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Generalized q-Difference Equation for Generalized Cigler's Polynomials

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ABSTRACT: This work establishes the generalised Cigler's polynomials and the generalised homogeneous q-shift operator. The q-difference equation is then utilized to demonstrate numerous polynomial q-identities, including the generating function and its extension, Rogers' formula and its extension, and Mehler's formula and its extension for the generalized Cigler's polynomials. Also presented a transformational identity involving generating functions for generalised Cigler's polynomials.

Key Words: Cigler's polynomials, homogeneous q-shift operator, q-difference equation, generating function, Mehler's formula, Rogers' formula, transformational identity.

Contents

1. Introduction

The notations and definitions used in [15] are followed here. It is assumed that $0<|q|<1$.

The q-shifted factorial is defined for $a\in\mathbb{C}$ as [13,14,15]:

$$(a;q)_0=1,\quad (a;q)_n=\prod_{k=0}^{n-1}(1-aq^k),\quad (a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k).$$

The multiple q-shifted factorials is given by [15]:

$$(a_1,a_2,\ldots,a_r;q)_m=(a_1;q)_m(a_2;q)_m\cdots(a_r;q)_m,$$

where $m\in\mathbb{Z}$ or ∞.

The basic hypergeometric series ${}_r\phi_s$ is presented as follows [15]:

$${}_r\phi_s\left(\begin{matrix}\alpha_1,\ldots,\alpha_r\\ \beta_1,\ldots,\beta_s\end{matrix};q,x\right)=\sum_{n=0}^{\infty}\frac{(\alpha_1,\ldots,\alpha_r;q)_n}{(q,\beta_1,\ldots,\beta_s;q)_n}\left[(-1)^nq^{\binom{n}{2}}\right]^{1+s-r}x^n,$$

Note that

$${}_{r+1}\phi_r\left(\begin{matrix}\alpha_1,\ldots,\alpha_{r+1}\\ \beta_1,\ldots,\beta_r\end{matrix};q,x\right)=\sum_{n=0}^{\infty}\frac{(\alpha_1,\ldots,\alpha_{r+1};q)_n}{(q,\beta_1,\ldots,\beta_r;q)_n}x^n,\quad |x|<1.$$

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The q-binomial coefficient is defined as [15]:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}} \quad for \quad 0 \leqslant k \leqslant n,$$

where $n, k \in \mathbb{N}$.

The Cauchy identity is given by [15,22]:

$$\sum_{m=0}^{\infty} \frac{(a;q)_m}{(q;q)_m}\, y^m = \frac{(ay;q)_\infty}{(y;q)_\infty}, \quad |y| < 1. \tag{1.1}$$

For $a = 0$, Cauchy identity (1.1) becomes Euler's identity [15,20]:

$$\sum_{m=0}^{\infty} \frac{y^m}{(q;q)_m} = \frac{1}{(y;q)_\infty}, \quad |y| < 1. \tag{1.2}$$

We will use the following identity in this paper [1]:

$$(q^{-n};q)_k = \frac{(q;q)_n}{(q;q)_{n-k}}(-1)^k q^{\binom{k}{2}-nk}. \tag{1.3}$$

The q-Chu-Vandermonde sum is [4,15]:

$${}_2\phi_1(q^{-n}, a; c; q, q) = \frac{(c/a;q)_n}{(c;q)_n} a^n. \tag{1.4}$$

The q-analog of the Chu-Vandermonde summation is [6]:

$$\begin{bmatrix} n+m \\ k \end{bmatrix} = \sum_{i=0}^{k} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} m \\ k-i \end{bmatrix} q^{(n-i)(k-i)}. \tag{1.5}$$

Transformation of ${}_2\phi_1$ series is [8]:

$${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix} ; q, z\right) = \frac{(abz/c;q)_\infty}{(az/c;q)_\infty}\, {}_3\phi_2\left(\begin{matrix} b, c/a, 0 \\ qc/az, c \end{matrix} ; q, q\right). \tag{1.6}$$

The Cauchy polynomials is defined by [21,23,24,27]

$$P_n(x,y) = (x-y)(x-qy)\cdots(x-q^{n-1}y) = (y/x;q)_n\, x^n,$$

which has the following generating function [2,3,12,16]:

$$\sum_{n=0}^{\infty} P_n(x,y) \frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt;q)_\infty}, \quad |xt| < 1. \tag{1.7}$$

The Mehler's formula for Cauchy polynomials $P_n(x,y)$ is [19,26]:

$$\sum_{n=0}^{\infty} P_n(x,y) P_n(u,v) \frac{t^n}{(q;q)_n} = \frac{(xvt;q)_\infty}{(xut;q)_\infty}\, {}_1\phi_1\left(\begin{matrix} v/u \\ xvt \end{matrix} ; q, yut\right), \quad |xut| < 1. \tag{1.8}$$

In 2010, Saad and Sukhi [25] introduced the homogeneous q-difference operator θ_{xy} as follows:

$$\theta_{xy}\{f(x,y)\} = \frac{f(q^{-1}x, y) - f(x, qy)}{q^{-1}x - y}.$$

Furthermore, the homogeneous q-shift operator $\mathbb{L}(z\theta_{xy})$ was introduced as follows [5,25]:

$$L(z\theta_{xy}) = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}(z\theta_{xy})^k}{(q;q)_k}.$$

Proposition 1.1 [25]. *We have*

$$\theta_{xy}^{k}\left\{P_n(y,x)\right\} = (-1)^k \frac{(q;q)_n}{(q;q)_{n-k}}\ P_{n-k}(y,x). \tag{1.9}$$

In 2014, Cao [9] defined another homogeneous q-diffreence operator:

$$\mathbb{E}(a,b;\theta_{xy}) = \sum_{k=0}^{\infty} \frac{(-1)^k (a;q)_k}{(q;q)_k} (z\theta_{xy})^k.$$

Theorem 1.1 [9]. *Let $f(x,y,z)$ be a four-variables analytic function is a neighbourhood of $(0,0,0) \in \mathbb{C}^3$ satisfying the q-difference equation*

$$\begin{aligned}&(x-q^{-1}y)\left[(f(a,x,y,z)-f(a,x,y,qz))\right]\\&= z\left[f(a,x,qy,z)-f(a,q^{-1}x,y,z)\right] - az\left[f(a,x,qy,qz)-f(a,q^{-1}x,y,qz)\right].\end{aligned}$$

