frac{\beta\overline{r}}{2}$ $-\alpha f(1+11p)+p^2(3-11\alpha)]\lesseqgtr 0$

If $\beta = 0$, then we have $\lambda = \frac{1}{\alpha-1}\Big[\Theta + \frac{2p}{3}\eta(\kappa) + \frac{\nu(\kappa)}{3} - \alpha f(1+11p) + p^2(3-11\alpha)\Big]$.

Corollary 6.2 *If an [f-KM] $M^3(g,\phi,\eta,\zeta)$ admits a Ricci Yamabe soliton with κ as a torse-forming vector field, then α-Ricci soliton is shrinking or steady or expanding accordingly as*
$\frac{1}{\alpha-1}\Big[\Theta + \frac{2p}{3}\eta(\kappa) + \frac{\nu(\kappa)}{3} - \alpha f(1+11p) + p^2(3-11\alpha)\Big] \gtreqless 0$,
unless $\frac{1}{\alpha-1}\Big[\Theta + \frac{2p}{3}\eta(\kappa) + \frac{\nu(\kappa)}{3} - \alpha f(1+11p) + p^2(3-11\alpha)\Big] = constant$.

κ	***Existence condition***	***Nature of solitons (expanding or steady or shrinking)***
torse-forming	$\frac{1}{\alpha-1}[\Theta + \frac{2p}{3}\eta(\kappa) + \frac{\nu(\kappa)}{3} - \alpha f(1+11p) + p^2(3-11\alpha)] = constant$	$\frac{1}{\alpha-1}[\Theta + \frac{2p}{3}\eta(\kappa) + \frac{\nu(\kappa)}{3} - \alpha f(1+11p) + p^2(3-11\alpha)] \lesseqgtr 0$
recurrent	$\frac{1}{\alpha-1}[\frac{2p}{3}\eta(\kappa) + \frac{\nu(\kappa)}{3} - \alpha f(1+11p) + p^2(3-11\alpha)] = constant$	$\frac{1}{\alpha-1}[\frac{2p}{3}\eta(\kappa) + \frac{\nu(\kappa)}{3} - \alpha f(1+11p) + p^2(3-11\alpha)] \lesseqgtr 0$
concircular	$\frac{1}{\alpha-1}[\Theta + \frac{2p}{3}\eta(\kappa) - \alpha f(1+11p) + p^2(3-11\alpha)] = constant$	$\frac{1}{\alpha-1}[\Theta + \frac{2p}{3}\eta(\kappa) - \alpha f(1+11p) + p^2(3-11\alpha)] \lesseqgtr 0$
concurrent	$\frac{1}{\alpha-1}[1 + \frac{2p}{3}\eta(\kappa) - \alpha f(1+11p) + p^2(3-11\alpha)] = constant$	$\frac{1}{\alpha-1}[1 + \frac{2p}{3}\eta(\kappa) - \alpha f(1+11p) + p^2(3-11\alpha)] \lesseqgtr 0$
parallel	$\frac{1}{\alpha-1}[\frac{2p}{3}\eta(\kappa) - \alpha f(1+11p) + p^2(3-11\alpha)] = constant$	$\frac{1}{\alpha-1}[\frac{2p}{3}\eta(\kappa) - \alpha f(1+11p) + p^2(3-11\alpha)] \lesseqgtr 0$
torqued	$\frac{1}{\alpha-1}[\Theta + \frac{2p}{3}\eta(\kappa) - \alpha f(1+11p) + p^2(3-11\alpha)] = constant$	$\frac{1}{\alpha-1}[\Theta + \frac{2p}{3}\eta(\kappa) - \alpha f(1+11p) + p^2(3-11\alpha)] \lesseqgtr 0$

If $\alpha = 0$, then we have $\lambda = \frac{\beta\bar{\tau}}{2} - [\Theta + \frac{2p}{3}\eta(\kappa) + \frac{\nu(\kappa)}{3} + 3p^2]$.

Corollary 6.3 *If an [f-KM] $M^3(g,\phi,\eta,\zeta)$ admits a Ricci Yamabe soliton with κ as a torse-forming vector field, then β-Yamabe soliton is shrinking or steady or expanding accordingly as*
$\Theta + \frac{2p}{3}\eta(\kappa) + \frac{\nu(\kappa)}{3} + 3p^2 \lesseqgtr \frac{\beta\bar{\tau}}{2}$, *unless* $\frac{\beta\bar{\tau}}{2} - [\Theta + \frac{2p}{3}\eta(\kappa) + \frac{\nu(\kappa)}{3} + 3p^2] = constant$.

