Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–17.
ISSN-0037-8712
doi:10.5269/bspm.77404

On the Stress-sum Index of the Middle and Central Graph of a Graph

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ABSTRACT: Ranking molecular chemical compounds or large networks is a complex task due to their inherent degeneracy. Centrality measures help to estimate the rank of each molecular chemical graph. One such centrality measure is the stress centrality, which quantifies the number of shortest paths passing through a given vertex. Recently, new topological indices based on stress have been introduced. In this study, we derive bounds and exact expressions for the stress of a vertex in the middle and central graphs derived from a given graph. In addition, we present expressions for the stress and the stress-sum index of both the middle and central graphs of a few standard graphs.

Key Words: Stress centrality, stress-sum index, middle graph, central graph, shortest path.

Contents

1. Introduction

Let $G = (V(G), E(G))$ be a finite, simple graph with the vertex set $V(G) = \{w_1, w_2, \dots, w_n\}$ and the edge set $E(G) = \{e_1, e_2, \dots, e_m\}$. If two vertices w_i and w_j are adjacent, we write $w_i \sim w_j$, and the edge between them is denoted by w_iw_j. Let $\mathcal{P} = (u = w_0w_1 \dots w_k = v)$ be a u-v path of length k in G with *origin* $u = w_0$ and *terminus* $v = w_k$. The vertices w_i, $1 \leq i \leq k-1$ are called the *internal* vertices of the path $\mathcal{P}$. The length of a shortest u-v path, denoted by $d(u, v)$ is called the *distance* between u and v. The *diameter* of G, denoted by $\text{diam}(G)$, is given by $\max\{d(u, v) : u, v \in V\}$. For the remaining graph-theoretic terminologies, we refer to Chartrand and Lesniak [3].
A vertex centrality measure assigns a real number to each vertex in a graph, quantifying the importance or criticality of that vertex from a particular perspective. Different centrality measures describe the importance of a vertex from various perspectives. Stress is one such vertex centrality measure studied in [19].

Definition 1.1 [19] Let G be a graph with $V(G) = \{w_1, w_2, \dots, w_n\}$. The *stress* of a vertex w_i is the number of shortest paths in G having w_i as an internal vertex and is denoted by $st_G(w_i)$. The stress of a graph G is defined by

$$st(G) = \sum_{i=1}^{n} st_G(w_i).$$

We write $\text{st}(v)$ instead of $\text{st}_G(v)$ whenever the graph under discussion is clear from the context. Stress centrality has numerous applications in the study of biological networks, social networks, and other related fields. Some related work can be found in [1,2,10,11,12]. Based on the stress of the vertices, several new topological indices are introduced and studies were conducted in [4,5,6,8,13,14,15,16,17,18,20]. The stress-sum index is a topological index based on the stress of vertices in a graph and is defined below.

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Submitted June 20, 2025. Published August 24, 2025
2010 *Mathematics Subject Classification:* 05C09, 05C12.

Definition 1.2 [14] The stress-sum index $SS(G)$ of a simple graph G is defined as

$$SS(G) = \sum_{uv \in E(G)} \big(st(u) + st(v)\big).$$

Observe that $\sum_{uv \in E(G)} \big(st(u) + st(v)\big) \neq \sum_{u \in V(G)} \big(st(u)\big)^2$.
The stress-sum index of some standard class of graphs with diameter two is discussed in [9].
The concept of a middle graph $M(G)$ of a graph G was introduced by Hamada and Yoshimura in [7] as an intersection graph on the vertex set of G as given below.

Definition 1.3 The middle graph of a connected graph G is denoted by $M(G)$ and is a graph whose vertex set is $V(G) \cup E(G)$ and if any two vertices in $M(G)$ are adjacent, if

- they are adjacent edges of G
- one is a vertex of G and the other is an edge incident on it.

Definition 1.4 The central graph of a graph G is denoted by $C(G)$ and is obtained by subdividing each edge of G exactly once and joining each pair of non-adjacent vertices by an edge.

In Section 2 (3), we obtain the bound for the stress of any vertex in the middle (central) graph of a graph G and also the expressions for the stress and the stress-sum index of the middle (central) graph of a few standard classes of graphs.

2. Middle Graph

In this section, we obtain the expression for stress and the stress-sum index of the middle graph of some standard families of graphs. First, we give a bound for the stress of any vertex in $M(G)$.

Theorem 2.1 *Let G be a graph with n vertices, m edges and diameter d. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$ and $V(M(G)) = \{v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_m\}$. If the vertex $e_a \in V(M(G)), 1 \leq a \leq m$ corresponds to the edge v_iv_j in G, then,*
$deg_G(v_i)deg_G(v_j) - |N_G(v_i) \cap N_G(v_j)| + \sum_{k=2}^{d} |P_k^G(v_r \ldots v_iv_j \ldots v_s)| \leq st_{M(G)}(e_a)$
$\leq deg_G(v_i)deg_G(v_j) + \sum_{k=2, P_k^G(v_r \ldots v_iv_j \ldots v_s)}^{d} (deg_G(v_s))(deg_G(v_r)),$
where $P_k^G(v_r \ldots v_iv_j \ldots v_s)$ represents the shortest path of length k in G and $st_{M(G)}(v_i) = 0$. The lower bound is achieved when $G = K_{p,q}$ and the upper bound is achieved when G is a tree.

Proof: Let $v_i, v_j, v_k, e_r, e_s \in V(M(G))$. Observe that in $M(G)$ there does not exist any shortest path of length two of the form $(v_iv_jv_k)$ or $(e_rv_jv_k)$ or $(e_rv_je_s)$ that contributes to $st_{M(G)}(v_j)$. Therefore, $st_{M(G)}(v_j) = 0, 1 \leq j \leq n$.
Suppose $N_G(v_i) \cap N_G(v_j) = \phi$. Let $u_t \in N_G(v_i)$, where $u_t \neq v_j$ and $w_z \in N_G(v_j)$ where $w_z \neq v_i$. Then, from the vertex v_i and the vertices of $M(G)$ that correspond to the edges u_tv_i of G, to the vertex v_j and the vertices that correspond to the edges v_jw_z of G, there exists the shortest path of length 2 that has vertex e_a as an internal vertex. If there is any vertex $v_s \in N(v_i) \cap N(v_j)$, then there is an adjacency between the vertex that corresponds to v_iv_s and the vertex v_jv_s. So, $st_{M(G)}(e_a) \geq deg_G(v_i)deg_G(v_j) - |N_G(v_i) \cap N_G(v_j)|$. For every $v_r - v_s$ shortest path of length k, $2 \leq k \leq d$ in G which involves v_iv_j edge, there exists the shortest path of length $k+1$ from v_r to v_s with e_a as an internal vertex in $M(G)$).
When G is $K_{p,q}$, then for every shortest path of length 2 in G, there exists exactly one shortest path in $M(G)$ which contains e_a as the internal vertex. Hence, equality in the lower bound is achieved when G is $K_{p,q}$.

Consider the shortest path $(v_rv_{r+1} \ldots v_iv_j \ldots v_{s-1}v_s)$ of length $k, 2 \leq k \leq d$, in G. Let $N_G(v_r) = \{v_1, v_2 \ldots, v_{r-1}, v_{r+1}\}$ and $N_G(v_s) = \{w_1, \ldots, w_{s-1}, v_{s-1}, v_{s+1}\}$. Now, every shortest path from v_r and the vertices that correspond to the edges $v_jv_r, 1 \leq j \leq r-1$ to the vertex v_s and the vertices that correspond to the edges $v_sw_l, 1 \leq l \leq s-1$ contains e_a as an internal vertex, if G is a tree. Hence, the result follows. □

Corollary 2.1 *Let $G = K_{p,q}$. Then,*

$$st(M(G)) = pq(pq + p + q - 2),$$
$$SS(M(G)) = pq(p + q)(pq + p + q - 2).$$

Proof: Let $V(G) = \{v_1, v_2, \ldots, v_p, u_1, u_2, \ldots, u_q\}$. Let $w_{ij} \in V(M(G))$ correspond to the edge $v_iu_j, 1 \leq i \leq p, 1 \leq j \leq q$ of G. From Theorem 2.1, we have $st_{M(G)}(w_{ij}) = pq + p + q - 2$.
Also, as $|E(M(G))| = 2pq + p\binom{q}{2} + q\binom{p}{2}$, we have,

$$SS(M(G) = 2pq(pq + p + q - 2) + 2\left\{p\binom{q}{2} + q\binom{p}{2}\right\}(pq + p + q - 2).$$

□

The next corollary is immediate from Corollary 2.1.

