Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44)** : 1–15.
ISSN-0037-8712
doi:10.5269/bspm.77383

A New Iterative Approach for Solving Fractal-Fractional Systems of Differential Equations

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ABSTRACT: This paper presents a new iterative technique for solving fractal-fractional differential equations (FFDEs) with different kernel functions, such as power-law, exponential and Mittag-Leffler, which are commonly used in mathematics and physics. Bhalekar and Gejji's method solves ordinary and partial fractal-fractional differential equations (ODEs and PDEs). Special algorithms are designed to make the methods accurate and efficient. These algorithms are implemented on FFDE systems and verified by error analysis and plots illustrating the fractal nature of the solutions. The findings show that the proposed algorithms are stable, reliable, and efficient and can be used to solve a broad variety of FFDE problems in science and engineering.

Key Words: Fractal - fractional derivatives, fractal - fractional integrals, fractal - fractional differential equation, new iterative method.

Contents

1. Introduction

Fractional calculus is a field of mathematical analysis that extends the principles of differentiation and integration to non-integer (fractional) orders [1,2,4,3,5]. Classical calculus is concerned with integer-order derivatives and integrals, which are commonly interpreted as the rate of change of a function or the area under a curve. However, fractional calculus extends these concepts to encompass derivatives and integrals of arbitrary, typically non-integer orders, resulting in a more adaptable mathematical framework.

This extension enables fractional calculus to model memory, hereditary characteristics, and higher-level dynamics of physical, biological and engineering systems that cannot be sufficiently represented by integer-order calculus. It has been applied in many areas like physics, engineering, economics and biology, where systems have anomalous diffusion, viscoelasticity and fractal-like structures.

Fractals are intricate geometric forms that have self-similarity at various scales and a fractional or non-integer dimension instead of an integer one. Developed by Mandelbrot [9] during the 1970s, fractals

* Corresponding author.
2020 *Mathematics Subject Classification:* 26A33, 35R11, 28A80, 65L03.
2020 *Mathematics Subject Classification:* 26A33, 35R11, 28A80, 65L03.
Submitted June 21, 2025. Published February 04, 2026

are observed in nature, such as coastlines, clouds and mountain ranges. They are generated using iterative processes and recursive algorithms and have complex recurring structures. Fractals are investigated with the help of tools like complex analysis, geometric measure theory and fractional calculus and have practical applications in fields such as physics, biology and computer graphics.

The fractal derivative concept, proposed in [10], extends classical calculus to deal with irregular, complex fractal structures. In contrast to classical derivatives which are applied to smooth functions the fractal derivative exploits fractal geometry concepts to grasp self-similarity across scales. The extension gives a more realistic description of fractal-like systems.

Atangana's seminal work [11] brought about the idea of fractal-fractional operators, which combine the ideas of fractal geometry and fractional calculus. These operators which generalize integration to non-integer orders, are best suited to describing the intricate irregular patterns seen in natural systems. Fractal geometry describes self-similar, complex patterns and when used in conjunction with the versatility of fractional calculus, it offers a sophisticated mathematical tool. This method is particularly useful in the study of dynamic systems that have fractal-like properties, where integer-order methods may not be able to capture the intricacies involved.

Fractal-fractional calculus is therefore a cutting-edge mathematics that reconciles fractal geometry and fractional calculus. Whereas fractional calculus classically addresses differentiating and integrating functions to non-integer orders in order to describe behaviors such as long-range dependencies and memory effects, fractal-fractional calculus extends this further by including fractal structures that are self-similar over various scales. This unification gives rise to novel operators such as the fractal-fractional derivative and integral to address systems with fractal memory in which history dictates present dynamics in non-linear and non-local manners. Implementations of this method cut across physics, biology and engineering, wherein it is utilized in modeling intricate systems whose description through conventional integer-order derivatives is challenging [21,22,23,24,20,25,8].

Daftardar-Gejji and Jafari [12] proposed a new iterative scheme for the solution of fractional differential equations. This scheme presents a novel and versatile method, improving the accuracy and efficiency of solving the equations. It presents a great advantage in mathematical modeling and analysis in specific areas. Their work represents a significant breakthrough, opening the door to further investigation and insight into intricate systems [13,14,16,17,18,19,15]. The goal of the present work is to extend the iterative approach introduced by Daftardar-Gejji and Jafari [12] to solve linear and nonlinear ordinary and partial differential equations of fractal and fractional order.

2. Mathematical Preliminaries

We provide a few definitions that are relevant to our investigation.

Definition 2.1 Let f be a continuous function and $\beta > 0$. The fractal derivative of f with order β is define as [10]:

$$\frac{df(t)}{dt^{\beta}} = \lim_{t\to t_0} \frac{f(t) - f(t_0)}{t^{\beta} - t_0^{\beta}} \tag{2.1}$$

Definition 2.2 Let f be a continuous function on (a, b) where $0 < \alpha \leq 1$ and $\beta > 0$. The fractal-fractional derivative of f with fractional order α and fractal dimension β, with power law kernel in the Caputo sense, is given as [11]:

$${}_{a}^{FFPC}D_t^{\alpha,\beta} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{df(y)}{dy^{\beta}} (t-y)^{-\alpha} dy. \tag{2.2}$$

Definition 2.3 If f is continuous on (a, b), then the fractal - fractional integral of f with fractional order α and fractal dimension β is defined as [11].

$${}_{0}^{FFP}I_t^{\alpha,\beta} f(t) = \frac{\beta}{\Gamma(\alpha)} \int_0^t y^{\beta-1} f(y)(t-y)^{\alpha-1} dy. \tag{2.3}$$

Definition 2.4 Let f be a continuous function on (a, b) where $0 < \alpha \leq 1$ and $\beta > 0$. The fractal-fractional derivative of f of fractional order α and fractal dimension β with exponential decay kernel in the Caputo sense is given as [11].

$$ {}_{a}^{FFEC}D_t^{\alpha,\beta} f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t \frac{df(y)}{dy^{\beta}} exp\left[-\frac{\alpha}{1-\alpha}(t-y)\right] dy. \tag{2.4} $$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$.

Definition 2.5 If f is a continuous function on (a, b), the fractal-fractional integral of f with fractional order α and fractal dimension β is defined as [11].

$$ {}_{0}^{FFE}I_t^{\alpha,\beta} f(t) = \frac{\beta\alpha}{M(\alpha)} \int_0^t y^{\beta-1} f(y) dy + \frac{\beta(1-\alpha)t^{\beta-1} f(t)}{M(\alpha)} \tag{2.5} $$

Definition 2.6 Let f be a continuous function on (a, b) where $0 < \alpha \leq 1$ and $\beta > 0$. The fractal-fractional derivative of f with fractional order α and fractal dimension β, with Mittag-Leffler kernel in the Caputo sense, is given as [11]:

$$ {}_{a}^{FFMC}D_t^{\alpha\ \beta} f(t) = \frac{AB(\alpha)}{1-\alpha} \int_a^t \frac{df(y)}{dy^{\beta}} E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-y)^{\alpha}\right] dy. \tag{2.6} $$

where $AB(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$.