κ	***Existence condition***	***Nature of solitons (expanding or steady or shrinking)***
torse-forming	$\frac{\beta\bar{\tau}}{2} - [\Theta + \frac{2p}{3}\eta(\kappa) + \frac{\nu(\kappa)}{3} + 3p^2] = constant$	$\Theta + \frac{2p}{3}\eta(\kappa) + \frac{\nu(\kappa)}{3} + 3p^2 \gtreqless \frac{\beta\bar{\tau}}{2}$
recurrent	$\frac{\beta\bar{\tau}}{2} - [\frac{2p}{3}\eta(\kappa) + \frac{\nu(\kappa)}{3} + 3p^2] = constant$	$\frac{2p}{3}\eta(\kappa) + \frac{\nu(\kappa)}{3} + 3p^2 \gtreqless \frac{\beta\bar{\tau}}{2}$
concircular	$\frac{\beta\bar{\tau}}{2} - [\Theta + \frac{2p}{3}\eta(\kappa) + 3p^2] = constant$	$\Theta + \frac{2p}{3}\eta(\kappa) + 3p^2 \gtreqless \frac{\beta\bar{\tau}}{2}$
concurrent	$\frac{\beta\bar{\tau}}{2} - [1 + \frac{2p}{3}\eta(\kappa) + 3p^2] = constant$	$1 + \frac{2p}{3}\eta(\kappa) + 3p^2 \gtreqless \frac{\beta\bar{\tau}}{2}$
parallel	$\frac{\beta\bar{\tau}}{2} - [\frac{2p}{3}\eta(\kappa) + 3p^2] = constant$	$\frac{2p}{3}\eta(\kappa) + 3p^2 \gtreqless \frac{\beta\bar{\tau}}{2}$
torqued	$\frac{\beta\bar{\tau}}{2} - [\Theta + \frac{2p}{3}\eta(\kappa) + 3p^2] = constant$	$\Theta + \frac{2p}{3}\eta(\kappa) + 3p^2 \gtreqless \frac{\beta\bar{\tau}}{2}$

7. Ricci Yamabe soliton whose potential vector field is torse-forming on CR-submanifold of [f-KM]

Let M is $\zeta - horizontal$ for every $U_1, U_2 \in \Gamma(G)$ and G is parallel with respect to $\overline{\overline{\nabla}}$, then using (4.7), we yield

$$\overline{\overline{\nabla}}_{U_1}U_2 = \overline{\nabla}_{U_1}U_2 + p[\eta(U_2)U_1 - g(U_1,U_2)\zeta] - q\eta(U_1)\phi U_2. \tag{7.1}$$

In view of (4.1)and (1.2), we conclude that the induced connection $\overline{\nabla}$ is a generalized symmetric metric connection.
This leads to the following theorem:

Theorem 7.1 *Let the CR-submanifold $\overline{M}$ of an [f-KM] $M^3(g,\phi,\eta,\zeta)$ admitting a generalized symmetric metric connection $\overset{\equiv}{\nabla}$ is ζ-horizontal (resp. ζ-horizontal) and G is parallel with respect to $\overset{\equiv}{\nabla}$. If (g,λ,κ) is a Ricci Yamabe soliton on $\overline{M}$ and κ is a torse-forming vector field, then (g,λ,κ) is shrinking or steady or expanding accordingly as*
$$\frac{1}{\alpha-1}\left[\Theta+\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)+p^2(3-11\alpha)+\frac{\nu(\kappa)}{3}-\frac{\beta\overline{r}}{2}+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)\right]\gtreqless 0.$$

In this continuation, we state the following corollaries:

Corollary 7.1 *Let the CR-submanifold $\overline{M}$ of an [f-KM] $M^3(g,\phi,\eta,\zeta)$ admitting a generalized symmetric metric connection $\overset{\equiv}{\nabla}$ is ζ-horizontal (resp. ζ-horizontal) and G is parallel with respect to $\overset{\equiv}{\nabla}$. If (g,λ,κ) is a Ricci Yamabe soliton on $\overline{M}$ and κ is a torse-forming vector field, then the following relations hold:*