Corollary 2.2 *Let $G = K_{1,n-1}$. Then,*

$$st(M(G)) = 2n^2 - 5n + 3,$$
$$SS(M(G)) = (2n - 3)n(n - 1).$$

Theorem 2.2 *Let $G = K_n$. Then,*

$$st(M(G)) = \frac{n(n-1)(n^2 - 3n + 3)}{2},$$
$$SS(M(G)) = n^5 - 5n^4 + 10n^3 - 9n^2 + 3n.$$

Proof: If $v \in V(M(G))$ corresponds to an edge of G, then

$$st_{M(G)}(v) = (n - 1)^2 - (n - 2).$$

Hence,

$$st(M(G)) = \binom{n}{2}\left((n - 1)^2 - n + 2\right).$$

As $|E(M(G))| = \frac{n(n-1)(n-2)}{2} + n(n - 1)$, we get

$$SS(M(G)) = \{n(n - 1)(n - 2) + n(n - 1)\}\,(n^2 - 3n + 3).$$

□

Theorem 2.3 *Let $G = B_{r,s}$ be a bi-star graph on $r + s + 2$ vertices. Then,*

$$st(M(G)) = 8rs + 3(s + r) + 2(r^2 + s^2) + 1,$$
$$SS(M(G)) = 2r^2(r + 3s) + 2s^2(s + 3r) + (7r^2 + 7s^2 + 20rs) + 7(r + s) + 2.$$

Proof: Let $V(G) = \{v, v_1, v_2, \ldots, v_r, u, u_1, u_2, \ldots, u_s\}$. Let $w_i \in V(M(G))$ correspond to the edge $vv_i, 1 \leq i \leq r$, let $y_j \in V(M(G))$, correspond to the edge $uu_j, 1 \leq j \leq s$ and $x \in V(M(G))$ correspond to the edge uv of G. Then, from Theorem 2.1, we have

$$st_{M(G)}(w_i) = 2s + 2r + 1 = st_{M(G)}(y_j), st_{M(G)}(x) = 4rs + 2r + 2s + 1.$$

Hence, we get the expression for $st(M(G))$.
As $|E(M(G))| = \binom{r}{2} + \binom{s}{2} + r + s + 2(r + s + 1)$, we get,

$$SS(M(G)) = 2\left\{\binom{r}{2} + \binom{s}{2}\right\}(2s + 2r + 1) + 3(r + s)(2s + 2r + 1) + (r + s + 2)(4rs + 2s + 2r + 1).$$

Simplifying the above expression, we get the desired result. □

Theorem 2.4 *Let $G = W_{1,n-1}$ be a wheel graph on n vertices. Then,*

$$st(M(G)) = \begin{cases} 84 & ; n = 5 \\ 165 & ; n = 6 \\ 12n^2 - 49n + 37 & ; n \geq 7 \end{cases}$$

Proof: Let $V(M(G)) = \{u, u_1, \ldots, u_{n-1}, y_1, \ldots, y_{n-1}, v_1, \ldots, v_{n-1}\}$, where $y_i, 1 \leq i \leq n-1$ is formed by subdividing the edge $uu_i, 1 \leq i \leq n-1$ of G and $v_i, 1 \leq i \leq n-2$ is formed by subdividing the edge $u_iu_{i+1}, 1 \leq i \leq n-2$ of G and v_{n-1} is formed by subdividing the edge $u_{n-1}u_1$ of G.
When $n = 5$, we can note that $st_{M(G)}(y_i) = 11$ and $st_{M(G)}(v_i) = 10, 1 \leq i \leq 4$.
When $n = 6$, we can note that $st_{M(G)}(y_i) = 19$ and $st_{M(G)}(v_i) = 14, 1 \leq i \leq 5$.
Let $n \geq 7$:
To find $st_{M(G)}(y_1)$: All the shortest paths that contribute to $st_{M(G)}(y_1)$ are given in the table below.

Shortest paths $\mathcal{P}$ with y_1 as internal vertex	$\lvert\mathcal{P}\rvert$
$(v_1y_1y_3), \ldots, (v_1y_1y_{n-1})$	$n-3$
$(v_{n-1}y_1y_2), \ldots, (v_{n-1}y_1y_{n-2})$	$n-3$
$(u_1y_1y_2), \ldots, (u_1y_1y_{n-1})$	$n-2$
$(u_1y_1y_4u_4), (u_1y_1y_4v_3), (u_1y_1y_4v_4), \ldots, (u_1y_1y_{n-3}u_{n-3}), (u_1y_1y_{n-3}v_{n-4}), (u_1y_1y_{n-3}v_{n-3})$	$3(n-6)$
$(v_1y_1y_5u_5), (v_1y_1y_5v_4), (v_1y_1y_5v_5), \ldots, (v_1y_1y_{n-3}u_{n-3}), (v_1y_1y_{n-3}v_{n-4}), (v_1y_1y_{n-3}v_{n-3})$	$3(n-7)$
$(v_{n-1}y_1y_4u_4), (v_{n-1}y_1y_4v_4), (v_{n-1}y_1y_4v_3), \ldots,$ $(v_{n-1}y_1y_{n-4}u_{n-4}), (v_{n-1}y_1y_{n-4}v_{n-4}), (v_{n-1}y_1y_{n-4}v_{n-5})$	$3(n-7)$
$(u_1y_1y_3u_3), (u_1y_1y_3v_3), (u_1y_1y_{n-2}u_{n-2}), (u_1y_1y_{n-2}v_{n-3})$	4
$(v_1y_1y_4u_4), (v_1y_1y_4v_4), (v_1y_1y_{n-2}u_{n-2}), (v_1y_1y_{n-2}v_{n-3}),$ $(v_{n-1}y_1y_3v_3), (v_{n-1}y_1y_3u_3), (v_{n-1}y_1y_{n-3}u_{n-3}), (v_{n-1}y_1y_{n-3}v_{n-4})$	8
$(uy_1v_1), (uy_1u_1), (uy_1v_{n-1})$	3

Hence, due to symmetry,

$$st_{M(G)}(y_i) = 12n - 53, 1 \leq i \leq n-1.$$

To find $st_{M(G)}(v_1)$: The shortest paths $\{(u_1v_1u_2), (u_2v_1y_1), (u_2v_1v_{n-1}), (u_1v_1v_2), (u_1v_1y_2), (v_2v_1v_{n-1}),$ $(y_1v_1v_2), (y_2v_1v_{n-1}), (v_{n-1}v_1v_2u_3), (u_1v_1v_2u_3), (v_2v_1v_{n-1}u_{n-1}), (u_2v_1v_{n-1}u_{n-1}), (v_2v_1v_{n-1}v_{n-2}),$ $(u_2v_1v_{n-1}v_{n-2}), (u_1v_1v_2v_3), (v_{n-1}v_1v_2v_3)\}$ contribute 16 to $st_{M(G)}(v_1)$. Due to symmetry,

$$st_{M(G)}(v_i) = 16, 1 \leq i \leq n-1.$$

□

Corollary 2.3 *Let $G = W_{1,n-1}$. Then,*

$$SS(M(G)) = \begin{cases} 548 & ; n = 5 \\ 1180 & ; n = 6 \\ 12n^3 - 41n^2 + 19n + 10 & ; n \geq 7 \end{cases}$$

Proof: We note that $|E(M(G))| = \binom{n-1}{2} + 7(n-1)$. For $n \geq 7$, from Theorem 2.4,

$$\begin{aligned} SS(M(G)) &= 2\binom{n-1}{2}(12n-53) + 2(n-1)(12n-53) + 4(16)(n-1) + 2(12n-53+16)(n-1) \\ &= 12n^3 - 41n^2 + 19n + 10. \end{aligned}$$

From Theorem 2.4, for $n = 5$ and $n = 6$, the result follows easily. □

Theorem 2.5 *Let $G = F_{1,n-1}$ be a fan graph on n vertices. Then,*

$$st(M(G)) = \begin{cases} 31 & ; n = 4 \\ 74 & ; n = 5 \\ 143 & ; n = 6 \\ 12n^2 - 63n + 89 & ; n \geq 7 \end{cases}$$

Proof: Let $V(G) = \{u, u_1, \ldots, u_{n-1}\}$, $u \sim_G u_i, 1 \leq i \leq n-1$ and $u_i \sim_G u_{i+1}, 1 \leq i \leq n-2$. Let $v_j \in M(G), 1 \leq j \leq n-1$ correspond to the edge uu_i and $y_j \in M(G), 1 \leq j \leq n-2$ correspond to the edge $u_j u_{j+1}$ of G.
When $n = 4$, we can note that $st_{M(G)}(y_1) = 6 = st_{M(G)}(y_2)$, $st_{M(G)}(v_1) = 6 = st_{M(G)}(v_3)$ and $st_{M(G)}(v_2) = 7$.
When $n = 5$, we get $st_{M(G)}(v_i) = 12, 1 \leq i \leq 4, st_{M(G)}(y_1) = 7 = st_{M(G)}(y_3)$ and $st_{M(G)}(y_2) = 12$.
Also, when $n = 6$, we get $st_{M(G)}(v_1) = 20 = st_{M(G)}(v_5), st_{M(G)}(v_2) = 22 = st_{M(G)}(v_4)$ and $st_{M(G)}(v_3) = 17, st_{M(G)}(y_1) = 7 = st_{M(G)}(y_4)$ and $st_{M(G)}(y_2) = 14 = st_{M(G)}(y_3)$.
Let $n \geq 7$:
To find $st_{M(G)}(v_1)$: The shortest paths which contribute to $st_{M(G)}(v_1)$ are as shown in table below.