Definition 2.7 If f is a continuous function on (a, b), the fractal-fractional integral of f with order α and fractal dimension β is defined as [11].

$$ {}_{0}^{FFM}I_t^{\alpha,\beta} f(t) = \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \int_0^t y^{\beta-1} u(y)(t-y)^{\alpha-1} dy + \frac{\beta(1-\alpha)t^{\beta-1}u(t)}{AB(\alpha)} \tag{2.7} $$

3. New Iterative Method

Daftardar - Gejji and Jafari [6] have introduced an new iterative method designed specifically to tackle nonlinear equations. This method is characterized by its simplicity, making it easy to comprehend and implement.
Assume that non - linear equations of the form,

$$ \phi(\bar{x}, t) = h(\bar{x}, t) + L(\phi(\bar{x}, t)) + N(\phi(\bar{x}, t)) \tag{3.1} $$

Let $\bar{x} = (x_1, x_2, x_3, \ldots, x_n)$ represent the variable vector. Where $h(\bar{x}, t)$ denotes the source term, and L and N indicate the linear and nonlinear operators, respectively. The solution to Equation (3.1), as obtained through the new iterative method, can be expressed in an expanded form as follows:

$$ \phi(\bar{x}, t) = \sum_{m=0}^{\infty} \phi_m(\bar{x}, t) \tag{3.2} $$

Due to the linearity of L, $\phi(\bar{x}, t)$ is expressed as,

$$ L\left(\sum_{m=0}^{\infty} \phi_m(\bar{x}, t)\right) = \sum_{m=0}^{\infty} L(\phi_m(\bar{x}, t)) \tag{3.3} $$

The non - linear operator N presented by Daftardar - Gejji and Jafari [6] is expressed as,

$$ N\left(\sum_{m=0}^{\infty} \phi_m(\bar{x}, t)\right) = N(\phi_0(\bar{x}, t)) + \sum_{m=1}^{\infty} \left\{ N\left(\sum_{j=0}^{i} \phi_j(\bar{x}, t)\right) - N\left(\sum_{j=0}^{i-1} \phi_j(\bar{x}, t)\right) \right\} \tag{3.4} $$

where $G_0 = N(\phi_0)$ and $G_j = \left\{ N\left(\sum_{j=0}^{i} \phi_j(\bar{x},t)\right) - N\left(\sum_{j=0}^{i-1} \phi_j(\bar{x},t)\right) \right\}, j \geq 1.$

equation (3.2) and (3.4) are substitute in equation (3.1) to obtain

$$\sum_{m=0}^{\infty} \phi_m(\bar{x},t) = h(\bar{x},t) + \sum_{m=0}^{\infty} L\left(\phi_m(\bar{x},t)\right) + N\left(\phi_0(\bar{x},t)\right) + \sum_{m=1}^{\infty} \left\{ N\left(\sum_{j=0}^{i} \phi_j(\bar{x},t)\right) - N\left(\sum_{j=0}^{i-1} \phi_j(\bar{x},t)\right) \right\} \tag{3.5}$$

Further define the recurrence relation as follows,

$$\begin{aligned} \phi_0(x,t) &= h, \\ \phi_1(x,t) &= L(\phi_0) + N(\phi_0), \\ \phi_2(x,t) &= L(\phi_1) + N(\phi_0+\phi_1) - N(\phi_0), \\ &\vdots \\ \phi_m(x,t) &= L(\phi_{m-1}) + N(\phi_0+\phi_1+\cdots+\phi_{m-1}) - N(\phi_0+\phi_1+\cdots+\phi_{m-2}), \\ & m = 1,2,3,\ldots, \end{aligned} \tag{3.6}$$

The n - term solution of Equations (3.1) and (3.2) is

$$\phi(\bar{x},t) = \phi_0 + \phi_1 + \phi_2 + \cdots + \phi_{n-1} \tag{3.7}$$

3.1. Convergence of New Iterative Method

We analyze the convergence of the new iterative approach [18] for solving a general functional equation (3.1). Let $e = \phi^* - \phi$, where ϕ^* is the exact answer, ϕ is the approximate solution, and e is the error associated with solving (3.1). Clearly, e fulfills 3.1. We begin with the functional equation

$$e(x) = f(x) + N(e(x)). \tag{3.8}$$

This leads to the following recurrence relation, as described in (3.6):

$$e_0 = f, \quad e_1 = N(e_0), \quad e_{n+1} = N\left(\sum_{i=0}^{n} e_i\right) - N\left(\sum_{i=0}^{n-1} e_i\right), \quad n = 1,2,\ldots \tag{3.9}$$

Assume the operator N satisfies the Lipschitz condition

$$\|N(x) - N(y)\| \leq k\|x-y\|, \quad \text{with} \quad 0 < k < 1.$$

Under this assumption, we can establish the following estimates for the sequence $\{e_n\}$. For the first term,

$$\|e_1\| = \|N(e_0)\| \leq k\|e_0\|. \tag{3.10}$$

For the second term,

$$\|e_2\| = \|N(e_0+e_1) - N(e_0)\| \leq k\|e_1\| \leq k^2\|e_0\|. \tag{3.11}$$

Similarly, for the third term,

$$\|e_3\| = \|N(e_0+e_1+e_2) - N(e_0+e_1)\| \leq k\|e_2\| \leq k^3\|e_0\|. \tag{3.12}$$

Proceeding in this manner,

$$\vdots$$

we find that

$$\|e_{n+1}\| = \left\| N\left(\sum_{i=0}^{n} e_i\right) - N\left(\sum_{i=0}^{n-1} e_i\right) \right\| \leq k\|e_n\| \leq k^{n+1}\|e_0\|, \tag{3.13}$$

for all $n = 0,1,2,\ldots$

This recursive inequality implies that $\|e_{n+1}\|$ approaches zero as n tends to infinity. Consequently, the iterative process converges, providing a solution to the functional equation (3.1). For a detailed analysis of this method and its convergence, refer to [7].

4. A New Iterative Approach for Solving Systems of Nonlinear Functional Equations

Consider the following system of nonlinear functional equations [6]:

$$\phi_i = f_i + N_i(\phi_1, \phi_2, \dots, \phi_n), \quad i = 1, 2, \dots, n. \tag{4.1}$$

Here, each f_i is a known function, and N_i represents a nonlinear operator.