κ	***Existence condition***	***Nature of solitons (expanding or steady or shrinking)***
torse-forming	$\frac{1}{\alpha-1}[\Theta+\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)$ $+p^2(3-11\alpha)+\frac{\nu(\kappa)}{3}-\frac{\beta\overline{r}}{2}$ $+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)]=constant$	$\frac{1}{\alpha-1}[\Theta+\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)$ $+p^2(3-11\alpha)+\frac{\nu(\kappa)}{3}-\frac{\beta\overline{r}}{2}$ $+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)]\lesseqgtr 0$
recurrent	$\frac{1}{\alpha-1}[\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)$ $+p^2(3-11\alpha)+\frac{\nu(\kappa)}{3}-\frac{\beta\overline{r}}{2}$ $+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)]=constant$	$\frac{1}{\alpha-1}[\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)$ $+p^2(3-11\alpha)+\frac{\nu(\kappa)}{3}-\frac{\beta\overline{r}}{2}$ $+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)]\lesseqgtr 0$
concircular	$\frac{1}{\alpha-1}[\Theta+\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)$ $+p^2(3-11\alpha)-\frac{\beta\overline{r}}{2}$ $+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)]=constant$	$\frac{1}{\alpha-1}[\Theta+\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)$ $+p^2(3-11\alpha)-\frac{\beta\overline{r}}{2}$ $+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)]\lesseqgtr 0$
concurrent	$\frac{1}{\alpha-1}[1+\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)$ $+p^2(3-11\alpha)-\frac{\beta\overline{r}}{2}$ $+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)]=constant$	$\frac{1}{\alpha-1}[1+\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)$ $+p^2(3-11\alpha)-\frac{\beta\overline{r}}{2}$ $+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)]\lesseqgtr 0$
parallel	$\frac{1}{\alpha-1}[\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)$ $+p^2(3-11\alpha)-\frac{\beta\overline{r}}{2}$ $+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)]=constant$	$\frac{1}{\alpha-1}[\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)$ $+p^2(3-11\alpha)-\frac{\beta\overline{r}}{2}$ $+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)]\lesseqgtr 0$
torqued	$\frac{1}{\alpha-1}[\Theta+\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)$ $+p^2(3-11\alpha)-\frac{\beta\overline{r}}{2}$ $+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)]=constant$	$\frac{1}{\alpha-1}[\Theta+\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)$ $+p^2(3-11\alpha)-\frac{\beta\overline{r}}{2}$ $+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)]\lesseqgtr 0$

If $\beta=0$, then we have
$$\lambda=\frac{1}{\alpha-1}\left[\Theta+\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)+p^2(3-11\alpha)+\frac{\nu(\kappa)}{3}+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)\right].$$

Corollary 7.2 *Let the CR-submanifold $\overline{M}$ of an [f-KM] $M^3(g,\phi,\eta,\zeta)$ admitting a generalized symmetric metric connection $\overset{\equiv}{\nabla}$ is ζ-horizontal (resp. ζ-horizontal) and G is parallel with respect to $\overset{\equiv}{\nabla}$ of type $(p,q)=(1,0)$. If (g,λ,κ) is a Ricci Yamabe soliton on $\overline{M}$ and κ is a torse-forming vector field, then the following results hold:*

κ	**Existence condition**	**Nature of solitons (expanding or steady or shrinking)**
torse-forming	$\frac{1}{\alpha-1}[\Theta+\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)$ $+p^2(3-11\alpha)+\frac{\nu(\kappa)}{3}$ $+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)]=constant$	$\frac{1}{\alpha-1}[\Theta+\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)$ $+p^2(3-11\alpha)+\frac{\nu(\kappa)}{3}$ $+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)]\lesseqgtr 0$
recurrent	$\frac{1}{\alpha-1}[\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)$ $+p^2(3-11\alpha)+\frac{\nu(\kappa)}{3}$ $+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)]=constant$	$\frac{1}{\alpha-1}[\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)$ $+p^2(3-11\alpha)+\frac{\nu(\kappa)}{3}$ $+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)]\lesseqgtr 0$
concircular	$\frac{1}{\alpha-1}[\Theta+\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)$ $+p^2(3-11\alpha)$ $+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)]=constant$	$\frac{1}{\alpha-1}[\Theta+\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)$ $+p^2(3-11\alpha)$ $+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)]\lesseqgtr 0$
concurrent	$\frac{1}{\alpha-1}[1+\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)$ $+p^2(3-11\alpha)$ $+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)]=constant$	$\frac{1}{\alpha-1}[1+\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)$ $+p^2(3-11\alpha)$ $+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)]\lesseqgtr 0$
parallel	$\frac{1}{\alpha-1}[\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)$ $+p^2(3-11\alpha)$ $+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)]=constant$	$\frac{1}{\alpha-1}[\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)$ $+p^2(3-11\alpha)$ $+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)]\lesseqgtr 0$
torqued	$\frac{1}{\alpha-1}[\Theta+\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)$ $+p^2(3-11\alpha)$ $+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)]=constant$	$\frac{1}{\alpha-1}[\Theta+\frac{4}{3}p\eta(\kappa)-\alpha f(1+11p)$ $+p^2(3-11\alpha)$ $+\frac{\alpha}{3}(f-\mu)+\frac{5}{6}(f+p)]\lesseqgtr 0$