Then we have

$$f(a,x,y,z) = \mathbb{E}(a,z\theta_{xy})\left\{f(a,x,y,0)\right\}.$$

In 2016, Cao and Niu [10] studied Cigler's polynomials

$$D_n^{(\alpha-n)}(x,b) = \sum_{k=0}^{n} \begin{bmatrix}\alpha\\k\end{bmatrix} q^{k^2-nk} b^k \frac{(q;q)_n}{(q;q)_{n-k}} x^{n-k}. \tag{1.10}$$

Theorem 1.2 [10]. *Let $f(\alpha,x,b)$ be an three-variable analytic function in a neighborhood of $f(\alpha,x,b) = (0,0,0) \in \mathbb{C}^3$. Then f can be expanded in terms of $D_n^{(\alpha-n)}(x,b)$ if and only if satisfies the q-difference equation*

$$\begin{aligned}&x\left[f(\alpha,x,b)-f(\alpha,x,qb)\right]\\&= qb\left[f(\alpha,q^{-1}x,qb)-f(\alpha,x,qb)\right] - bq^{\alpha+1}\left[f(\alpha,q^{-1}x,b)-f(\alpha,x,b)\right].\end{aligned}$$

Wang and Cao [29] in 2018 introduced an extension of Cigler's polynomials as follows:

$$D_n^{(\alpha-n)}(x,y,b) = (-1)^n q^{-\binom{n}{2}} \sum_{k=0}^{n} \begin{bmatrix}\alpha\\k\end{bmatrix} (-1)^k q^{\binom{k}{2}} b^k \frac{(q;q)_n}{(q;q)_{n-k}} P_{n-k}(y,x). \tag{1.11}$$

- Setting $y=0$ in (1.11), we get Cigler's polynomials $D_n^{(\alpha-n)}(x,b)$ defined in (1.10).

Theorem 1.3 [29]. *Let $f(a,b,c,d)$ be an three-variable analytic function in a neighborhood of $f(a,b,c,d) = (0,0,0,0) \in \mathbb{C}^4$. Then f can be expanded in terms of $D_n^{(\alpha-n)}(x,y,b)$ if and only if satisfies the q-difference equation*

$$\begin{aligned}&x\left[f(\alpha,x,y,b)-f(\alpha,x,y,qb)\right]\\&= qb\left[f(\alpha,q^{-1}x,y,qb)-f(\alpha,x,qy,qb)\right] - bq^{\alpha+1}\left[f(\alpha,q^{-1}x,y,b)-f(\alpha,x,qy,b)\right].\end{aligned}$$

Recently, Srivastava et al. [28] introduced the homogeneous q-shift operator

$$\tilde{L}(a,b;\theta_{xy}) = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}(a;q)_k}{(q;q)_k} (z\theta_{xy})^k.$$

Theorem 1.4 [7]. *Let $f(a,x,y,z)$ be a four-variables analytic function is a neighbourhood of $(0,0,0,0) \in \mathbb{C}^4$ satisfying the q-difference equation*

$$\begin{aligned}&(x-q^{-1}y)\left[(f(a,x,y,z)-f(a,x,y,qz))\right]\\&= z\left[f(a,q^{-1}x,y,qz)-f(a,x,qy,qz)\right] + az\left[f(a,x,qy,q^2z)-f(a,q^{-1}x,y,q^2z)\right].\end{aligned}$$

Then we have

$$f(a,x,y,z) = \tilde{L}(a,b;\theta_{xy})\left\{f(a,x,y,0)\right\}.$$

In 2021, Cao et al. [11] built the generalized homogeneous q-difference equations for the q-polynomials $\zeta_n^{\binom{a,b,c}{d,e}}(x,y,z|q)$.

$$\zeta_n^{\binom{a,b,c}{d,e}}(x,y,z|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{q^{\binom{k}{2}}(a,b,c;q)_k}{(d,e;q)_k}P_{n-k}(y,x)z^k.$$

Theorem 1.5 [11]. *Let $f(a,b,c,d,e,x,y,z)$ be an eight-variable analytic function in a neighborhood of $f(a,b,c,d,e,x,y,z)=(0,0,0,0,0,0,0,0)\in\mathbb{C}^8$. Then f can be expanded in terms of $\zeta_n^{\binom{a,b,c}{d,e}}(x,y,z|q)$ if and only if satisfies the q-difference equation*

$$\begin{aligned}(q^{-1}x-y)\Big\{&[f(a,b,c,d,e,x,y,z)-f(a,b,c,d,e,x,y,qz)]\\&-(d+e)q^{-1}\left[f(a,b,c,d,e,x,y,qz)-f(a,b,c,d,e,x,y,q^2z)\right]\\&+deq^{-2}\left[f(a,b,c,d,e,x,y,q^2z)-f(a,b,c,d,e,x,y,q^3z)\right]\Big\}\\=z\Big\{&\left[f(a,b,c,d,e,q^{-1}x,y,qz)-f(a,b,c,d,e,x,qy,qz)\right]\\&-(a+b+c)\left[f(a,b,c,d,e,q^{-1}x,y,q^2z)-f(a,b,c,d,e,x,qy,q^2z)\right]\\&+(ab+bc+ac)\left[f(a,b,c,d,e,q^{-1}x,y,q^3z)-f(a,b,c,d,e,x,qy,q^3z)\right]\\&-abc\left[f(a,b,c,d,e,x,q^{-1}x,y,q^4z)-f(a,b,c,d,e,x,qy,q^4z)\right]\Big\}.\end{aligned}$$

The structure of the paper is as follows: In section 2, we'll involve the generalized homogeneous q-shift operator $E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})$; subsequently, an equation for q-difference will be formulated. For the q-shift operator $E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})$, we provide operator identities. The q-difference equation method is used to establish these identities. In section 3, we provide generating functions and its extension for $\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$ by employing q-difference equation method. In section 4, we derive the Rogers type formula and its extension for $\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$ by q-difference equations technique. Mehler's formula and its extension for $\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$ are obtained in section 5 by q-difference equation approach, then we obtain a transformational identity involving generating functions for $\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$ in section 6.

2. q-Difference Equation for the Generalized Homogeneous q-Shift Operator Identities

In this section, we introduce the generalised homogeneous q-shift operator

$$E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy}).$$

Also, we define the generalized Cigler's polynomials $\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$. Then we construct a q-difference equation. We offer operator identities for the generalized homogeneous q-shift operator. These identities are established using q-difference equations approach.