Let us denote by $\bar{\phi} = (\phi_1, \dots, \phi_n)$ the solution of the system (4.1). Assume that each component ϕ_i can be expressed as an infinite series of the form

$$\phi_i = \sum_{j=0}^{\infty} \phi_{i,j}, \quad i = 1, 2, \dots, n. \tag{4.2}$$

The nonlinear operator N_i can be decomposed as follows:

$$\begin{aligned} N_i(\bar{\phi}) &= N_i\left(\sum_{j=0}^{\infty} \phi_{1,j}, \dots, \sum_{j=0}^{\infty} \phi_{n,j}\right) \\ &= N_i(\phi_{1,0}, \dots, \phi_{n,0}) \\ &\quad + \sum_{k=1}^{\infty}\left[N_i\left(\sum_{j=0}^{k} \phi_{1,j}, \dots, \sum_{j=0}^{k} \phi_{n,j}\right) - N_i\left(\sum_{j=0}^{k-1} \phi_{1,j}, \dots, \sum_{j=0}^{k-1} \phi_{n,j}\right)\right]. \end{aligned} \tag{4.3}$$

Making use of (4.2) and (4.3), the system (4.1) becomes equivalent to

$$\begin{aligned} \sum_{j=0}^{\infty} \phi_{i,j} &= f_i + N_i(\phi_{1,0}, \dots, \phi_{n,0}) \\ &\quad + \sum_{k=1}^{\infty}\left[N_i\left(\sum_{j=0}^{k} \phi_{1,j}, \dots, \sum_{j=0}^{k} \phi_{n,j}\right) - N_i\left(\sum_{j=0}^{k-1} \phi_{1,j}, \dots, \sum_{j=0}^{k-1} \phi_{n,j}\right)\right], \end{aligned} \tag{4.4}$$

for $i = 1, 2, \dots, n$. We now introduce a recurrence relation for the components $\phi_{i,j}$, given by

$$\begin{aligned} \phi_{i,0} &= f_i, \\ \phi_{i,1} &= N_i(\phi_{1,0}, \dots, \phi_{n,0}), \\ \phi_{i,m+1} &= N_i\left(\sum_{j=0}^{m} \phi_{1,j}, \dots, \sum_{j=0}^{m} \phi_{n,j}\right) - N_i\left(\sum_{j=0}^{m-1} \phi_{1,j}, \dots, \sum_{j=0}^{m-1} \phi_{n,j}\right), \quad m = 1, 2, \dots \end{aligned} \tag{4.5}$$

Thus, the solution ϕ_i can be represented as the infinite series

$$\phi_i = \sum_{j=0}^{\infty} \phi_{i,j}, \quad i = 1, 2, \dots, n. \tag{4.6}$$

For practical computations, the k-th order approximation of ϕ_i is given by

$$\phi_i^{(k)} = \sum_{j=0}^{k-1} \phi_{i,j}. \tag{4.7}$$

This framework provides an iterative method for solving the system of nonlinear functional equations. To illustrate the procedure, we consider the application of this method to the system defined in (4.1).
Initial Step:

$$\phi_{i,0} = f_i, \quad i = 1, 2, \dots, n.$$

First Iteration:

$$\begin{aligned}
\phi_{1,1} &= N_1(\phi_{1,0}, \phi_{2,0}, \ldots, \phi_{n,0}),\\
\phi_{2,1} &= N_2(\phi_{1,0} + \phi_{1,1}, \phi_{2,0}, \ldots, \phi_{n,0}),\\
\phi_{3,1} &= N_3(\phi_{1,0} + \phi_{1,1}, \phi_{2,0} + \phi_{2,1}, \phi_{3,0}, \ldots, \phi_{n,0}),\\
&\vdots\\
\phi_{n,1} &= N_n(\phi_{1,0} + \phi_{1,1}, \phi_{2,0} + \phi_{2,1}, \ldots, \phi_{n-1,0} + \phi_{n-1,1}, \phi_{n,0}).
\end{aligned}$$

k-th Iteration ($k = 2, 3, \ldots$):

$$\begin{aligned}
\phi_{1,k} &= N_1\left(\sum_{i=0}^{k-1}\phi_{1,i}, \ldots, \sum_{i=0}^{k-1}\phi_{n,i}\right) - N_1\left(\sum_{i=0}^{k-2}\phi_{1,i}, \ldots, \sum_{i=0}^{k-2}\phi_{n,i}\right),\\
\phi_{2,k} &= N_2\left(\sum_{i=0}^{k}\phi_{1,i}, \sum_{i=0}^{k-1}\phi_{2,i}, \ldots, \sum_{i=0}^{k-1}\phi_{n,i}\right) - N_2\left(\sum_{i=0}^{k-1}\phi_{1,i}, \sum_{i=0}^{k-2}\phi_{2,i}, \ldots, \sum_{i=0}^{k-2}\phi_{n,i}\right),\\
&\vdots\\
\phi_{j,k} &= N_j\left(\sum_{i=0}^{k}\phi_{1,i}, \ldots, \sum_{i=0}^{k}\phi_{j-1,i}, \sum_{i=0}^{k-1}\phi_{j,i}, \ldots, \sum_{i=0}^{k-1}\phi_{n,i}\right)\\
&\quad - N_j\left(\sum_{i=0}^{k-1}\phi_{1,i}, \ldots, \sum_{i=0}^{k-1}\phi_{j-1,i}, \sum_{i=0}^{k-2}\phi_{j,i}, \ldots, \sum_{i=0}^{k-2}\phi_{n,i}\right),\\
&\vdots\\
\phi_{n,k} &= N_n\left(\sum_{i=0}^{k}\phi_{1,i}, \ldots, \sum_{i=0}^{k}\phi_{n-1,i}, \sum_{i=0}^{k-1}\phi_{n,i}\right) - N_n\left(\sum_{i=0}^{k-1}\phi_{1,i}, \ldots, \sum_{i=0}^{k-1}\phi_{n-1,i}, \sum_{i=0}^{k-2}\phi_{n,i}\right).
\end{aligned}$$

As a result, the operator $N_i(\bar{\phi})$ can be expressed as

$$N_i(\bar{\phi}) = N_i\left(\sum_{j=0}^{\infty}\phi_{1,j}, \ldots, \sum_{j=0}^{\infty}\phi_{n,j}\right) = \sum_{j=1}^{\infty}\phi_{i,j}. \tag{4.8}$$