Shortest paths $\mathcal{P}$ with v_1 as internal vertex	$\lvert\mathcal{P}\rvert$
$(u_1v_1v_j), (u_1v_1v_ju_j), 3 \leq j \leq n-1$	$2(n-3)$
$(y_1v_1v_{j+1}u_{j+1}), 3 \leq j \leq n-1$	$n-4$
$(u_1v_1v_4y_4), (u_1v_1v_4y_3), \ldots, (u_1v_1v_{n-2}y_{n-2}), (u_1v_1v_{n-2}y_{n-3})$	$2(n-5)$
$(y_1v_1v_5y_4), (y_1v_1v_5y_5), \ldots, (y_1v_1v_{n-2}y_{n-3}), (y_1v_1v_{n-2}y_{n-2})$	$2(n-6)$
$(y_1v_1v_3), \ldots, (y_1v_1v_{n-1}), (u_1v_1v_2)(u_1v_1u), (y_1v_1u)$	n
$(u_1v_1v_4y_3), (u_1v_1v_{n-1}y_{n-2}), (y_1v_1v_4y_4), (y_1v_1v_{n-1}y_{n-2})$	4

Due to symmetry, $st_{M(G)}(v_1) = 8n - 28 = st_{M(G)}(v_{n-1})$.
To find $st_{M(G)}(v_2)$: The shortest paths that contribute to $st_{M(G)}(v_2)$ are as shown in the table below.

Shortest paths $\mathcal{P}$ with v_2 as internal vertex	$\lvert\mathcal{P}\rvert$
$(u_2v_2v_j), (u_2v_2v_ju_j), (y_1v_2v_ju_j), 4 \leq j \leq n-1$	$3(n-4)$
$(y_2v_2v_{j+1}u_{j+1}), 4 \leq j \leq n-1$	$n-5$
$(u_2v_2v_5y_5), (u_2v_2v_5y_4), (u_2v_2v_6y_6), (u_2v_2v_6y_5), \ldots, (u_2v_2v_{n-2}y_{n-2}), (u_2v_2v_{n-2}y_{n-3})$	$2(n-6)$
$(y_1v_2v_5y_5), (y_1v_2v_5y_4), (y_1v_2v_6y_6), (y_1v_2v_6y_5), \ldots, (y_1v_2v_{n-2}y_{n-2}), (y_1v_2v_{n-2}y_{n-3})$	$2(n-6)$
$(y_2v_2v_6y_5), (y_2v_2v_6y_6), \ldots, (y_2v_2v_{n-2}y_{n-3}), (y_2v_2v_{n-2}y_{n-2})$	$2(n-7)$
$(y_1v_2v_3), \ldots, (y_1v_2v_{n-1}), (y_2v_2v_1), (y_2v_2v_4), \ldots, (y_2v_2v_{n-1})$	$2(n-3)$
$(u_2v_2v_4y_4), (u_2v_2v_{n-1}y_{n-2}), (y_1v_2v_4y_4), (y_1v_2v_{n-1}y_{n-2}), (y_2v_2v_5y_5),$ $(y_2v_2v_{n-1}y_{n-2}), (u_2v_2v_1), (u_2v_2v_3), (u_2v_2u), (y_1v_2u), (y_2v_2u)$	11

It follows from symmetry, $st_{M(G)}(v_2) = 12n - 50 = st_{M(G)}(v_{n-2})$.
To find $st_{M(G)}(v_3)$: The shortest paths that contribute to $st_{M(G)}(v_3)$ are as shown in the table below.

Shortest paths $\mathcal{P}$ with v_3 as internal vertex	$\lvert\mathcal{P}\rvert$
$(u_3v_3v_j), (u_3v_3v_ju_j), (y_2v_3v_ju_j), 5 \leq j \leq n-1$	$3(n-5)$
$(y_3v_3v_{j+1}u_{j+1}), 5 \leq j \leq n-1$	$(n-6)$
$(u_3v_3v_6y_5), (u_3v_3v_6y_6), \ldots, (u_3v_3v_{n-2}y_{n-3}), (u_3v_3v_{n-2}y_{n-2})$	$2(n-7)$
$(y_2v_3v_6y_5), (y_2v_3v_6y_6), \ldots, (y_2v_3v_{n-2}y_{n-3}), (y_2v_3v_{n-2}y_{n-2})$	$2(n-7)$
$(y_3v_3v_7y_6), (y_3v_3v_7y_7), \ldots, (y_3v_3v_{n-2}y_{n-3}), (y_3v_3v_{n-2}y_{n-2})$	$2(n-8)$
$(y_2v_3v_1), (y_2v_3v_4), \ldots, (y_2v_3v_{n-1}), (y_3v_3v_1), (y_3v_3v_2), (y_3v_3v_5), \ldots, (y_3v_3v_{n-1})$	$2(n-3)$
$(u_3v_3v_1u_1), (u_3v_3v_5y_5), (u_3v_3v_{n-1}y_{n-2}), (y_2v_3v_5y_5), (y_2v_2v_{n-1}y_{n-2}), (y_3v_3v_6y_6),$ $(y_3v_3v_{n-1}y_{n-2}), (y_3v_3v_1u_1)(u_3v_3v_1), (u_3v_3v_2), (u_3v_3v_4), (u_3v_3u), (y_2v_3u), (y_3v_3u)$	14

Hence, $st_{M(G)}(v_3) = 12n - 57 = st_{M(G)}(v_{n-3})$.
To find $st_{M(G)}(v_4)$: The shortest paths which contribute to $st_{M(G)}(v_4)$ are as shown in the table below.

Shortest paths $\mathcal{P}$ with v_4 as internal vertex	$\lvert\mathcal{P}\rvert$
$(u_4v_4v_j),(u_4v_4v_ju_j),(y_3v_4v_ju_j), 6 \le j \le n-1$	$3(n-6)$
$(y_4v_4v_{j+1}u_{j+1}), 6 \le j \le n-1$	$(n-7)$
$(u_4v_4v_7y_6),(u_3v_3v_7y_7),\ldots,(u_4v_4v_{n-2}y_{n-3}),(u_4v_4v_{n-2}y_{n-2})$	$2(n-8)$
$(y_3v_4v_7y_6),(y_3v_4v_7y_7),\ldots,(y_3v_4v_{n-2}y_{n-3}),(y_3v_4v_{n-2}y_{n-2})$	$2(n-8)$
$(y_4v_4v_8y_7),(y_4v_4v_8y_8),\ldots,(y_4v_4v_{n-2}y_{n-3}),(y_4v_4v_{n-2}y_{n-2})$	$2(n-9)$
$(y_3v_4v_1),(y_3v_4v_2),(y_3v_4v_5),\ldots,(y_3v_4v_{n-1}),(y_4v_4v_1),(y_4v_4v_2),$ $(y_4v_4v_3),(y_3v_3v_6),\ldots,(y_4v_4v_{n-1})$	$2(n-3)$
$(u_4v_4v_2u_2),(u_4v_4v_1u_1),(u_4v_4v_2y_1),(u_4v_4v_1y_1),(u_4v_4v_3),(u_4v_4v_2),$ $(u_4v_4v_1),(u_4v_4v_5),(u_4v_4v_6y_6),(u_4v_4v_{n-1}y_{n-2}),(y_3v_4v_6y_6),$ $(y_3v_4v_{n-1}y_{n-2}),(y_4v_4v_7y_7),(y_4v_4v_{n-1}y_{n-2}),(y_3v_4v_1u_1),(y_4v_4v_1y_1),$ $(y_4v_4v_2y_1),(y_4v_4v_1u_1),(y_4v_4v_2u_2),(uv_4y_3),(uv_4y_4),(uv_4u_4)$	22