Definition 2.1 We define the generalized homogeneous q-shift operator as follows:

$$\begin{aligned}&E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\\&=\sum_{k=0}^{\infty}\begin{bmatrix}\alpha\\k\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_k}{(\rho_1,\cdots,\rho_s;q)_k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}(-z\theta_{xy})^k.\end{aligned}\tag{2.1}$$

Definition 2.2 We define the generalised Cigler's polynomials as follows:

$$\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$$

$$= (-1)^n q^{-\binom{n}{2}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} \alpha \\ k \end{bmatrix} \frac{(\sigma_1, \cdots, \sigma_r; q)_k}{(\rho_1, \cdots, \rho_s; q)_k} \; (q;q)_k \; \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k P_{n-k}(y,x), \tag{2.2}$$

where $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_r)$,$\boldsymbol{\rho} = (\rho_1, \cdots, \rho_s)$.

Theorem 2.1 *Let $g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, z)$ be a $r+s+4$-variable analytic function is a neighbourhood of $(0, ..., 0) \in \mathbb{C}^{r+s+4}$ satisfying the q-difference equation*

$$\begin{aligned}
&(q^{-1}x - y) \left[\sum_{j=0}^{s} (-q)^{-j} B_j \left(g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, zq^j) - g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, zq^{j+1}) \right) \right] \\
&= z(-1)^{1+s-r} \sum_{j=0}^{r} (-1)^j A_j \left[q^\alpha \left(g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, q^{-1}x, y, zq^{j+s-r}) - g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, qy, zq^{j+s-r}) \right) \right. \\
&\qquad \left. - \left(g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, q^{-1}x, y, zq^{j+1+s-r}) - g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, qy, zq^{j+1+s-r}) \right) \right],
\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
&\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_r), \boldsymbol{\rho} = (\rho_1, \cdots, \rho_s), B_0 = A_0 = 1, B_1 = \sum_{i=1}^{s} \rho_i, B_2 = \sum_{1 \le i < j \le s} \rho_i \rho_j, \\
&B_3 = \sum_{1 \le i < j < k \le s} \rho_i \rho_j \rho_k, ..., B_s = \rho_1 \rho_2 ... \rho_s, \\
&A_1 = \sum_{i=1}^{r} \sigma_i, A_2 = \sum_{1 \le i < j \le r} \sigma_i \sigma_j, A_3 = \sum_{1 \le i < j < k \le r} \sigma_i \sigma_j \sigma_k, ..., A_r = \sigma_1 \, \sigma_2 ... \sigma_r
\end{aligned} \tag{2.4}$$

Then, we have

$$g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, z) = E(q^{-\alpha}, \sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s; q, q^\alpha z \theta_{xy}) \left\{ g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, 0) \right\}. \tag{2.5}$$

Proof: From the theory of several complex variable and Hartoge's theorem [17,18], we assume that

$$g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, z) = \sum_{n=0}^{\infty} \hbar_n(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y) z^n. \tag{2.6}$$

Substitute (2.6) into (2.3), we have

$$\begin{aligned}
&(q^{-1}x - y) \left[\sum_{j=0}^{s} (-q)^{-j} B_j \left(\sum_{n=0}^{\infty} \hbar_n(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y)(zq^j)^n - \sum_{n=0}^{\infty} \hbar_n(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y)(zq^{j+1})^n \right) \right] \\
&= z(-1)^{1+s-r} \sum_{j=0}^{r} (-1)^j A_j \left[q^\alpha \left(\sum_{n=0}^{\infty} \hbar_n(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, q^{-1}x, y)(zq^{j+s-r})^n \right. \right. \\
&\quad \left. - \sum_{n=0}^{\infty} \hbar_n(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, qy)(zq^{j+s-r})^n \right) \\
&\quad \left. - \left(\sum_{n=0}^{\infty} \hbar_n(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, q^{-1}x, y)(zq^{j+1+s-r})^n - \sum_{n=0}^{\infty} \hbar_n(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, qy)(zq^{j+1+s-r})^n \right) \right].
\end{aligned}$$

Which is equal to

$$(q^{-1}x - y) \left[\sum_{j=0}^{s} (-q)^{-j} B_j \sum_{n=0}^{\infty} (1 - q^n) \hbar_n(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y)(q^{nj}) \right] z^n$$

$$= (-1)^{1+s-r}\sum_{j=0}^{r}(-1)^j A_j\Bigg[q^{\alpha}\Bigg(\sum_{n=0}^{\infty}\hbar_n(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},q^{-1}x,y)-\sum_{n=0}^{\infty}\hbar_n(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,qy)(q^{j+s-r})^n\Bigg)$$
$$-\Bigg(\sum_{n=0}^{\infty}\hbar_n(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},q^{-1}x,y)-\sum_{n=0}^{\infty}\hbar_n(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,qy)\Bigg)(q^{j+1+s-r})^n\Bigg]z^{n+1}.$$

Equating the coefficients of z^n, we get

$$(q^{-1}x-y)\left[\sum_{j=0}^{s}(-1)^j B_j(1-q^n)q^{(n-1)j}\hbar_n(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y)\right]$$
$$=(-1)^{1+s-r}\sum_{j=0}^{r}(-1)^j A_j q^{(n-1)(j+s-r)}(q^{\alpha}-q^{n-1})\Big[\hbar_{n-1}(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},q^{-1}x,y)-\hbar_{n-1}(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,qy)\Big].$$

For each $n\geqslant 1$, we get

$$\hbar_n(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y)$$
$$=\frac{(-1)^{1+s-r}\left[\sum_{j=0}^{r}(-1)^j A_j q^{(j+s-r)(n-1)}(q^{\alpha}-q^{n-1})\right]}{(1-q^n)\sum_{j=0}^{s}(-1)^j B_j q^{(n-1)j}}$$
$$\times\frac{(\hbar_{n-1}(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},q^{-1}x,y)-\hbar_{n-1}(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,qy))}{(q^{-1}x-y)}$$
$$=\frac{(-1)^{1+s-r}q^{(n-1)(s-r)}(q^{\alpha}-q^{n-1})\sum_{j=0}^{r}(-1)^j A_j q^{(n-1)j}}{(1-q^n)\sum_{j=0}^{s}(-1)^j B_j q^{(n-1)j}}\,\theta_{xy}\left\{\hbar_{n-1}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y)\right\}$$
$$=\frac{(-1)^{1+s-r}q^{(n-1)(s-r)}q^{\alpha}(1-q^{-\alpha+n-1})\sum_{j=0}^{r}(-1)^j A_j q^{(n-1)j}}{(1-q^n)\sum_{j=0}^{s}(-1)^j B_j q^{(n-1)j}}\,\theta_{xy}\left\{\hbar_{n-1}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y)\right\}.$$