If $\alpha=0$ then we have $\lambda=\frac{\beta\overline{r}}{2}-[\Theta+\frac{4}{3}p\eta(\kappa)+\frac{\nu(\kappa)}{3}+3p^2+\frac{5}{6}(f+p)]$.

Corollary 7.3 *Let the CR-submanifold $\overline{M}$ of an [f-KM] $M^3(g,\phi,\eta,\zeta)$ admitting a generalized symmetric metric connection $\overline{\overline{\nabla}}$ is ζ-horizontal (resp. ζ-horizontal) and G is parallel with respect to $\overline{\overline{\nabla}}$ of type $(p,q)=(0,1)$. If (g,λ,κ) is a Ricci Yamabe soliton on $\overline{M}$ and κ is a torse-forming vector field, then the following results hold:*

κ	**Existence condition**	**Nature of solitons (expanding or steady or shrinking)**
torse-forming	$\frac{\beta\overline{r}}{2}-[\Theta+\frac{4}{3}p\eta(\kappa)+\frac{\nu(\kappa)}{3}$ $+3p^2+\frac{5}{6}(f+p)]=constant$	$[\Theta+\frac{4}{3}p\eta(\kappa)+\frac{\nu(\kappa)}{3}$ $+3p^2+\frac{5}{6}(f+p)]\gtreqless\frac{\beta\overline{r}}{2}$
recurrent	$\frac{\beta\overline{r}}{2}-[\frac{4}{3}p\eta(\kappa)+\frac{\nu(\kappa)}{3}$ $+3p^2+\frac{5}{6}(f+p)]=constant$	$[\frac{4}{3}p\eta(\kappa)+\frac{\nu(\kappa)}{3}$ $+3p^2+\frac{5}{6}(f+p)]\gtreqless\frac{\beta\overline{r}}{2}$
concircular	$\frac{\beta\overline{r}}{2}-[\Theta+\frac{4}{3}p\eta(\kappa)$ $+3p^2+\frac{5}{6}(f+p)]=constant$	$[\Theta+\frac{4}{3}p\eta(\kappa)$ $+3p^2+\frac{5}{6}(f+p)]\gtreqless\frac{\beta\overline{r}}{2}$
concurrent	$\frac{\beta\overline{r}}{2}-[1+\frac{4}{3}p\eta(\kappa)$ $+3p^2+\frac{5}{6}(f+p)]=constant$	$[1+\frac{4}{3}p\eta(\kappa)$ $+3p^2+\frac{5}{6}(f+p)]\gtreqless\frac{\beta\overline{r}}{2}$
parallel	$\frac{\beta\overline{r}}{2}-[\frac{4}{3}p\eta(\kappa)+3p^2+\frac{5}{6}(f+p)]=constant$	$[\frac{4}{3}p\eta(\kappa)+3p^2+\frac{5}{6}(f+p)]\gtreqless\frac{\beta\overline{r}}{2}$
torqued	$\frac{\beta\overline{r}}{2}-[\Theta+\frac{4}{3}p\eta(\kappa)$ $+3p^2+\frac{5}{6}(f+p)]=constant$	$[\Theta+\frac{4}{3}p\eta(\kappa)$ $+3p^2+\frac{5}{6}(f+p)]\gtreqless\frac{\beta\overline{r}}{2}$

8. Conclusion

This study establishes that Ricci Yamabe solitons on 3-dimensional f-Kenmotsu manifolds admitting a generalized symmetric metric connection, with particular focus on solitons whose potential vector fields are torse-forming. We derived explicit conditions under which the solitons are expanding, steady, or shrinking, considering various types of torse-forming vector fields. The study was further extended to CR-submanifolds of such manifolds, leading to comprehensive scalar criteria that describe the nature of the solitons. Also provide a unified framework for studying Ricci Yamabe solitons in contact geometry. Future research should explore higher-dimensional f-Kenmotsu manifolds, other types of geometric solitons, or different ambient structures such as LP-Sasakian and trans-Sasakian manifolds.

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