Therefore, the complete solution ϕ_i is given by

$$\phi_i = \sum_{j=0}^{\infty}\phi_{i,j}. \tag{4.9}$$

5. Implementation of New Iterative Method

Example 5.1 Consider the system of fractal-fractional differential equations with a power-law kernel:

$$\begin{aligned}
{}^{FFP}D_t^{\alpha,\beta}x(t) &= y(t) + t,\\
{}^{FFP}D_t^{\alpha,\beta}y(t) &= -x(t) + t^2
\end{aligned} \tag{5.1}$$

subject to the initial conditions $x(0) = 1$ and $y(0) = 2$.
Solution: Applying the fractal-fractional integral with power-law kernel (2.3) to system (5.1), and using the fundamental theorem of fractal-fractional calculus, we obtain

$$\begin{aligned}
x(t) &= x(0) + {}^{FFP}I_t^{\alpha,\beta}\left(y(t) + t\right),\\
y(t) &= y(0) + {}^{FFP}I_t^{\alpha,\beta}\left(-x(t) + t^2\right).
\end{aligned} \tag{5.2}$$

Using the proposed algorithm, the initial approximations are:

$$x_{1,0} = 1, \quad y_{1,0} = 2.$$

By applying the new iterative method, we construct the following iterative scheme:

$$\begin{aligned}
x_{1,0} &= 1, \quad y_{1,0} = 2, \\
x_{1,1} &= 2 \cdot \frac{\beta\Gamma(\beta)}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1} + \frac{\beta\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}, \\
y_{1,1} &= -\frac{\beta\Gamma(\beta)}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1} + \frac{\beta\Gamma(\beta+2)}{\Gamma(\alpha+\beta+2)} t^{\alpha+\beta+1}, \\
x_{1,2} &= -\frac{\beta^2\Gamma(\beta)}{\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(2\beta+\alpha-1)}{\Gamma(2\alpha+2\beta-1)} t^{2\alpha+2\beta-2} + \frac{\beta^2\Gamma(\beta+2)}{\Gamma(\alpha+\beta+2)} \cdot \frac{\Gamma(\alpha+2\beta+1)}{\Gamma(2\alpha+2\beta+1)} t^{2\alpha+2\beta}, \\
y_{1,2} &= \frac{\beta^3\Gamma(\beta)\Gamma(2\beta+\alpha-1)}{\Gamma(\alpha+\beta)\Gamma(2\alpha+2\beta-1)} \cdot \frac{\Gamma(2\alpha+3\beta-2)}{\Gamma(3\alpha+3\beta-2)} t^{3\alpha+3\beta-3} \\
&\quad - \frac{\beta^3\Gamma(\beta+2)\Gamma(\alpha+2\beta+1)}{\Gamma(\alpha+\beta+2)\Gamma(2\alpha+2\beta+1)} \cdot \frac{\Gamma(2\alpha+3\beta)}{\Gamma(3\alpha+3\beta)} t^{3\alpha+3\beta-1}.
\end{aligned}$$

Therefore, the three-term approximate solutions for $x(t)$ and $y(t)$ are given by:

$$\begin{aligned}
x(t) = 1 &+ 2 \cdot \frac{\beta\Gamma(\beta)}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1} + \frac{\beta\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta} \\
&- \frac{\beta^2\Gamma(\beta)}{\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(2\beta+\alpha-1)}{\Gamma(2\alpha+2\beta-1)} t^{2\alpha+2\beta-2} \\
&+ \frac{\beta^2\Gamma(\beta+2)}{\Gamma(\alpha+\beta+2)} \cdot \frac{\Gamma(\alpha+2\beta+1)}{\Gamma(2\alpha+2\beta+1)} t^{2\alpha+2\beta},
\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}
y(t) = 2 &- \frac{\beta\Gamma(\beta)}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1} + \frac{\beta\Gamma(\beta+2)}{\Gamma(\alpha+\beta+2)} t^{\alpha+\beta+1} \\
&+ \frac{\beta^3\Gamma(\beta)\Gamma(2\beta+\alpha-1)}{\Gamma(\alpha+\beta)\Gamma(2\alpha+2\beta-1)} \cdot \frac{\Gamma(2\alpha+3\beta-2)}{\Gamma(3\alpha+3\beta-2)} t^{3\alpha+3\beta-3} \\
&- \frac{\beta^3\Gamma(\beta+2)\Gamma(\alpha+2\beta+1)}{\Gamma(\alpha+\beta+2)\Gamma(2\alpha+2\beta+1)} \cdot \frac{\Gamma(2\alpha+3\beta)}{\Gamma(3\alpha+3\beta)} t^{3\alpha+3\beta-1}.
\end{aligned} \tag{5.4}$$

Example 5.2 Consider the following system of two linear fractal-fractional differential equations with an exponential kernel:

$$\begin{aligned}
{}^{FFE}D_t^{\alpha,\beta} x(t) &= x(t) + y(t), \\
{}^{FFE}D_t^{\alpha,\beta} y(t) &= -x(t) + y(t).
\end{aligned} \tag{5.5}$$

These equations are subject to the initial conditions:

$$x(0) = 0, \quad y(0) = 1.$$

Solution:

By applying the fundamental theorem of fractal-fractional calculus and using the proposed iterative algorithm, we begin with the initial approximations:

$$x_{1,0} = 0, \quad y_{1,0} = 1.$$

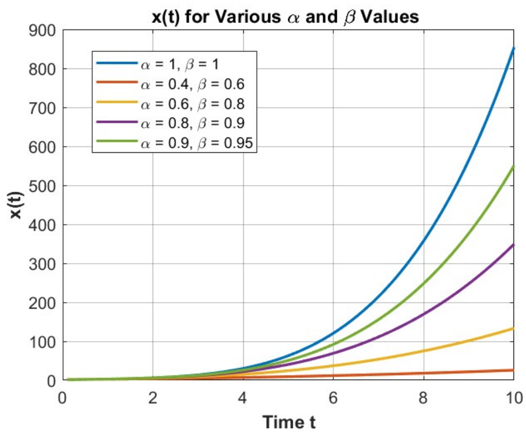

Figure 1: Graphical representation of the three-term iterative solutions $x(t)$ for various values of α and β.

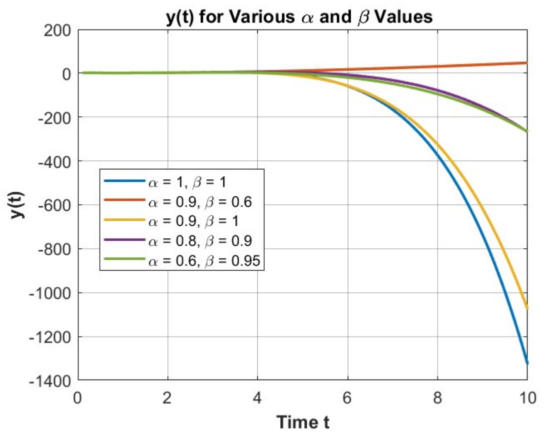

Figure 2: Graphical representation of the three-term iterative solutions $y(t)$ for various values of α and β.