By iteration, we obtain

$$\hbar_n(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y)=\frac{(-q^{\alpha})^n(q^{-\alpha},\sigma_1,\cdots,\sigma_r;q)_n}{(q,\rho_1,\cdots,\rho_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{s-r}\theta_{xy}^n\left\{\hbar_0(\boldsymbol{\sigma},\boldsymbol{\rho},x,y)\right\}. \tag{2.7}$$

Setting $z=0$ in (2.6), we have

$$g(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,0)=\hbar_0(\boldsymbol{\sigma},\boldsymbol{\rho},x,y). \tag{2.8}$$

Substituting (2.8) into (2.7), we get

$$\hbar_n(\boldsymbol{\sigma},\boldsymbol{\rho},x,y)=\frac{(-q^{\alpha})^n(q^{-\alpha},\sigma_1,\cdots,\sigma_r;q)_n}{(q,\rho_1,\cdots,\rho_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{s-r}\theta_{xy}^n\left\{g(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,0)\right\}. \tag{2.9}$$

Substituting (2.9) into (2.6), we obtain

$$g(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)=\sum_{n=0}^{\infty}\frac{(q^{-\alpha},\sigma_1,\cdots,\sigma_r;q)_n}{(q,\rho_1,\cdots,\rho_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{s-r}(-zq^{\alpha})^n\,\theta_{xy}^n\left\{g(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,0)\right\}$$

$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\left\{g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,0)\right\}.$$

□

- Setting $\alpha\to\infty$, $r=1$, $s=0$, $\sigma_1=a$ in Theorem 2.3, we recover Theorem 1.1.
- Setting $\alpha\to\infty$, $r=s=1$, $\sigma_1=a$, $\rho_1=0$ in Theorem 2.3, we recover Theorem 1.4.

Theorem 2.2 *Let the operator $E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})$ be defined as in (2.1), then*

$$\begin{aligned}&E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\left\{(-1)^nq^{-\binom{n}{2}}P_n(y,x)\right\}\\&=\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z).\end{aligned}\tag{2.10}$$

Proof: Let $g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$ be RHS of equation (2.10).

$$g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)=\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z).$$

Now we check that $g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$ satisfies equation (2.3).

$$\begin{aligned}&(q^{-1}x-y)\sum_{j=0}^{s}(-q)^{-j}B_j\Big(g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,zq^j)-g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,zq^{j+1})\Big)\\
&=(q^{-1}x-y)\sum_{j=0}^{s}(-q)^{-j}B_j\Big(\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,zq^j)-\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,zq^{j+1})\Big)\\
&=(q^{-1}x-y)\sum_{j=0}^{s}(-q)^{-j}B_j\Bigg((-1)^nq^{-\binom{n}{2}}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}\alpha\\k\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_k}{(\rho_1,\cdots,\rho_s;q)_k}\,(q;q)_k\,\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\\
&\quad\times(zq^j)^kP_{n-k}(y,x)-(-1)^nq^{-\binom{n}{2}}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}\alpha\\k\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_k}{(\rho_1,\cdots,\rho_s;q)_k}\,(q;q)_k\,\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\\
&\qquad\times(zq^{j+1})^kP_{n-k}(y,x)\Bigg)\quad\text{(by using (2.2))}\\
&=(q^{-1}x-y)\sum_{j=0}^{s}(-q)^{-j}B_j\Bigg((-1)^nq^{-\binom{n}{2}}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}\alpha\\k\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_k}{(\rho_1,\cdots,\rho_s;q)_k}\,(q;q)_k\,\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\\
&\qquad\times P_{n-k}(y,x)((zq^j)^k-(zq^{j+1})^k)\Bigg)\\
&=(q^{-1}x-y)\sum_{j=0}^{s}(-q)^{-j}B_j\Bigg((-1)^nq^{-\binom{n}{2}}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}\alpha\\k\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_k}{(\rho_1,\cdots,\rho_s;q)_k}\,(q;q)_k\,\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\\
&\qquad\times P_{n-k}(y,x)(zq^j)^k(1-q^k)\Bigg)\\
&=(q^{-1}x-y)\sum_{j=0}^{s}(-q)^{-j}B_j\Bigg((-1)^nq^{-\binom{n}{2}}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(q^{-\alpha},\sigma_1,\cdots,\sigma_r;q)_k}{(q,\rho_1,\cdots,\rho_s;q)_k}\,(q;q)_k\,\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\\
&\qquad\times(-1)^kq^{-\binom{k}{2}+\alpha k}P_{n-k}(y,x)(zq^j)^k(1-q^k)\Bigg)\quad\text{(by using (1.3))}\\
&=(q^{-1}x-y)\sum_{j=0}^{s}(-q)^{-j}B_j\Bigg((-1)^nq^{-\binom{n}{2}}\sum_{k=0}^{n}\frac{(q^{-\alpha},\sigma_1,\cdots,\sigma_r;q)_k}{(q,\rho_1,\cdots,\rho_s;q)_k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\\
&\qquad\times(zq^j)^kq^{-\binom{k}{2}+\alpha k}\frac{(q;q)_n}{(q;q)_{n-k}}P_{n-k}(y,x)(1-q^k)\Bigg)\end{aligned}$$