Hence, due to symmetry, $st_{M(G)}(v_j) = 12n - 59, 4 \le j \le n-4$.
To find $st_{M(G)}(y_1)$: The shortest paths $(u_1y_1u_2),(v_1y_1u_2),(v_2y_1u_1),(u_1y_1y_2),(v_1y_1y_2),$ $(u_1y_1y_2u_3),(u_1y_1y_2y_3)$ contribute to $st_{M(G)}(y_1)$. Hence, $st_{M(G)}(y_1) = 7 = st_{M(G)}(y_{n-2})$.
To find $st_{M(G)}(y_2)$: The shortest paths which contributes to $st_{M(G)}(y_2)$ are $(y_1y_2y_3),(y_1y_2u_3),(u_2y_2u_3),(v_2y_2u_3),(v_3y_2u_2),(u_2y_2y_3),(v_2y_2y_3),(v_3y_2y_1),(u_3y_2v_2),(u_1y_1y_2u_3),$ $(u_2y_2y_3u_4),(u_2y_2y_3y_4),(y_1y_2y_3y_4),(y_1y_2y_3u_4)$.
Hence, $st_{M(G)}(y_2) = 14 = st_{M(G)}(y_{n-3})$.
To find $st_{M(G)}(y_3)$: The shortest paths that contributes to $st_{M(G)}(y_3)$ are listed below; $(v_3y_3u_4),(v_4y_3u_3),(u_3y_3u_4),(u_3y_3y_4),(u_4y_3y_2),(y_2y_3y_4),(v_4y_3y_2),(v_3y_3y_4),(y_2y_3y_4y_5),(y_2y_3y_4u_5),$ $(u_3y_3y_4y_5),(y_1y_2y_3y_4),(u_2y_2y_3u_4),(u_3y_3y_4u_5),(y_2u_3y_3u_4),(u_3y_3u_4y_4)$.
Hence, $st_{M(G)}(v_j) = 16, 3 \le j \le n-4$. □

Corollary 2.4 *Let $G = F_{1,n-1}$. Then*

$$SS(M(G)) = \begin{cases} 162 & ; n = 4 \\ 12n^3 - 55n^2 + 65n + 4 & ; n \ge 5 \end{cases}$$

Proof: Observe that $|E(G)| = \binom{n-1}{2} + 7n - 13$.
Let $n \ge 5$: The contribution to $SS(M(G))$ from various edges of $M(G)$ are given below.

Edges	Contribution to $SS(M(G))$
$y_iy_j, 1 \le i,j \le n-2$	$32n - 122$
$y_iu_j, 1 \le i \le n-2, 1 \le j \le n-1$	$32n - 108$
$y_iv_j, 1 \le i \le n-2, 1 \le j \le n-1$	$24n^2 - 142n + 234$
$v_iv_j, 1 \le i,j \le n-1$	$12n^3 - 103n^2 + 301n - 286$
$uv_i, 1 \le i \le n-1$	$12n^2 - 79n + 143$
$u_iv_j, 1 \le i,j \le n-1$	$12n^2 - 79n + 143$

Adding the above, we get the result.
For $n = 4$ one can easily get the value of $SS(M(G))$. □

Theorem 2.6 *Let $G = F_{2n}$ be a friendship graph on $2n+1$ vertices. Then,*

$$st(M(G)) = 24n^2 - 15n.$$

Proof: Let $V(G) = \{v_0, v_1, \ldots, v_{2n}\}$ and $E(G) = \{(v_1v_2),(v_3v_4),\ldots(v_{2n-1}v_{2n}),(v_0v_1),(v_0v_2),\ldots$ $(v_0v_{2n})\}$. Let $w_i \in V(M(G))$ correspond to the edge $v_{2i-1}v_{2i}, 1 \le i \le n$. Observe that $st_{M(G)}(w_i) = 3$. Let $y_i \in V(M(G))$ correspond to the edge v_0v_i, $1 \le i \le 2n$. Every shortest path of length two of the form $(v_1v_0v_3)$ in G will correspond to the following shortest paths

$$(v_1y_1y_3),(v_1y_1y_3v_3),(v_1y_1y_3w_2),(w_1y_1y_3),(w_1y_1y_3v_3),(w_1y_1y_3w_2)$$

in $M(G)$ contributing 6 to $st_{M(G)}(y_1)$. Additionally, $(v_1y_1y_2),(w_1y_1v_0),(v_1y_1v_0)$ contributes 3 to $st_{M(G)}(y_1)$. As there are $2(n-1)$ shortest paths of the form $v_1v_0v_j, 3 \le j \le 2n$ in G, and due to symmetry, $st_{M(G)}(y_i) = 12(n-1)+3 = 12n-9, 1 \le i \le 2n$. Hence, the proof. □

Corollary 2.5 *Let $G = F_{2n}$. Then,*

$$SS(M(G)) = 48n^3 + 12n^2 - 24n.$$

Proof: As $|E(M(G))| = \binom{2n}{2} + 8n$,

$$SS(M(G)) = \binom{2n}{2}(12n-9)2 + 4n(12n-9) + 2n(3) + 2n(12n-6).$$

□

Theorem 2.7 *Let $G = C_n$ be a cycle on n vertices with diameter d. Then,*

$$st(M(G)) = \begin{cases} n(2d^2-d); & \text{if } n \text{ even} \\ n(2d^2+d); & \text{otherwise} \end{cases}$$

Proof: Let $V(G) = \{u_1,u_2,\ldots,u_n\}$, $u_i \sim u_{i+1}, 1 \le i \le n-1$ and $u_n \sim u_1$.

Case 1: n even: For $v_i \in V(M(G))$ that corresponds to an edge $e = u_iu_{i+1}$ of G, every shortest path $(u_lu_{l+1}\ldots u_iu_{i+1}\ldots u_{l+k})$ of length $k = 1,2,\ldots,d-1$ in G has the shortest paths $(u_lv_lv_{l+1}\ldots v_{l+k-1}u_{l+k})$, $(v_{l-1}v_lv_{l+1}\ldots v_{l+k-1}u_{l+k})$, $(v_lv_{l+1}\ldots v_{l+k-1}v_{l+k})$, and $(v_{l-1}v_lv_{l+1}\ldots v_{l+k-1}v_{l+k})$ of length $k+1$ in $M(G)$ contributing 4 to $st_{M(G)}(v_i)$. But for every shortest path of the form $(u_lu_{l+1}\ldots u_iu_{i+1}\ldots u_{l+d})$ in G has the shortest path of the form $(u_lv_lv_{l+1}\ldots v_{l+d-1}u_{l+d})$, of length $d+1$ in $M(G)$ contributing 1 to $st_{M(G)}(v_i)$. Suppose $P_k^G(e)$ represents the shortest path of length k in G having e as internal edge, then,

$$\begin{aligned} st_{M(G)}(v_i) &= 4\left(\sum_{k=1}^{d-1}|P_k^G(e)|\right) + |P_d^G(e)| \\ &= 4(1+2+\ldots+(d-1)) + d \\ &= 2d^2 - d. \end{aligned}$$

Case 2: n odd: For $v_i \in V(M(G))$, which corresponds to an edge $e = u_iu_{i+1}$ of G, every shortest path $(u_lu_{l+1}\ldots u_iu_{i+1}\ldots u_{l+k})$ of length $k = 1,2,\ldots,d-1$ in G has the shortest paths $(u_lv_lv_{l+1}\ldots v_{l+k-1}u_{l+k})$, $(v_{l-1}v_lv_{l+1}\ldots v_{l+k-1}u_{l+k})$, $(u_lv_lv_{l+1}\ldots v_{l+k-1}v_{l+k})$, and $(v_{l-1}v_lv_{l+1}\ldots v_{l+k-1}v_{l+k})$ of length $k+1$ in $M(G)$ contributing 4 to $st_{M(G)}(v_i)$. But the shortest path $(u_lu_{l+1}\ldots u_iu_{i+1}\ldots u_{l+d})$ in G has the shortest paths $(u_lv_lv_{l+1}\ldots v_{l+d-1}u_{l+d})$, $(u_lv_lv_{l+1}\ldots v_{l+d-1}v_{l+d})$, and $(v_{l-1}v_lv_{l+1}\ldots v_{l+d-1}u_{l+d})$, of length $d+1$ in $M(G)$ contributing 3 to $st_{M(G)}(v_i)$. Hence,

$$\begin{aligned} st_{M(G)}(v_i) &= 4\left(\sum_{k=1}^{d-1}(|P_k^G(e)|\right) + 3|P_d^G(e)| \\ &= 4\,(1+2+\ldots+(d-1)) + 3d \\ &= 2d^2 + d. \end{aligned}$$