Now, applying the fractal-fractional integral operator with an exponential kernel to the system (5.5), we get:

$$\begin{aligned} x(t) &= x(0) + {}^{FFE}I_t^{\alpha,\beta}\left(x(t)+y(t)\right), \\ y(t) &= y(0) + {}^{FFE}I_t^{\alpha,\beta}\left(-x(t)+y(t)\right). \end{aligned} \tag{5.6}$$

Using the iterative process, we obtain the two-term approximate solutions for $x(t)$ and $y(t)$ as follows:

$$x(t) = \frac{\beta\alpha}{M(\alpha)\beta}t^{\beta} + \frac{\beta(1-\alpha)}{M(\alpha)}t^{\beta-1}. \tag{5.7}$$

$$\begin{aligned} y(t) = 1 &- \frac{\beta\alpha}{M(\alpha)\beta}\left\{\frac{\beta\alpha}{M(\alpha)}\cdot\frac{t^{2\beta}}{2\beta} + \frac{\beta(1-\alpha)}{M(\alpha)}t^{2\beta-1}\right\} \\ &+ \frac{\beta(1-\alpha)}{M(\alpha)}\left\{\frac{\beta\alpha}{M(\alpha)}\cdot\frac{t^{2\beta-1}}{2\beta-1} + \frac{\beta(1-\alpha)}{M(\alpha)}t^{2\beta-2}\right\} \\ &+ \frac{\beta\alpha}{M(\alpha)\beta}t^{\beta} + \frac{\beta(1-\alpha)}{M(\alpha)}t^{\beta-1}. \end{aligned} \tag{5.8}$$

Example 5.3 consider the following system of two linear fractal - fractional differential equations with Mittag -Leffler kernel

$$\begin{aligned} {}^{FFM}D_t^{\alpha,\beta}x(t) &= x(t), \\ {}^{FFM}D_t^{\alpha,\beta}y(t) &= x(t) - y(t) \end{aligned} \tag{5.9}$$

Subject to initial condition $x(0) = 1$ and $y(0) = 2$

Solution: By applying the fundamental theorem of fractal-fractional calculus and utilizing the proposed algorithm, we obtain the results.

$$\begin{aligned} x(t) &= x(0) + {}^{FFM}I_t^{\alpha,\beta}\left(x(t)\right) \\ y(t) &= y(0) + {}^{FFM}I_t^{\alpha,\beta}\left(x(t)-y(t)\right) \end{aligned} \tag{5.10}$$

we get the two term solution x(t) and y(t) respectively is,

$$x(t) = 1 + \frac{\alpha\beta\Gamma(\beta)}{AB(\alpha)\Gamma(\alpha+\beta)}t^{\alpha+\beta-1} + \frac{\beta(1-\alpha)}{AB(\alpha)}t^{\beta-1}. \tag{5.11}$$

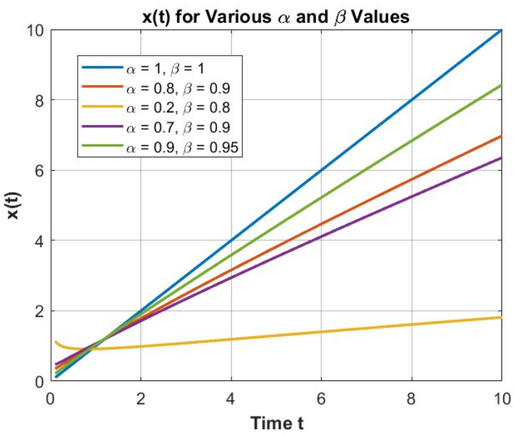

Figure 3: Graphical representation of the three-term iterative solutions $x(t)$ for various values of α and β.

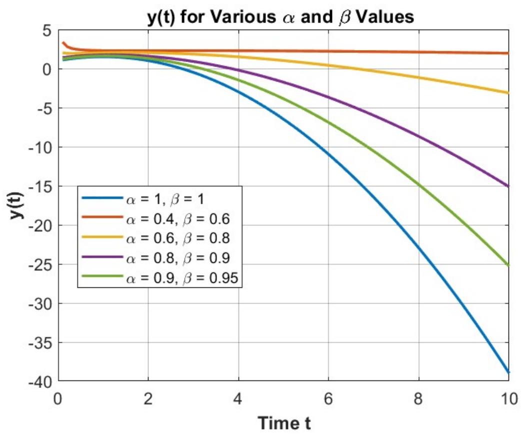

Figure 4: Graphical representation of the three-term iterative solutions $y(t)$ for various values of α and β.

and

$$y(t) = 2 + \frac{\alpha\beta\Gamma(\beta)}{AB(\alpha)\Gamma(\alpha+\beta)}\left[\frac{\alpha\beta\Gamma(\alpha+2\beta-1)}{AB(\alpha)\Gamma(2\alpha+2\beta-1)}t^{2\alpha+2\beta-2} + \frac{\beta(1-\alpha)}{AB(\alpha)}t^{\alpha+2\beta-2}\right] + \frac{\beta(1-\alpha)}{AB(\alpha)}\left[\frac{\alpha\beta\Gamma(2\beta-1)}{AB(\alpha)\Gamma(\alpha+2\beta-1)}t^{\alpha+2\beta-2} + \frac{\beta(1-\alpha)}{AB(\alpha)}t^{2\beta-2}\right] - \frac{\alpha\beta\Gamma(\beta)}{AB(\alpha)\Gamma(\alpha+\beta)}t^{\alpha+\beta-1} - \frac{\beta(1-\alpha)}{AB(\alpha)}t^{\beta-1} \tag{5.12}$$

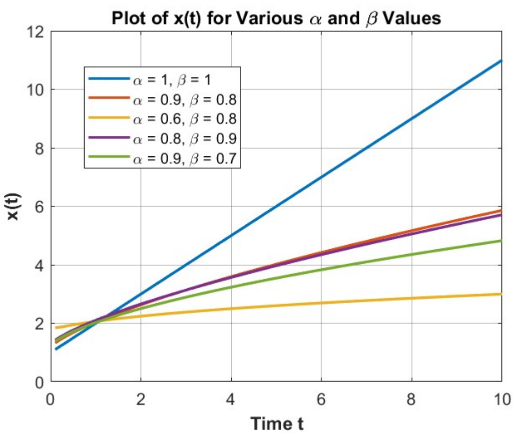

Figure 5: Graphical representation of the two-term iterative solutions $x(t)$ for various values of α and β.