$$
\begin{aligned}
&= (q^{-1}x-y)\sum_{j=0}^{s}(-q)^{-j}B_j\Bigg((-1)^n q^{-\binom{n}{2}}\sum_{k=0}^{n}\frac{(q^{-\alpha},\sigma_1,\cdots,\sigma_r;q)_k}{(q,\rho_1,\cdots,\rho_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\\
&\qquad\qquad \times (zq^j)^k(-1)^k q^{-\binom{k}{2}+\alpha k}(1-q^k)\frac{(q;q)_n}{(q;q)_{n-k}}P_{n-k}(y,x)\Bigg)\\
&= (q^{-1}x-y)\sum_{j=0}^{s}(-q)^{-j}B_j\Bigg((-1)^n q^{-\binom{n}{2}}\sum_{k=0}^{n}\frac{(q^{-\alpha},\sigma_1,\cdots,\sigma_r;q)_k}{(q,\rho_1,\cdots,\rho_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\\
&\qquad\qquad \times (zq^j)^k q^{-\binom{k}{2}+\alpha k}(1-q^k)(-1)^k\frac{(q;q)_n}{(q;q)_{n-k}}P_{n-k}(y,x)\Bigg)\\
&= (q^{-1}x-y)\sum_{j=0}^{s}(-q)^{-j}B_j\Bigg((-1)^n q^{-\binom{n}{2}}\sum_{k=0}^{n}\frac{(q^{-\alpha},\sigma_1,\cdots,\sigma_r;q)_k}{(q,\rho_1,\cdots,\rho_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\\
&\qquad\qquad \times (zq^j)^k q^{-\binom{k}{2}+\alpha k}(1-q^k)\theta_{xy}^k\Big\{P_n(y,x)\Big\}\Bigg)\quad \text{(by using (1.9))}\\
&= (q^{-1}x-y)\Bigg((-1)^n q^{-\binom{n}{2}}\sum_{k=0}^{n}\frac{(q^{-\alpha},\sigma_1,\cdots,\sigma_r;q)_k}{(q,\rho_1,\cdots,\rho_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} q^{-\binom{k}{2}+\alpha k}(1-q^k)\\
&\qquad\qquad \times z^k\theta_{xy}^k\Big\{P_n(y,x)\Big\}\sum_{j=0}^{s}(-)^j B_j q^{j(k-1)}\Bigg)\\
&= (q^{-1}x-y)\Bigg((-1)^n q^{-\binom{n}{2}}\sum_{k=0}^{n}\frac{(q^{-\alpha},\sigma_1,\cdots,\sigma_r;q)_k}{(q,\rho_1,\cdots,\rho_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} q^{-\binom{k}{2}+\alpha k}(1-q^k)\\
&\qquad\qquad \times z^k\theta_{xy}^k\Big\{P_n(y,x)\Big\}(1-B_1q^{k-1}+...+(-1)^sB_sq^{k-1})\Bigg)\\
&= (q^{-1}x-y)\Bigg((-1)^n q^{-\binom{n}{2}}\sum_{k=0}^{n}\frac{(q^{-\alpha},\sigma_1,\cdots,\sigma_r;q)_k}{(q,\rho_1,\cdots,\rho_s;q)_k}(1-\rho_1q^{k-1})...(1-\rho_sq^{k-1})\\
&\qquad\qquad \times \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k(1-q^k)q^{-\binom{k}{2}+\alpha k}\theta_{xy}^k\Big\{P_n(y,x)\Big\}\Bigg)\\
&= (q^{-1}x-y)\Bigg((-1)^n q^{-\binom{n}{2}}\sum_{k=1}^{n}\frac{(\sigma_1,\cdots,\sigma_r;q)_{k-1}}{(q,\rho_1,\cdots,\rho_s;q)_{k-1}}(q^{-\alpha};q)_k(1-\sigma_1q^{k-1})...(1-\sigma_rq^{k-1})\\
&\qquad\qquad \times \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k q^{-\binom{k}{2}+\alpha k}\theta_{xy}^k\Big\{P_n(y,x)\Big\}\Bigg)\\
&= (q^{-1}x-y)\Bigg((-1)^n q^{-\binom{n}{2}}\sum_{k=0}^{n+1}\frac{(q^{-\alpha},\sigma_1,\cdots,\sigma_r;q)_k}{(q,\rho_1,\cdots,\rho_s;q)_k}(1-\sigma_1q^{k})...(1-\sigma_rq^{k})(1-q^{-\alpha+k})\\
&\qquad\qquad \times \left[(-1)^{(k+1)} q^{\binom{k+1}{2}}\right]^{1+s-r} z^{k+1} q^{-\binom{k+1}{2}+\alpha(k+1)}\theta_{xy}^{k+1}\Big\{P_n(y,x)\Big\}\Bigg)\\
&= (q^{-1}x-y)\Bigg((-1)^n q^{-\binom{n}{2}}\sum_{k=0}^{n+1}\frac{(q^{-\alpha},\sigma_1,\cdots,\sigma_r;q)_k}{(q,\rho_1,\cdots,\rho_s;q)_k}(1-\sigma_1q^{k})...(1-\sigma_rq^{k})q^{-\alpha}(q^{\alpha}-q^{k})\\
&\qquad\qquad \times \left[(-1)^{(k+1)} q^{\binom{k+1}{2}}\right]^{1+s-r} z^{k+1} q^{-\binom{k+1}{2}+\alpha(k+1)}\theta_{xy}^{k+1}\Big\{P_n(y,x)\Big\}\Bigg)\\
&= (-1)^{1+s-r}(q^{-1}x-y)\Bigg((-1)^n q^{-\binom{n}{2}}\sum_{k=0}^{n+1}\frac{(q^{-\alpha},\sigma_1,\cdots,\sigma_r;q)_k}{(q,\rho_1,\cdots,\rho_s;q)_k}(1-\sigma_1q^{k})...(1-\sigma_rq^{k})(q^{\alpha}-q^{k})\\
&\quad \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^{k+1} q^{-\binom{k}{2}+\alpha k+k(s-r)}\,\theta_{xy}^{k+1}\Big\{P_n(y,x)\Big\}\Bigg)
\end{aligned}
$$

$$= (-1)^{1+s-r}\, z(q^{-1}x-y)\Bigg((-1)^n q^{-\binom{n}{2}} \sum_{k=0}^{n+1} \begin{bmatrix}\alpha\\k\end{bmatrix} \frac{(\sigma_1,\cdots,\sigma_r;q)_k}{(q,\rho_1,\cdots,\rho_s;q)_k}(1-\sigma_1 q^k)...(1-\sigma_r q^k)$$

$$\times (q^\alpha - q^k)\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k (-1)^k q^{k(s-r)}\, \theta_{xy}^{k+1}\Big\{P_n(y,x)\Big\}\Bigg) \quad \text{(by using (1.3))}$$

$$= (-1)^{1+s-r}\, z(q^{-1}x-y)\Bigg((-1)^n q^{-\binom{n}{2}} \sum_{k=0}^{n+1} \begin{bmatrix}\alpha\\k\end{bmatrix} \frac{(\sigma_1,\cdots,\sigma_r;q)_k}{(q,\rho_1,\cdots,\rho_s;q)_k} \sum_{j=0}^{r}(-1)^j A_j q^{k(j-1)}$$

$$\times (q^\alpha - q^k)\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k (-1)^k q^{k(s-r)}\, \theta_{xy}^{k+1}\Big\{P_n(y,x)\Big\}\Bigg)$$