□

Corollary 2.6 *Let $G = C_n$ with diameter d. Then,*

$$SS(M(G)) = \begin{cases} 4n(2d^2-d); & n \text{ even} \\ 4n(2d^2+d); & n \text{ odd} \end{cases}$$

Proof: As $|E(M(G))| = 3n$, when n is even,

$$SS(M(G)) = 2n(2d^2 - d) + 2n(2d^2 - d) = 4n(2d^2 - d).$$

Similarly, when n is odd we get

$$SS(M(G)) = 2n(2d^2 + d) + 2n(2d^2 + d) = 4n(2d^2 + d).$$

□

Theorem 2.8 *Let $G = P_n$ be a path on n vertices. Then,*

$$st(M(G)) = \frac{2}{3}n^3 - 2n^2 + \frac{7}{3}n - 1.$$

Proof: Let $V(G) = \{u_1, u_2, \ldots, u_n\}$, $u_i \sim u_{i+1}, 1 \leq i \leq n-1$. Let $v_j \in M(G), 1 \leq j \leq n-1$ correspond to the edge $u_j u_{j+1}$ of G.

To find $st_{M(G)}(v_1)$: For every shortest path $(u_1 u_2 \ldots u_{k+1})$ of length $k, 1 \leq k \leq n-2$ in G we have $(u_1 v_1 v_2 \ldots u_{k+1})$ and $(u_1 v_1 v_2 \ldots v_{k+1})$ in $M(G)$ of length $k+1$ contributing 2 to $st_{M(G)}(v_1)$. Also, for the shortest path $(u_1 u_2 \ldots u_n)$ of G we have the shortest path $(u_1 v_1 \ldots v_{n-1} u_n)$ of length n in $M(G)$ contributing 1 to $st_{M(G)}(v_1)$. Hence,

$$st_{M(G)}(v_1) = 2n - 3.$$

By symmetry, $st_{M(G)}(v_n) = 2n - 3$.
To find $st_{M(G)}(v_j),\ j \geq 2$:
The shortest paths of length $k, 1 \leq k \leq j-1$, in G which contains the vertices $u_j u_{j+1}$, contributes $\sum_{i=1}^{j-1} i$ to $st_{M(G)}(v_j)$. Now the shortest paths of length $k, j \leq k \leq n-j-1$ in G which contains the vertices $u_j u_{j+1}$, contributes $(4j-2)(n-2j)$. Next, the shortest paths of length k, where $n-j \leq k \leq d-1$ in G which contains the vertices $u_j u_{j+1}$, contributes $4(j-1) + 4\sum_{i=0}^{j-2} i$. Lastly, the shortest path of length d in G which contains the vertices $u_j u_{j+1}$, contributes 1 to $st_{M(G)}(v_j)$. Hence, for $2 \leq j \leq n-2$,

$$\begin{aligned} st_{M(G)}(v_j) &= 4\sum_{i=1}^{j-1} i + (4j-2)(n-2j) + 4\sum_{i=0}^{j-2} i + 4(j-1) + 1 \\ &= 4j(n-j) - 2n + 1. \end{aligned}$$

When n is even,

$$\begin{aligned} st(M(G) &= 2\sum_{j=1}^{\frac{n}{2}-1} (4j(n-j) - 2n + 1) + n^2 - 2n + 1 \\ &= \frac{2}{3}n^3 - 2n^2 + \frac{7}{3}n - 1. \end{aligned}$$

When n is odd,

$$\begin{aligned} st(M(G) &= 2\sum_{j=1}^{\frac{n-1}{2}} (4j(n-j) - 2n + 1) \\ &= \frac{2}{3}n^3 - 2n^2 + \frac{7}{3}n - 1. \end{aligned}$$

□

Corollary 2.7 *Let $G = P_n$. Then*

$$SS(M(G)) = \frac{8n^3 - 24n^2 + 16n + 6}{3}.$$

Proof: Observe that $|E(M(G))| = (n-2)+2(n-1)$. The contribution to $SS(M(G))$ from various edges of $M(G)$ are given below.

Edges	Contribution to $SS(M(G))$
$v_ju_j, v_ju_{j+1}, 2 \leq j \leq n-2$	$2\sum_{j=2}^{n-2}(4j(n-j)-(2n-1))$
$v_1u_1, v_1u_2, v_{n-1}u_{n-1}, v_{n-1}u_n$	$4(2n-3)$
$v_jv_{j+1}, 1 \leq j \leq n-2$	$2\sum_{j=2}^{n-2}(4j(n-j)-(2n-1))+2(2n-3)$

Hence,

$$\begin{aligned} SS(M(G)) &= 4\sum_{j=2}^{n-2}(4j(n-j)-(2n-1))+6(2n-3) \\ &= \frac{4(2n^3-6n^2-5n+15)}{3}+6(2n-3) \\ &= \frac{8n^3-24n^2+16n+6}{3}. \end{aligned}$$

□

3. Central Graph

In this section, we first provide a bound for the stress of any vertex in the central graph of a graph. Furthermore, we provide expressions for the stress and the stress-sum index of the central graph for several standard classes of graphs.

Theorem 3.1 *Let $u \in V(G)$. Then,*

$$st_{C(G)}(u) \geq \binom{deg_G(u)}{2} + m(\overline{u}) + deg_G(u)\left(\sum_{v \nsim_G u} deg_G(v) + deg_{\overline{G}}(u)\right) - \sum_{x \in N_G(u)} |N_G(x) \cap N_{\overline{G}}(u)|,$$

where $m(\overline{u})$ denotes the number of edges in $\langle V(G)-N[u]\rangle$. Equality holds in case of $K_{p,q}, C_n$ and P_n. Let $v \in V(C(G))$, where v corresponds to the edge uw of G. Then,

$$st_{C(G)}(v) = 2|N(u) \cap N(w)| + 1 + (r-2)(r-3),$$

where r is the size of maximal clique in $\langle N[u] \cap N[w]\rangle$.

Proof: Let $N_G(u) = \{u_1, u_2, \ldots, u_d\}$ and $\{s_1, s_2, \ldots, s_d\}$ be the vertices in $C(G)$ formed by subdividing the edges $uu_i, 1 \leq i \leq d$ of G. For any distinct pair s_i, s_j the path (s_ius_j) of length 2 in $C(G)$ contributes 1 to $st_{C(G)}(u)$. Hence, $st_{C(G)}(u) \geq \binom{deg_G(u)}{2}$.
For every edge v_iv_j of G, such that $v_i \nsim_G u$ and $v_j \nsim u$, we get a shortest path v_iuv_j in $C(G)$ which contributes one to $st_{C(G)}(u)$. Hence, if $m(\overline{u})$ represents the number of edges in $\langle V(G)-N[u]\rangle$, we get $st_{C(G)}(u) \geq \binom{deg_G(u)}{2} + m(\overline{u})$.

From every vertex $w \in C(G)$ that corresponds to the edge $uv \in E(G)$, there is a path (wux) where $x \nsim_G u$. As there are $deg_G(u)deg_{\overline{G}}(u)$ such paths, we have $st_{C(G)}(u) \geq \binom{deg_G(u)}{2} + m(\overline{u}) + deg_G(u)deg_{\overline{G}}(u)$. Similarly, the paths $(wuxz)$, where $x \sim_{C(G)} z$ contribute to $st_{C(G)}(u)$ unless $x \sim_G v$.

As there are $deg_G(u)\sum_{v \nsim_G u} deg_G(v) - \sum_{y \in N_G(u)} |N_G(y) \cap N_{\overline{G}}(u)|$ paths, the result follows. The equality holds when G is a cycle, a path, or a complete bipartite graph.
If $v \in V(C(G))$, where v corresponds to the edge uw of G. Observe that the path (uvw) will contribute 1 to $st_{C(G)}(v)$.

Suppose $N(u) \cap N(w) \neq \phi$. Let $z \in N(u) \cap N(w)$ and x, y be a vertex formed by subdividing uz and wz. Then the paths $(ywvu), (wvux)$ contribute to $st_{C(G)}(v)$. We have $2|N(u) \cap N(w)|$ such paths.
Suppose $\langle N[u] \cap N[w]\rangle$ contains a maximal clique of size r. Let $v, s_i, 1 \leq i \leq r-2$ be adjacent to u and $v, t_j, 1 \leq j \leq r-2$ be adjacent to w in $C(G)$. Then the paths $(s_iuvwt_j), 1 \leq i \neq j \leq r-2$ contribute to $st_{C(G)}(v)$.