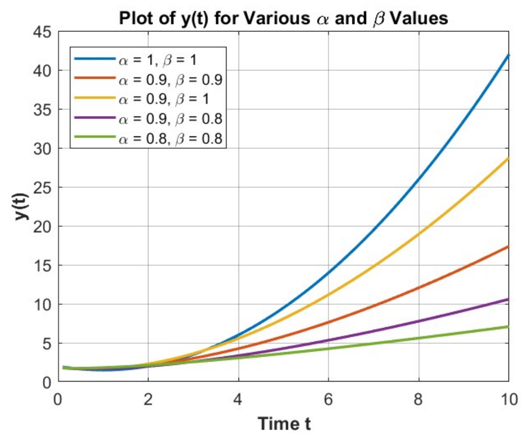

Figure 6: Graphical representation of the two-term iterative solutions $y(t)$ for various values of α and β.

Example 5.4 Consider the nonlinear fractional shallow water (FSW) coupled system with power law kernel

$$\begin{aligned} {}^{FFP}D_t^{\alpha,\beta}u(x,t) &= -vu_x - uv_x, \\ {}^{FFP}D_t^{\alpha,\beta}v(x,t) &= -vv_x - 2u_x \end{aligned} \tag{5.13}$$

Subject to initial condition $u(x,0) = \frac{1}{9}(x^2 - 2x + 1)$ and $v(x,0) = \frac{2}{3}(1-x)$.
Solution: Applying fractal - fractional power law integral (2.3) and using proposed algorithm we get the value of $u_{1,0} = \frac{1}{9}(x^2 - 2x + 1)$, $v_{1,0} = \frac{2}{3}(1-x)$. We get after applying fundamental theory of fractal - fractional calculus,

$$u(x,t) = u(x,0) + {}^{FFP}I_t^{\alpha,\beta}\left(-vu_x - uv_x\right)$$

and

$$v(x,t) = v(x,0) + {}^{FFP}I_t^{\alpha,\beta}\left(-vv_x - 2u_x\right).$$

Applying proposed algorithm, We get the two term solution u(x,t) and v (x,t) respectively

$$u(x,t) = \frac{1}{9}(x^2 - 2x + 1) + \frac{3}{27}(x-1)^2 \frac{\beta\Gamma(\beta)}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1}. \tag{5.14}$$

and

$$\begin{aligned} v(x,t) = {} & \frac{2}{3}(1-x) + \frac{4}{9}(1-x)\frac{\beta\Gamma(\beta)}{\Gamma(\alpha+\beta)}t^{\alpha+\beta-1} - \frac{2}{9}(x-1)^2\frac{\beta\Gamma(\beta)}{\Gamma(\alpha+\beta)}t^{\alpha+\beta-1} \\ & - \frac{2}{9}(x-1)^2\frac{\beta^2\Gamma(\beta)\Gamma(\alpha+2\beta-1)}{\Gamma(\alpha+\beta)\Gamma(2\alpha+2\beta-1)}t^{2\alpha+2\beta-2}. \end{aligned} \tag{5.15}$$

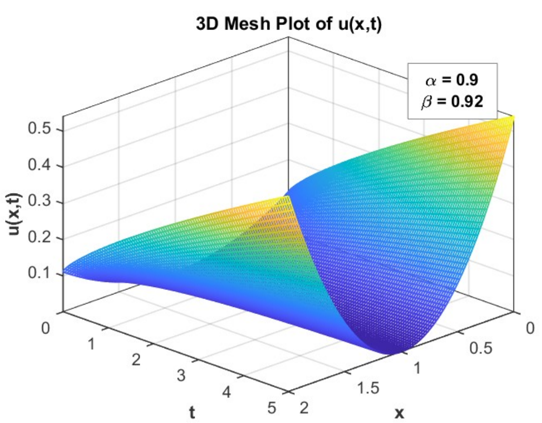

Figure 7: Graphical representation of the two-term iterative solutions $u(x,t)$ for different values of $\alpha = 0.9$ and $\beta = 0.92$.

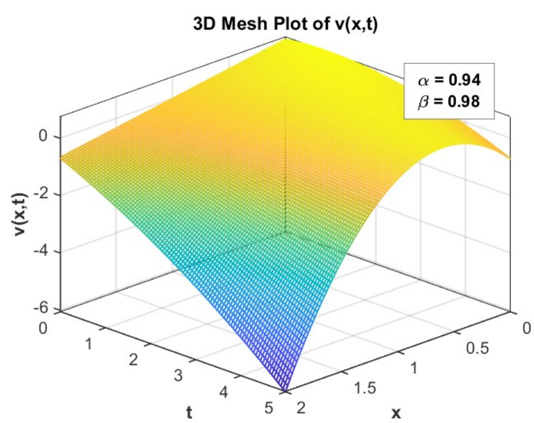

Figure 8: Graphical representation of the two-term iterative solutions $v(x,t)$ for different values of $\alpha = 0.94$ and $\beta = 0.98$.

Example 5.5 Consider the system of nonlinear partial differential equations with exponential kernel

$$\begin{aligned} {}^{FFE}D_t^{\alpha,\beta}u(x,t) + vu_x + u &= 1, \\ {}^{FFE}D_t^{\alpha,\beta}v(x,t) - uv_x - v &= 1 \end{aligned} \tag{5.16}$$

Subject to initial condition $u(x,0) = e^x$ and $v(x,0) = e^{-x}$.
Solution: Using new iterative algorithm we get two term solution $u(x,t)$ and $v(x,t)$ respectively is,

$$u(x,t) = e^x - e^x\left[\frac{\alpha}{M(\alpha)}t^{\beta} + \frac{\beta(1-\alpha)}{M(\alpha)}t^{\beta-1}\right]. \tag{5.17}$$

and

$$v(x,t) = e^{-x} + (2+e^{-x})\left[\frac{\alpha}{M(\alpha)}t^{\beta} + \frac{\beta(1-\alpha)}{m(\alpha)}t^{\beta-1}\right]$$
$$-\frac{\alpha}{M(\alpha)}\left[\frac{\beta\alpha}{M(\alpha)2\beta}t^{2\beta} + \frac{\beta(1-\alpha)}{M(\alpha)}t^{2\beta-1}\right] + \frac{\beta(1-\alpha)}{M(\alpha)}\left[\frac{\beta\alpha}{M(\alpha)(2\beta-1)}t^{2\beta-1} + \frac{\beta(1-\alpha)}{M(\alpha)}t^{2\beta-2}\right]. \tag{5.18}$$

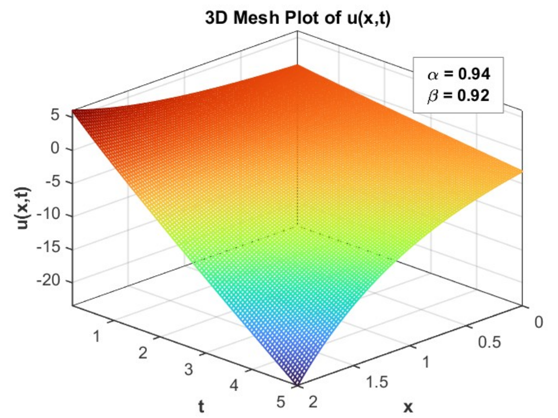

Figure 9: Graphical representation of the two-term iterative solutions $u(x,t)$ for different values of $\alpha = 0.94$ and $\beta = 0.92$.