$$= (-1)^{1+s-r}\, z(q^{-1}x-y)\Bigg((-1)^n q^{-\binom{n}{2}} \sum_{j=0}^{r}(-1)^j A_j \sum_{k=0}^{n+1} \begin{bmatrix}\alpha\\k\end{bmatrix} \frac{(\sigma_1,\cdots,\sigma_r;q)_k}{(q,\rho_1,\cdots,\rho_s;q)_k}(q^\alpha - q^k)$$

$$\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k (-1)^k q^{k(j+s-r)}\, \theta_{xy}^{k+1}\Big\{P_n(y,x)\Big\}\Bigg)$$

$$= (-1)^{1+s-r}\, z(q^{-1}x-y)\Bigg((-1)^n q^{-\binom{n}{2}} \sum_{j=0}^{r}(-1)^j A_j \sum_{k=0}^{n+1} \begin{bmatrix}\alpha\\k\end{bmatrix} \frac{(\sigma_1,\cdots,\sigma_r;q)_k}{(q,\rho_1,\cdots,\rho_s;q)_k}(q^\alpha - q^k)$$

$$\times \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k (-1)^k q^{k(j+s-r)}\, \theta_{xy}\Big\{(-1)^k \frac{(q;q)n}{(q;q)_{n-k}} P_{n-k}(y,x)\Big\}\Bigg) \quad \text{(by using (1.9))}$$

$$= (-1)^{1+s-r}\, z\Bigg((-1)^n q^{-\binom{n}{2}} \sum_{j=0}^{r}(-1)^j A_j \sum_{k=0}^{n+1} \begin{bmatrix}\alpha\\k\end{bmatrix}\begin{bmatrix}n\\k\end{bmatrix} \frac{(\sigma_1,\cdots,\sigma_r;q)_k}{(q,\rho_1,\cdots,\rho_s;q)_k}(q^\alpha - q^k)$$

$$\times \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k q^{k(j+s-r)}\Big(P_{n-k}(y,q^{-1}x) - P_{n-k}(qy,x)\Big)\Bigg)$$

$$= (-1)^{1+s-r}\, z \sum_{j=0}^{r}(-1)^j A_j \Bigg[(-1)^n q^{-\binom{n}{2}} \sum_{k=0}^{n+1} \begin{bmatrix}\alpha\\k\end{bmatrix}\begin{bmatrix}n\\k\end{bmatrix} \frac{(\sigma_1,\cdots,\sigma_r;q)_k}{(q,\rho_1,\cdots,\rho_s;q)_k}(q^\alpha - q^k)$$

$$\times \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} (zq^{j+s-r})^k \Big(P_{n-k}(y,q^{-1}x) - P_{n-k}(qy,x)\Big)\Bigg]$$

$$= (-1)^{1+s-r}\, z \sum_{j=0}^{r}(-1)^j A_j \Bigg[q^\alpha\Big(\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},q^{-1}x,y,zq^{j+s-r}) - \mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,qy,zq^{j+s-r})\Big)$$

$$-\Big(\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},q^{-1}x,y,zq^{j+1+s-r}) - \mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,qy,zq^{j+1+s-r})\Big)\Bigg]$$

$$= (-1)^{1+s-r}\, z \sum_{j=0}^{r}(-1)^j A_j \Bigg[q^\alpha\Big(g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,zq^{j+s-r}) - g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,zq^{j+s-r})\Big)$$

$$-\Big(g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,zq^{j+1+s-r}) - g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,zq^{j+1+s-r})\Big)\Bigg].$$

Therefore, the q-difference equation (2.3) is satisfied by $g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$, and by using (2.5), we obtain

$$\begin{aligned} g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z) &= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\{g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,0)\} \\ &= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\Big\{(-1)^n q^{-\binom{n}{2}} P_n(y,x)\Big\}. \end{aligned}$$

□

Theorem 2.3 *Let the operator $E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})$ be defined as in equation (2.1), then*

$$E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{\frac{(xt;q)_\infty}{(yt;q)_\infty}\right\}$$

$$=\frac{(xt;q)_\infty}{(yt;q)_\infty}\sum_{k=0}^{\infty}\begin{bmatrix}\alpha\\k\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_k}{(\rho_1,\cdots,\rho_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}(zt)^k,\quad |yt|<1. \tag{2.11}$$

Proof: Denoting RHS of equation (2.11) by $g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$. We can check that $g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$ satisfies the q-difference equation (2.3) by the same method used in Theorem 2.2. Note that

$$g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,0)=\frac{(xt;q)_\infty}{(yt;q)_\infty}.$$

Hence, by using (2.5), we get

$$g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)=E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{\frac{(xt;q)_\infty}{(yt;q)_\infty}\right\}.$$

□

Theorem 2.4 *Let the operator $E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})$ be defined as in equation (2.1), then*

$$\begin{aligned}&E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{\frac{P_k(y,x)}{(xt;q)_k}\frac{(xt;q)_\infty}{(yt;q)_\infty}\right\}\\&=\frac{(xt;q)_\infty}{(yt;q)_\infty}t^{-k}\sum_{j=0}^{k}\frac{(q^{-k},yt;q)_j}{(q,xt;q)_j}q^j\sum_{\varrho=0}^{\infty}\begin{bmatrix}\alpha\\\varrho\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_\varrho}{(\rho_1,\cdots,\rho_s;q)_\varrho}\left[(-1)^\varrho q^{\binom{\varrho}{2}}\right]^{1+s-r}(ztq^j)^\varrho,\end{aligned} \tag{2.12}$$

provided that $|yt|<1$.

Proof: Denoting RHS of equation (2.12) by $g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$. We can check that $g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$ satisfies the q-difference equation (2.3) by the same method used in Theorem 2.2, so we have

$$\begin{aligned}g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,0)&=\frac{(xt;q)_\infty}{(yt;q)_\infty}t^{-k}\sum_{j=0}^{k}\frac{(q^{-k},yt;q)_j}{(q,xt;q)_j}q^j\\&=\frac{P_k(y,x)}{(xt;q)_k}\frac{(xt;q)_\infty}{(yt;q)_\infty}.\quad\text{(by using (1.4))}\end{aligned}$$

By using (2.5), the proof is completed. □

Theorem 2.5 *Let $g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$ be a $r+s+4$-variable analytic function is a neighbourhood of $(0,...,0)\in\mathbb{C}^{r+s+4}$ satisfying (2.3) and $g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$ has the following equation:*

$$g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,0)=\sum_{n=0}^{\infty}\eta_n P_n(y,x). \tag{2.13}$$

Then, we have

$$g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)=\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\eta_n\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z). \tag{2.14}$$