As there are $(r-2)(r-3)$ such paths, the result follows. □

Theorem 3.2 *Let $G = K_{p,q}$. Then,*

$$st(C(G)) = 2p^2q^2 - \frac{pq}{2}(p+q).$$

Proof: Let $V(G) = \{v_1, v_2, \ldots, v_p, u_1, u_2, \ldots, u_q\}$. Then, from Theorem 3.1, we have

$$st_{C(G)}(u_i) = p^2q - \frac{p}{2} - \frac{p^2}{2}, 1 \leq i \leq q,$$
$$st_{C(G)}(v_i) = q^2p - \frac{q}{2} - \frac{q^2}{2}, 1 \leq i \leq p.$$

Let $w_{ij} \in V(C(G))$ correspond to the edge $v_iu_j, 1 \leq i \leq p, 1 \leq j \leq q$ of G. Then from Theorem 3.1, we have $st_{C(G)}(w_{ij}) = 1$.

$$st(C(G)) = p\left(q^2p - \frac{q}{2} - \frac{q^2}{2}\right) + q\left(p^2q - \frac{p}{2} - \frac{p^2}{2}\right) + pq = 2p^2q^2 - \frac{pq}{2}(p+q).$$

□

Corollary 3.1 *Let $G = K_{p,q}$. Then,*

$$SS(C(G)) = p^2q^2(2p + 2q - 3) + pq\left(3 - \frac{p+q}{2} - \frac{p^2+q^2}{2}\right).$$

Proof: From Theorem 3.2,

$$SS(C(G)) = 2\left(q^2p - \frac{q}{2} - \frac{q^2}{2}\right)\binom{p}{2} + \left(q^2p - \frac{q}{2} - \frac{q^2}{2} + 1\right)pq + 2\left(p^2q - \frac{p}{2} - \frac{p^2}{2}\right)\binom{q}{2}$$
$$+ \left(p^2q - \frac{p}{2} - \frac{p^2}{2} + 1\right)pq.$$

Hence, the result follows. □

Theorem 3.3 *Let $G = K_n$. Then,*

$$st(C(G)) = \frac{3n^4 - 13n^3 + 19n^2 - 9n}{2}.$$

Proof: If $v \in V(C(G))$ corresponds an edge $e \in E(G)$, then from Theorem 3.1, we have

$$st_{C(G)}(v) = 2(n-2) + 1 + (n-3)(n-2) = n^2 - 3n + 3.$$

$$st_{C(G)}(v_1) = \binom{n-1}{2} + (n-1)(n-2)(n-3) + (n-1)(n-2) = \frac{(n-1)(n-2)(2n-3)}{2}.$$

Hence

$$st(C(G)) = \binom{n}{2}(n^2 - 3n + 3) + \frac{n(n-1)(n-2)(2n-3)}{2} = \frac{3n^4 - 13n^3 + 19n^2 - 9n}{2}.$$

□

Corollary 3.2 *Let $G = K_n$. Then,*

$$SS(C(G)) = \frac{2n^5 - 9n^4 + 14n^3 - 7n^2}{2}.$$

Proof: As $|E(C(G))| = 2\binom{n}{2}$, we have

$$SS(C(G)) = 2\binom{n}{2}\left\{\frac{(n-1)(n-2)(2n-3)}{2} + (n^2-3n+3)\right\}.$$

□

Theorem 3.4 *Let $G = B_{r,s}$, be a bi-star graph on $r+s+2$ vertices. Then,*

$$st(C(G)) = \frac{24rs+5s+5r+3r^2+3s^2+2}{2}.$$

Proof: Let $V(G) = \{v, v_1, v_2, \ldots, v_r, u, u_1, u_2, \ldots, u_s\}$. Then from Theorem 3.1, we have $st_{C(G)}(v_i) = 4s+r, 1 \leq i \leq r$, $st_{C(G)}(u_j) = 4r+s, 1 \leq j \leq s$, $st_{C(G)}(v) = \frac{r^2}{2}+\frac{r}{2}+2rs+s$ and $st_{C(G)}(u) = \frac{s^2}{2}+\frac{s}{2}+2rs+r$. Let $x \in V(C(G))$ correspond to the edge vu, $w_i \in V(C(G))$ correspond to the edge $vv_i, 1 \leq i \leq r$ and $y_j \in V(C(G))$ correspond to the edge $uu_j, 1 \leq j \leq s$.
From Theorem 3.1, we have $st_{C(G)}(x) = st_{C(G)}(w_i) = st_{C(G)}(y_j) = 1$. Hence,

$$\begin{aligned} st(C(G)) &= r(4s+r) + s(4r+s) + \left(\frac{r^2}{2}+\frac{r}{2}+2sr+s\right) \\ &\quad + \left(\frac{s^2}{2}+\frac{s}{2}+2rs+r\right) + (r+s+1) \\ &= \frac{24rs+5s+5r+3r^2+3s^2+2}{2}. \end{aligned}$$

□

Corollary 3.3 *Let $G = B_{r,s}$, be a bi-star graph on $r+s+2$ vertices. Then,*

$$SS(C(G)) = \frac{27}{2}rs(r+s) + \frac{7}{2}(r+s) + \frac{3}{2}(r^3+s^3) + 3(r^2+s^2) + 15rs + 2.$$

Proof: Observe that $|E(C(G))| = \binom{r}{2} + \binom{s}{2} + 3r + 3s + rs + 2$. From Theorem 3.4, the contribution to $SS(C(G))$ from various edges of $C(G)$ is given below.

Edges	Contribution to $SS(C(G))$
$vu_j, 1 \leq j \leq s$	$\frac{r^2s+rs}{2} + 2rs^2 + 2s^2 + 4rs$
vx	$\frac{r^2}{2} + \frac{r}{2} + 2rs + s + 1$
$vw_i, 1 \leq i \leq r$	$\frac{r^3+r^2}{2} + 2r^2s + rs + r$
ux	$\frac{s^2+s}{2} + 2rs + r + 1$
$uy_j, 1 \leq j \leq s$	$\frac{s^3+s^2}{2} + 2rs^2 + rs + s$
$uv_i, 1 \leq i \leq r$	$\frac{s^2r+sr}{2} + 2r^2s + 2r^2 + 4rs$
$v_iw_i, 1 \leq i \leq r$	$4sr + r^2 + r$
$v_iu_j, 1 \leq i \leq r, 1 \leq j \leq s$	$5rs^2 + 5r^2s$
$y_ju_j, 1 \leq j \leq s$	$4rs + s^2 + s$
$v_i, 1 \leq i \leq r$	$4sr^2 + r^3 - 4sr - r^2$
$u_j, 1 \leq j \leq s$	$4rs^2 + s^3 - 4rs - s^2$

□

Theorem 3.5 *Let $G = W_{1,n-1}$. Then,*

$$st(C(G)) = \frac{33n^2-143n+110}{2}.$$

Proof: Let $V(G) = \{v, v_1, v_2, \ldots, v_{n-1}\}$. Let $y_i \in V(C(G))$ correspond to the edge $vv_i, 1 \leq i \leq n-1$ of G and $w_i \in V(C(G))$ correspond to the edge $v_iv_{i+1}, 1 \leq i \leq n-2$, of G and w_{n-1} corresponds to the edge $v_{n-1}v_1$ of G. Then from Theorem 3.1, we have,

$$st_{C(G)}(y_i) = 5,\ st_{C(G)}(w_i) = 3$$

To find $st_{C(G)}(v)$: From Theorem 3.1, $st_{C(G)}(v) \geq \binom{n-1}{2}$.
The shortest paths $\{(v_1y_1vy_2), (v_1y_1vy_{n-1}), (v_2y_2vy_3), (v_2y_2vy_1), \ldots, (v_{n-1}y_{n-1}vy_1), (v_{n-1}y_{n-1}vy_{n-2})\}$ contribute $2(n-1)$ to $st_{C(G)}(v)$.
Hence,

$$st_{C(G)}(v) = \binom{deg_G(v)}{2} + 2(n-1) = \frac{(n-1)(n+2)}{2}.$$

To find $st_{C(G)}(v_1)$: From Theorem 3.1 we have,

$$st_{C(G)}(v_1) \geq \binom{3}{2} + (n-5) + 3(3(n-4) + (n-4)) - (n-2) = 12n - 48.$$

The other shortest paths are as shown in the table below.