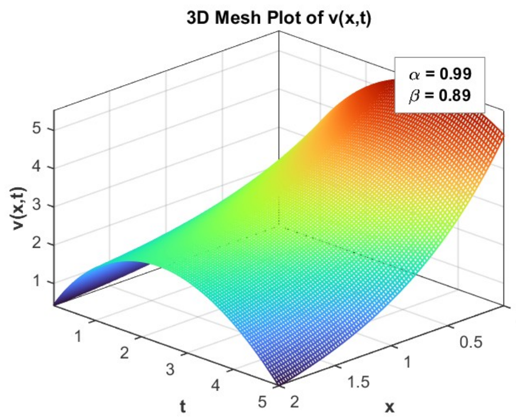

Figure 10: Graphical representation of the two-term iterative solutions $v(x,t)$ for different values of $\alpha = 0.99$ and $\beta = 0.89$.

Example 5.6 Consider the system of linear first-order hyperbolic partial differential equations with Mittag - Leffler kernel

$$\begin{aligned} {}^{FFM}D_t^{\alpha,\beta}u(x,t) &= u_x, \\ {}^{FFM}D_t^{\alpha,\beta}v(x,t) &= v_x \end{aligned} \tag{5.19}$$

Subject to initial condition $u(x,0) = sin(x)$ and $v(x,0) = cos(x)$
Solution: Using a new iterative algorithm, a three-term approximate solution for $u(x,t)$ and $v(x,t)$ is obtained, respectively.

$$\begin{aligned} u(x,t) &= sin(x) + cos(x)\left[\frac{\alpha\beta\Gamma(\beta)}{AB(\alpha)\Gamma(\alpha+\beta)}t^{\alpha+\beta-1} + \frac{\beta(1-\alpha)}{AB(\alpha)}t^{\beta-1}\right] \\ &+ cos(x)\left[\frac{\alpha\beta\Gamma(\beta)}{AB(\alpha)\Gamma(\alpha+\beta)}t^{\alpha+\beta-1} + \frac{\beta(1-\alpha)}{AB(\alpha)}t^{\beta-1}\right] \\ &- sin(x)\frac{\alpha\beta\Gamma(\beta)}{AB(\alpha)\Gamma(\alpha+\beta)}\left[\frac{\alpha\beta\Gamma(\alpha+2\beta-1)}{AB(\alpha)\Gamma(2\alpha+2\beta-1)}t^{2\alpha+2\beta-2} + \frac{\beta(1-\alpha)}{AB(\alpha)}t^{\alpha+2\beta-2}\right] \\ &- sin(x)\frac{\beta(1-\alpha)}{AB(\alpha)}\left[\frac{\alpha\beta\Gamma(2\beta-1)}{AB(\alpha)\Gamma(\alpha+2\beta-1)}t^{\alpha+2\beta-2} + \frac{\beta(1-\alpha)}{AB(\alpha)}t^{2\beta-2}\right] \\ &- cos(x)\left[\frac{\alpha\beta\Gamma(\beta)}{AB(\alpha)\Gamma(\alpha+\beta)}t^{\alpha+\beta-1} + \frac{\beta(1-\alpha)}{AB(\alpha)}t^{\beta-1}\right]. \end{aligned} \tag{5.20}$$

and

$$v(x,t) = cos(x) - sin(x)\left[\frac{\alpha\beta\Gamma(\beta)}{AB(\alpha)\Gamma(\alpha+\beta)}t^{\alpha+\beta-1} + \frac{\beta(1-\alpha)}{AB(\alpha)}t^{\beta-1}\right]$$
$$- cos(x)\frac{\alpha\beta\Gamma(\beta)}{AB(\alpha)\Gamma(\alpha+\beta)}\left[\frac{\alpha\beta\Gamma(\alpha+2\beta-1)}{AB(\alpha)\Gamma(2\alpha+2\beta-1)}t^{2\alpha+2\beta-2} + \frac{\beta(1-\alpha)}{AB(\alpha)}t^{\alpha+2\beta-2}\right]$$
$$- cos(x)\frac{\beta(1-\alpha)}{AB(\alpha)}\left[\frac{\alpha\beta\Gamma(2\beta-1)}{AB(\alpha)\Gamma(\alpha+2\beta-1)}t^{\alpha+2\beta-2} + \frac{\beta(1-\alpha)}{AB(\alpha)}t^{2\beta-2}\right]. \tag{5.21}$$

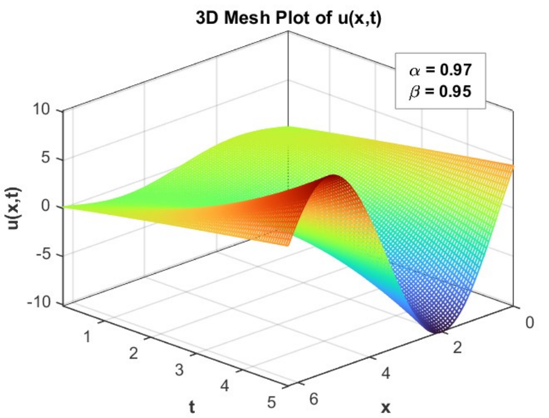

Figure 11: Graphical representation of the three-term iterative solutions $u(x,t)$ for different values of $\alpha = 0.97$ and $\beta = 0.95$.

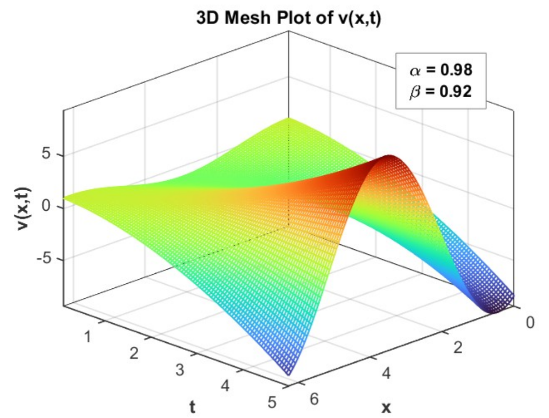

Figure 12: Graphical representation of the three-term iterative solutions $v(x,t)$ for different values of $\alpha = 0.98$ and $\beta = 0.92$.