Proof: From Theorem 2.1, we have

$$\begin{aligned}&g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)\\&=E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\{g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,0)\}\end{aligned}$$

$$= E(q^{-\alpha}, \sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s; q, q^{\alpha} z\theta_{xy}) \left\{ \sum_{n=0}^{\infty} \eta_n P_n(y, x) \right\} \quad \text{(by using (2.13))}$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} \eta_n E(q^{-\alpha}, \sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s; q, q^{\alpha} z\theta_{xy}) \left\{ (-1)^n q^{-\binom{n}{2}} P_n(y, x) \right\}$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} \eta_n \mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, z). \quad \text{(by using (2.10))}$$

□

- For $r = s = 0$, $z = b$ and $y = 0$ in Theorem 2.5, we get Theorem 1.2.
- For $r = s = 0$, $z = b$ in in Theorem 2.5, we find Theorem 1.3.
- For $\alpha \to \infty$, $r = s = 3$, $(\sigma_1, \sigma_2, \sigma_3) = (d, e, f)$, $(\rho_1, \rho_2) = (g, h)$ and $c = z$ in Theorem 2.5, we obtain Theorem 1.5.

3. The Generating Function for $\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, z)$

In this section, q-difference equation technique is utilised to derive the generating function and its extension for the polynomials $\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, z)$.

Theorem 3.1 (Generating function for $\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, z)$). *We have*

$$\sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} \mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, z) \frac{t^n}{(q;q)_n} = \frac{(xt;q)_\infty}{(yt;q)_\infty}$$
$$\times \sum_{k=0}^{\infty} \begin{bmatrix} \alpha \\ k \end{bmatrix} \frac{(\sigma_1, \cdots, \sigma_r; q)_k}{(\rho_1, \cdots, \rho_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} (zt)^k, \quad |yt| < 1. \tag{3.1}$$

Proof:

Let RHS of (3.1) be represented by $g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, z)$.

$$g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, z) = \frac{(xt;q)_\infty}{(yt;q)_\infty} \sum_{k=0}^{\infty} \begin{bmatrix} \alpha \\ k \end{bmatrix} \frac{(\sigma_1, \cdots, \sigma_r; q)_k}{(\rho_1, \cdots, \rho_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} (zt)^k.$$

We now confirm that $g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, z)$ satisfies (2.3).

$$(q^{-1}x - y) \sum_{j=0}^{s} (-q)^{-j} B_j \Big(g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, zq^j) - g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, zq^{j+1}) \Big)$$

$$= (q^{-1}x - y) \sum_{j=0}^{s} (-q)^{-j} B_j \Bigg(\frac{(xt;q)_\infty}{(yt;q)_\infty} \sum_{k=0}^{\infty} \begin{bmatrix} \alpha \\ k \end{bmatrix} \frac{(\sigma_1, \cdots, \sigma_r; q)_k}{(\rho_1, \cdots, \rho_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} (zq^j t)^k$$
$$- \frac{(xt;q)_\infty}{(yt;q)_\infty} \sum_{k=0}^{\infty} \begin{bmatrix} \alpha \\ k \end{bmatrix} \frac{(\sigma_1, \cdots, \sigma_r; q)_k}{(\rho_1, \cdots, \rho_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} (zq^{j+1} t)^k \Bigg)$$

$$= (q^{-1}x - y) \frac{(xt;q)_\infty}{(yt;q)_\infty} \sum_{j=0}^{s} (-q)^{-j} B_j \sum_{k=0}^{\infty} \begin{bmatrix} \alpha \\ k \end{bmatrix} \frac{(\sigma_1, \cdots, \sigma_r; q)_k}{(\rho_1, \cdots, \rho_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r}$$
$$\times \Big((zq^j t)^k - (zq^{j+1} t)^k \Big)$$

$$= (q^{-1}x - y) \frac{(xt;q)_\infty}{(yt;q)_\infty} \sum_{j=0}^{s} (-q)^{-j} B_j \sum_{k=0}^{\infty} \begin{bmatrix} \alpha \\ k \end{bmatrix} \frac{(\sigma_1, \cdots, \sigma_r; q)_k}{(\rho_1, \cdots, \rho_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} (zq^j t)^k$$