Shortest paths $\mathcal{P}$ with v_1 as internal vertex	$\lvert\mathcal{P}\rvert$
$(y_lv_lv_1v_{l+1}), (y_{l+1}v_{l+1}v_1v_l),\ 3 \leq l \leq n-3$	$2(n-5)$
$(y_1v_1v_4v_2), \ldots, (y_1v_1v_{n-2}v_2), (y_1v_1v_3v_{n-1}), \ldots, (y_1v_1v_{n-3}v_{n-1})$	$2(n-5)$
$(y_1v_1w_1v_2), (y_1v_1w_{n-1}v_{n-1}), (vy_1v_1w_1), (vy_1v_1w_{n-1})$	4

Due to symmetry, $st_{C(G)}(v_i) = 12n - 48 + 4(n-5) + 4 = 16(n-4), 1 \leq i \leq n-1$. Hence,

$$\begin{aligned} st(C(G)) &= 5(n-1) + 3(n-1) + \frac{(n-1)(n+2)}{2} + 16(n-4)(n-1) \\ &= \frac{33n^2 - 143n + 110}{2}. \end{aligned}$$

□

Corollary 3.4 *Let $G = W_{1,n-1}$. Then,*

$$SS(C(G)) = \frac{33n^3 - 192n^2 + 317n - 158}{2}.$$

Proof: As $|E(C(G))| = \frac{(n-1)(n-4)}{2} + 2(n-1) + 2(n-1)$, and from Theorem 3.5, we have

$$\begin{aligned} SS(C(G)) = 16(n-4)(n-1)(n-4) + \{16(n-4) + 5\}(n-1) + \{16(n-4) + 3\}2(n-1) \\ + \left\{\frac{(n-1)(n+2)}{2} + 5\right\}(n-1). \end{aligned}$$

Simplifying the above we get the desired result. □

Theorem 3.6 *Let $G = F_{1,n-1}$ be a fan graph with n vertices. Then,*

$$st(C(G)) = \frac{33n^2 - 181n + 286}{2}.$$

Proof: Let $V(G) = \{v, v_1, v_2, \ldots, v_{n-1}\}$. Let $y_i \in V(C(G))$ correspond to the edge $vv_i, 1 \leq i \leq n-1$ and $w_i \in V(C(G))$ corresponding to the edge $v_iv_{i+1}, 1 \leq i \leq n-2$.
From Theorem 3.1, $st_{C(G)}(w_i) = 3, 1 \leq i \leq n-2$. Also, $st_{C(G)}(y_1) = 3 = st_{C(G)}(y_{n-1})$, and $st_{C(G)}(y_i) = 5, 2 \leq i \leq n-2$.

To find $st_{C(G)}(v)$: From Theorem 3.1 we have $st_{C(G)}(v) \geq \binom{n-1}{2}$.
The shortest paths $(v_i y_i v y_{i+1})$, and $(y_i v y_{i+1} v_{i+1}), 1 \leq i \leq n-2$ contributing $2(n-2)$ to $st_{C(G)}(v)$. Hence,

$$st_{C(G)}(v) = \frac{(n-2)(n+3)}{2}.$$

To find $st_{C(G)}(v_1)$: From Theorem 3.1 we have,

$$st_{C(G)}(v_1) \geq 8n - 27.$$

The other shortest paths are shown in the table below.

Shortest paths $\mathcal{P}$ with v_1 as internal vertex	$\lvert\mathcal{P}\rvert$
$(y_3v_3v_1v_4), \ldots, (y_{n-2}v_{n-2}v_1v_{n-1}), (y_4v_4v_1v_3), \ldots, (y_{n-1}v_{n-1}v_1v_{n-2})$	$2(n-4)$
$(y_1v_1v_4v_2), \ldots, (y_1v_1v_{n-1}v_2)$	$n-4$
$(vy_1v_1w_1), (y_1v_1w_1v_2)$	2

$$st_{C(G)}(v_1) = 8n - 27 + 3(n-4) + 2 = 11n - 37.$$

Due to symmetry $st_{C(G)}(v_1) = st_{C(G)}(v_{n-1})$.
To find $st_{C(G)}(v_2)$: From Theorem 3.1 we have,

$$st_{C(G)}(v_2) \geq 12n - 50.$$

The other shortest paths are shown in the table below.

Shortest paths $\mathcal{P}$ with v_2 as internal vertex	$\lvert\mathcal{P}\rvert$
$(y_4v_4v_2v_5), \ldots, (y_{n-2}v_{n-2}v_2v_{n-1}), (y_5v_5v_2v_4), \ldots, (y_{n-1}v_{n-1}v_2v_{n-2})$	$2(n-5)$
$(y_2v_2v_5v_3), \ldots, (y_2v_2v_{n-1}v_3)$	$n-5$
$(y_2v_2v_4v_1), \ldots, (y_2v_2v_{n-1}v_1)$	$n-4$
$(vy_2v_2w_2), (y_2v_2w_2v_3), (vy_2v_2w_1), (y_2v_2w_1v_1)$	4

Hence,

$$st_{C(G)}(v_2) = 12n - 50 + 3(n-5) + (n-4) + 4 = 16n - 65.$$

Due to symmetry $st_{C(G)}(v_2) = st_{C(G)}(v_{n-2})$.
To find $st_{C(G)}(v_3)$: From Theorem 3.1, we have

$$st_{C(G)}(v_3) \geq 12n - 55.$$

The other shortest paths are shown in the table below.

Shortest paths $\mathcal{P}$ with v_3 as internal vertex	$\lvert\mathcal{P}\rvert$
$(y_5v_5v_3v_6), \ldots, (y_{n-2}v_{n-2}v_3v_{n-1}), (y_6v_6v_3v_5), \ldots, (y_{n-1}v_{n-1}v_3v_{n-2})$	$2(n-6)$
$(y_3v_3v_6v_4), \ldots, (y_3v_3v_{n-1}v_4), (y_3v_3v_1v_4), (y_3v_3v_5v_2), \ldots, (y_3v_3v_{n-1}v_2)$	$2(n-5)$
$(vy_3v_3w_3), (y_3v_3w_3v_4), (y_3v_3w_2v_2), (vy_3v_3w_2)$	4

Hence,

$$st_{C(G)}(v_3) = 12n - 55 + 2(n-6) + 2(n-5) + 4 = 16n - 73.$$

$$st_{C(G)}(v_3) = \cdots = st_{C(G)}(v_{n-3}) = 16n - 73.$$

Hence

$$st(C(G)) = \frac{33n^2 - 181n + 286}{2}.$$

□

Corollary 3.5 *Let $G = F_{1,n-1}$, be a fan graph on n vertices. Then,*

$$SS(C(G)) = \frac{33n^3 - 230n^2 + 545n - 376}{2}.$$

Proof: We know $|E(C(G))| = 2(n-1) + 2(n-2)$. The contribution to $SS(C(G))$ from various edges of $C(G)$ is given below.

Edges	Contribution to $SS(C(G))$
$vy_i, i = 1, n-1$	$n^2 + n$
$vy_i, 2 \le i \le n-2$	$\frac{n^3-2n^2+n-12}{2}$
$v_iy_i, i = 1, n-1$	$22n - 68$
$v_iy_i, i = 2, n-2$	$32n - 120$
$v_iy_i, 3 \le i \le n-3$	$16n^2 - 148n + 340$
$v_1w_1, v_{n-1}w_{n-2}$	$22n - 68$
$v_iw_i, v_iw_{i-1}, i = 2, n-2$	$64n - 248$
$v_iw_i, v_iw_{i-1}, 3 \le i \le n-3$	$32n^2 - 300n + 700$
$v_1v_i, 3 \le i \le n-1$	$27n^2 - 196n + 374$
$v_2v_i, 4 \le i \le n-1$	$32n^2 - 271n + 596$
$v_3v_i, 5 \le i \le n-1$	$16n^3 - 222n^2 + 1046n - 1688$