6. Error Analysis

The error analysis shows that the proposed iterative method achieves high accuracy, with absolute errors between successive approximations decreasing steadily. Here, we provide two numerical examples to demonstrate the error behavior and convergence of the method. The results confirm its reliability and effectiveness for solving fractal-fractional differential equations.

Table 1: Absolute error for solution $x(t)$ of (5.1) between consecutive approximations for $\alpha = 0.9$ and $\beta = 0.85$

Value of t	$x_{1,0}$	$x_{1,1}$	$x_{1,2}$	$\lvert x_{1,1} - x_{1,0}\rvert$	$\lvert x_{1,2} - x_{1,1}\rvert$
0.02	1.0000	0.1100	-0.0002	0.8900	0.1102
0.04	1.0000	0.1858	-0.0005	0.8142	0.1864
0.06	1.0000	0.2531	-0.0010	0.7469	0.2541
0.08	1.0000	0.3156	-0.0015	0.6844	0.3171
0.10	1.0000	0.3748	-0.0021	0.6252	0.3769

Table 2: Absolute error for solution $y(t)$ of (5.1) between consecutive approximations for $\alpha = 0.9$ and $\beta = 0.7$

Value of t	$y_{1,0}$	$y_{1,1}$	$y_{1,2}$	$\lvert y_{1,1} - y_{1,0}\rvert$	$\lvert y_{1,2} - y_{1,1}\rvert$
0.02	2.0000	-0.0972	0.0002	2.0972	0.0974
0.04	2.0000	-0.1473	0.0007	2.1473	0.1481
0.06	2.0000	-0.1878	0.0015	2.1878	0.1893
0.08	2.0000	-0.2230	0.0025	2.2230	0.2255
0.10	2.0000	-0.2547	0.0037	2.2547	0.2584

Table 3: Absolute error for solution $u(x,t)$ of (5.19) between consecutive approximations for $\alpha = 0.8$ and $\beta = 0.9$

Value of x	Value of t	$u_{1,0}$	$u_{1,1}$	$u_{1,2}$	$\lvert u_{1,1} - u_{1,0}\rvert$	$\lvert u_{1,2} - u_{1,1}\rvert$
0.02	0.02	0.0200	0.3617	0.3504	0.3417	0.0113
0.04	0.04	0.0400	0.3799	0.3569	0.3399	0.0230
0.06	0.06	0.0600	0.4013	0.3653	0.3413	0.0360
0.08	0.08	0.0799	0.4227	0.3724	0.3428	0.0504
0.10	0.10	0.0998	0.4437	0.3776	0.3438	0.0660

Table 4: Absolute error for solution $v(x,t)$ of (5.19) between consecutive approximations for $\alpha = 0.9$ and $\beta = 0.8$

Value of x	Value of t	$v_{1,0}$	$v_{1,1}$	$v_{1,2}$	$\lvert v_{1,1} - v_{1,0}\rvert$	$\lvert v_{1,2} - v_{1,1}\rvert$
0.02	0.02	0.9998	-0.0050	-0.8507	1.0048	0.8457
0.04	0.04	0.9992	-0.0106	-0.8513	1.0098	0.8408
0.06	0.06	0.9982	-0.0171	-0.8916	1.0153	0.8745
0.08	0.08	0.9968	-0.0246	-0.9411	1.0214	0.9165
0.10	0.10	0.9950	-0.0329	-0.9928	1.0279	0.9599

The error analysis demonstrates the effectiveness of the proposed method through the numerical examples provided. As observed in the tables, the absolute errors between consecutive terms decrease progressively, indicating that the upcoming terms contribute diminishingly to the solution. This behavior confirms that the method is convergent and reliable for solving fractal-fractional differential equations.

7. Advantages of the Proposed Method

The research on fractal-fractional differential equations (FFDEs) has attracted considerable attention because FFDEs can represent complicated systems that have memory and anomalous diffusion. No classical analytical method has been established for solving general FFDEs, so numerical and iterative methods become inevitable. The Extended New Iterative Method (ENIM) introduced herein provides a direct computational method of solving FFDEs without specialized software. Differing from conventional numerical solutions that involve exhaustive programming and hardware, the current method can easily be applied by hand, as it is easily accessible and highly efficient. Also, the method convergence is illustrated in a straightforward manner using tables of error analysis, where successive approximations have diminishing absolute errors. This proves the accuracy and reliability of the method for both ordinary and partial FFDEs. The suggested method is of significant benefit to researchers and practitioners looking for a simple yet effective way of solving complicated differential systems.

8. Concluding Remark

The suggested iterative schemes successfully solve fractal-fractional differential equations (FFDEs) with various kernel functions, including power-law, exponential, and Mittag-Leffler. The schemes are applicable to both ordinary and partial differential equations, demonstrating their versatility. The schemes correctly capture the fractal nature of the solutions, which is critical for simulating systems with memory effects or anomalous diffusion. Error analysis validates their high accuracy, and the graphs of the numerical and exact solutions give consistent and reliable results. The computational speed of these schemes, enhanced employing the Bhalekar and Gejji methodology, renders them feasible for computing complex and big FFDE systems. This predictability renders them applicable to most scientific and engineering applications, such as materials science, biology, and fluid mechanics. There are still some challenges, such as the influence of kernel selection on convergence and computation. Adaptative or hybrid algorithms can be developed in the future to optimize performance and extend these approaches to higher-dimensional and more complicated FFDE systems.

Finally, the suggested methods present a stable, precise, and efficient method of solving FFDEs, rendering them useful in mathematical modeling and practical applications.

9. Declarations

Funding: The authors received no specific funding for this work.

Conflicts of Interests/Competing Interests: The authors declare that they have no conflicts of interest or competing interests that could influence the work reported in this paper.

Data Availability Statement: No Data associated in the manuscript.

Computational Tools and Software:
All numerical simulations and graphical analyses were performed using **MATLAB** R2023a. Symbolic computations, including the determination of equilibrium points and eigenvalue analysis, were conducted using **Wolfram Mathematica** version 11.0.1.0.

Acknowledgment
The author(s) gratefully acknowledge the financial support received through the **SHODH – Scheme of Developing High Quality Research Fellowship provided by the Government of Gujarat**. This support has been instrumental in carrying out the present research work and in facilitating the resources required for its successful completion. The author(s) sincerely thank the authorities for this encouragement.

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