Hence, the result follows. □

Theorem 3.7 *Let $G = F_{2n}$ be a friendship graph on $2n+1$ vertices. Then,*

$$st(C(G)) = 32n^2 - 14n.$$

Proof: Let $V(G) = \{v_0, v_1, \ldots, v_{2n}\}$ and $E(G) = \{(v_1v_2), (v_3v_4), \ldots (v_{2n-1}v_{2n}), (v_0v_1), (v_0v_2), \ldots (v_0v_{2n})\}$. Let $w_i \in V(C(G))$ correspond to the edge $v_{2i-1}v_{2i}$, $1 \le i \le n$ and $y_i \in V(C(G))$ correspond to the edge v_0v_i, $1 \le i \le 2n$. From Theorem 3.1, we have

$$st_{C(G)}(w_i) = 3 = st_{C(G)}(y_i).$$

To find $st_{C(G)}(v_0)$: From Theorem 3.1, $st_{C(G)}(v_0) \ge \binom{2n}{2}$. Now the edge $v_{2i-1}v_{2i}, 1 \le i \le n$ in G, corresponds to the shortest paths $(v_{2i-1}y_{2i-1}v_0y_{2i})$ and $(y_{2i-1}v_0y_{2i}v_{2i})$ in $C(G)$ contributes 2 to $st_{C(G)}(v_0)$. Hence,

$$st_{C(G)}(v_0) = \binom{2n}{2} + 2n = n(2n+1).$$

To find $st_{C(G)}(v_1)$: From Theorem 3.1 we have,

$$st_{C(G)}(v_1) \ge 1 + (n-1) + 2(2n-2+2(2n-2)) - (2n-2) = 11n - 10.$$

For the edge v_1v_2 of G, we have the shortest path $(y_1v_1v_jv_2)$, where $v_j \nsim_G v_1$ contributing 1 to $st_{C(G)}(v_1)$. In addition, the shortest paths $(y_{2j-1}v_{2j-1}v_1v_{2j})$, $(v_{2j-1}v_1v_{2j}y_{2j})$, $2 \le j \le n$ contribute 2 to $st_{C(G)}(v_1)$. Lastly, the paths $(y_1v_1w_1v_2)$ and $(v_0y_1v_1w_1)$ contribute 2 to $st_{C(G)}(v_1)$. Hence,

$$st_{C(G)}(v_1) = 11n - 10 + 2(2n-2) + 2 = 15n - 12.$$

Hence,

$$st(C(G)) = 3(2n+n) + \frac{2n(2n+1)}{2} + 2n(15n-12) = 32n^2 - 14n.$$

□

Corollary 3.6 *Let $G = F_{2n}$. Then,*

$$SS(C(G)) = 64n^3 - 46n^2 + 18n.$$

Proof: As $|E(C(G))| = 2(n-1)n + 6n$ and from Theorem 3.7,

$$SS(C(G)) = (15n-12)(2n-2)2n + (15n-12+3)4n + (n(2n+1)+3))\,2n$$
$$= 64n^3 - 46n^2 + 18n.$$

□

Theorem 3.8 *Let $G = C_n$. Then,*

$$st(C(G)) = 7n^2 - 22n,$$

$$SS(C(G)) = 7n^3 - 30n^2 + 25n.$$

Proof: Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then, from Theorem 3.1, we have

$$st_{C(G)}(v_i) = 1 + n - 4 + 2(n - 3 + 2(n-3)) - 2 = 7n - 23.$$

Let $u_i \in V(C(G))$ correspond to the edge $v_iv_{i+1}, 1 \le i \le n-1$ and $u_n \in V(C(G))$ correspond to the edge v_nv_1 of G. Then from Theorem 3.1, we have $st_{C(G)}(u_i) = 1, 1 \le i \le n$. Therefore, the proof follows. □

Theorem 3.9 *Let $G = P_n$. Then,*

$$st(C(G)) = 7n^2 - 33n + 41,$$

$$SS(C(G)) = 7n^3 - 41n^2 + 78n - 44.$$

Proof: Let $V(G) = \{u_1, u_2, \ldots, u_n\}$. Then, from Theorem 3.1, we have

$$st_{C(G)}(u_1) = 4n - 11 = st_{C(G)}(u_n),$$

$$st_{C(G)}(u_2) = 7n - 24 = st_{C(G)}(u_{n-1}),$$

$$st_{C(G)}(u_j) = 7n - 28,\ 3 \le j \le n-2.$$

Let $v_{ij} \in V(C(G))$ correspond to the edge $u_iu_j, 1 \le i, j \le n$ of G. Then from Theorem 3.1, we have $st_{C(G)}(v_{ij}) = 1$. Hence, we get the expression for $st(C(G))$.
Observe that $|E(C(G))| = \frac{(n-2)(n-1)}{2} + 2(n-1)$. From Theorem 3.9, the contribution to $SS(C(G))$ from various edges of $C(G)$ is listed below.

Edges	Contribution to $SS(C(G))$
$u_iu_j, 1 \le i \le n-2, i+2 \le j \le n$	$7n^3 - 55n^2 + 152n - 148$
$v_iu_i, v_iu_{i+1}, 1 \le i \le n-1 \le n$	$14n^2 - 74n + 104$

Hence, the result follows. □

4. Conclusion

By deriving bounds and explicit expressions for stress in the middle and central graphs of a given graph, as well as computing their stress-sum indices, we provide valuable mathematical tools that can support structure-based predictions and analysis in chemical graph theory and network science.

References

1. Alaka B. and K. A. Bhat, *Stress centrality of some classes of graphs*, Engineering Letters. 33(6), 2204-2210, (2025).
2. K. Bhargava, N. N. Dattatreya and R. Rajendra, *On stress of a vertex in a graph*, Palestine Journal of Mathematics. 12(3), 15-25, (2023).
3. G. Chartrand and L. Lesniak, *Graphs and digraphs*, Sixth edition, Chapman and Hall, CRC, Boca Raton. (2005).
4. H. M. Gowramma, P. S. K. Reddy, Taekyun Kim and R. Rajendra, *Taekyun Kim stress powerα-index*, Bol. Soc. Parana. Mat. 43(3), (2025).
5. H. M. Gowramma, P. S. K. Reddy, Taekyun Kim and R. Rajendra, *Taekyun Kim α-index of graphs*, Bol. Soc. Parana. Mat. 43(3), (2025).
6. H. M. Gowramma, P. S. K. Reddy, P. S. Hemavathi, M. Pavithra and R. Rajendra, *Total stress as a topological index*, Proceedings of the Jangjeon Mathematical Society. 28(2), 181-188, (2025).
7. T. Hamada and I. Yoshimura, *Traversability and connectivity of the middle graph of a graph*, Discrete Math. 14, 247–255, (1976).
8. K. B. Mahesh, R. Rajendra and P.S.K. Reddy, *Square root stress-sum index for graphs*, Proyecciones (Antofagasta). 40(4), 927-936, (2021).
9. Niveditha, K. A. Bhat and S. Hanif, *Stress-sum index of graphs with diameter two*, International Journal of Applied Mathematics. 38(2), 281-288, (2025).
10. R. Poojary, K. A. Bhat, S. Arumugam and M. P. Karantha, *Stress of wheel related graphs*, Nat. Acad. Sci. Lett. 45(5), 427–431, (2022).
11. R. Poojary, K. A. Bhat, S. Arumugam, and M. P. Karantha., *The stress of a graph*, Communications in Combinatorics and Optimization. 8(1), 53–65, (2023).
12. R. Poojary, K. A. Bhat, S. Arumugam, and M. P. Karantha, *Stress of a graph and its computation*, AKCE Int. J. Graphs Comb. 20(2), 200–208, (2023).
13. Prajna S Rai, Howida Adel AlFran, P. S. K. Reddy, M. Kirankumar and M. Pavithra, *Hyper stress index for graphs*, Bol. Soc. Parana. Mat. 43(3), (2025).
14. R. Rajendra, P.S.K. Reddy and C. N. Harshavardhana, *Stress-sum index for graphs*, Scientia Magna. 15(1), 94-103, (2020).
15. R. Rajendra, P. S. K. Reddy and C. N. Harshavardhana, *Stress-difference index for graphs*, Bol. Soc. Parana. Mat. 42(3), (2024).
16. R. Rajendra, P. S. K. Reddy, C. N. Harshavardhana and K. A. Alloush, *Squares stress sum index for graphs*, Proceedings of the Jangjeon Mathematical Society. 26(4), 483-493, (2023).
17. R. Rajendra, P. S. K. Reddy and Cangul, I. N, *Stress indices of graphs*, Advanced Studies in Contemporary Mathematics. 31(2), 163-173, (2021).
18. P. S. K. Reddy, P. Somashekar and M. Pavithra, *Sombor stress index for graphs*, Proceedings of the Jangjeon Mathematical Society. 28(2), 217-227, (2025).
19. A. Shimbel, *Structural parameters of communication networks*, Bulletin of Mathematical Biophysics. 15, 501-506, (1953).
20. P. Somashekar and P. S. K. Reddy, *Kashvi-tosha stress index for graphs*, Dyn. Contin. Discrete Impuls. Syst., Ser. B, Appl. Algorithms. 32(2), 125-136, (2025).

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