$$
\begin{aligned}
&\times (1-q^k)\\
&= (q^{-1}x-y)\frac{(xt;q)_\infty}{(yt;q)_\infty}\sum_{k=0}^{\infty}\begin{bmatrix}\alpha\\k\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_k}{(\rho_1,\cdots,\rho_s;q)_k}\ \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}(zq^jt)^k(1-q^k)\\
&\times \sum_{j=0}^{s}(-1)^j B_j q^{j(k-1)}\\
&= (q^{-1}x-y)\frac{(xt;q)_\infty}{(yt;q)_\infty}\sum_{k=0}^{\infty}\begin{bmatrix}\alpha\\k\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_k}{(\rho_1,\cdots,\rho_s;q)_k}\ \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}(zq^jt)^k(1-q^k)\\
&\times \left(1-B_1q^{k-1}+\cdots+(-1)^s B_s q^{s(k-1)}\right)\\
&= (q^{-1}x-y)\frac{(xt;q)_\infty}{(yt;q)_\infty}\sum_{k=0}^{\infty}\begin{bmatrix}\alpha\\k\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_k}{(\rho_1,\cdots,\rho_s;q)_k}\ \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}(zq^jt)^k(1-q^k)\\
&\times (1-\rho_1q^{k-1})\cdots(1-\rho_sq^{k-1})\\
&= (q^{-1}x-y)\frac{(xt;q)_\infty}{(yt;q)_\infty}\sum_{k=0}^{\infty}\frac{(q^{-\alpha},\sigma_1,\cdots,\sigma_r;q)_k}{(q,\rho_1,\cdots,\rho_s;q)_k}\ \left[(-1)^k q^{\binom{k}{2}}\right]^{s-r}q^{\alpha k}(zq^jt)^k(1-q^k)\\
&\times (1-\rho_1q^{k-1})\cdots(1-\rho_sq^{k-1})\quad \text{(by using (1.3))}\\
&= (q^{-1}x-y)\frac{(xt;q)_\infty}{(yt;q)_\infty}\sum_{k=1}^{\infty}\frac{(q^{-\alpha},\sigma_1,\cdots,\sigma_r;q)_{k-1}}{(q,\rho_1,\cdots,\rho_s;q)_{k-1}}\ \left[(-1)^k q^{\binom{k}{2}}\right]^{s-r}q^{\alpha k}(zq^jt)^k\\
&\times (1-q^{-\alpha+k-1})(1-\sigma_1q^{k-1})\cdots(1-\sigma_rq^{k-1})\\
&= (q^{-1}x-y)\frac{(xt;q)_\infty}{(yt;q)_\infty}\sum_{k=0}^{\infty}\frac{(q^{-\alpha},\sigma_1,\cdots,\sigma_r;q)_k}{(q,\rho_1,\cdots,\rho_s;q)_k}\ \left[(-1)^{k+1} q^{\binom{k+1}{2}}\right]^{s-r}q^{\alpha(k+1)}(zq^jt)^{k+1}\\
&\times (1-q^{-\alpha+k})(1-\sigma_1q^{k})\cdots(1-\sigma_rq^{k})\\
&= (-1)^{s-r}(q^{-1}x-y)(zt)\frac{(xt;q)_\infty}{(yt;q)_\infty}\sum_{k=0}^{\infty}\frac{(q^{-\alpha},\sigma_1,\cdots,\sigma_r;q)_k}{(q,\rho_1,\cdots,\rho_s;q)_k}\ \left[(-1)^k q^{\binom{k}{2}}\right]^{s-r}q^{\alpha(k+1)}(zt)^kq^{j(k+1)}\\
&\times q^{k(s-r)}q^{-\alpha}(q^\alpha-q^k)(1-\sigma_1q^{k})\cdots(1-\sigma_rq^{k})\\
&= (-1)^{s-r}(q^{-1}x-y)(zt)\frac{(xt;q)_\infty}{(yt;q)_\infty}\sum_{k=0}^{\infty}\begin{bmatrix}\alpha\\k\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_k}{(\rho_1,\cdots,\rho_s;q)_k}\ \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}(zt)^kq^{j(k+1)}\\
&\times q^{k(s-r)}(q^\alpha-q^k)(1-\sigma_1q^{k})\cdots(1-\sigma_rq^{k})\quad \text{(by using (1.3))}\\
&= (-1)^{s-r}(q^{-1}x-y)(zt)\frac{(xt;q)_\infty}{(yt;q)_\infty}\sum_{k=0}^{\infty}\begin{bmatrix}\alpha\\k\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_k}{(\rho_1,\cdots,\rho_s;q)_k}\ \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}(zt)^k\\
&\times q^{k(s-r)}(q^\alpha-q^k)\sum_{j=0}^{r}(-1)^jA_jq^{kj}\\
&= (-1)^{s-r}z(q^{-1}xt-yt)\frac{(xt;q)_\infty}{(yt;q)_\infty}\sum_{j=0}^{r}(-1)^jA_j\sum_{k=0}^{\infty}\begin{bmatrix}\alpha\\k\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_k}{(\rho_1,\cdots,\rho_s;q)_k}\ \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}(zt)^k\\
&\times q^{k(j+s-r)}(q^\alpha-q^k)\\
&= (-1)^{1+s-r}z\left(\frac{(q^{-1}xt;q)_\infty}{(yt;q)_\infty}-\frac{(xt;q)_\infty}{(qyt;q)_\infty}\right)\sum_{j=0}^{r}(-1)^jA_j\sum_{k=0}^{\infty}\begin{bmatrix}\alpha\\k\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_k}{(\rho_1,\cdots,\rho_s;q)_k}\\
&\times\ \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}(ztq^{j+s-r})^k(q^\alpha-q^k)
\end{aligned}
$$

$$= (-1)^{1+s-r}\, z \sum_{j=0}^{r} (-1)^j A_j \Bigg[q^{\alpha} \Big(g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, zq^{j+s-r}) - g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, zq^{j+s-r}) \Big)$$
$$- \Big(g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, zq^{j+1+s-r}) - g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, zq^{j+1+s-r}) \Big) \Bigg].$$

Hence $g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, z)$ satisfies the q-difference equation (2.3). Note that

$$\begin{aligned} g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, 0) &= \frac{(xt;q)_\infty}{(yt;q)_\infty} \\ &= \sum_{n=0}^{\infty} P_n(y,x) \frac{t^n}{(q;q)_n}. \quad \text{(by using (1.1))} \end{aligned}$$

By using (2.14), we obtain

$$g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, c) = \sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} \mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, z) \frac{t^n}{(q;q)_n}.$$

Hence, we get the required result. $\square$

Theorem 3.2 (Extended generating function for $\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, z)$). *We have*

$$\begin{aligned} &\sum_{n=0}^{\infty} (-1)^{n+k} q^{\binom{n+k}{2}} \mathbb{D}_{n+k}^{(\alpha-n-k)}(\boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, z) \frac{t^n}{(q;q)_n} \\ &\quad = \frac{(xt;q)_\infty}{(yt;q)_\infty} t^{-k} \sum_{j=0}^{k} \frac{(q^{-k}, yt; q)_j}{(q, xt; q)_j}\, q^j \sum_{\varrho=0}^{\infty} \begin{bmatrix} \alpha \\ \varrho \end{bmatrix} \frac{(\sigma_1, \cdots, \sigma_r; q)_\varrho}{(\rho_1, \cdots, \rho_s; q)_\varrho} \left[(-1)^{\varrho} q^{\binom{\varrho}{2}} \right]^{1+s-r} (ztq^j)^{\varrho}, \end{aligned} \tag{3.2}$$

where $|yt| < 1$.

Proof: Denoting RHS of equation (3.2) by $g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, z)$. We can check that $g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, z)$ satisfies the q-difference equation (2.3) by the same method used in Theorem 3.1, so we have

$$\begin{aligned} g(\boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, 0) &= \frac{(xt;q)_\infty}{(yt;q)_\infty}\, t^{-k} \sum_{j=0}^{k} \frac{(q^{-k}, yt; q)_j}{(q, xt; q)_j} q^j \\ &= \frac{(xt;q)_\infty}{(yt;q)_\infty} \frac{(x/y;q)_k}{(xt;q)_k} y^k \quad \text{(by using (1.4))} \\ &= P_k(y,x) \frac{(q^k xt;q)_\infty}{(yt;q)_\infty} \\ &= P_k(y,x) \sum_{n=0}^{\infty} P_n(y, q^k x) \frac{t^n}{(q;q)_n} \quad \text{(by using (1.7))} \\ &= \sum_{n=0}^{\infty} P_{n+k}(y,x) \frac{t^n}{(q;q)_n}. \end{aligned}$$

By using equation (2.14), we get

$$g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, z) = \sum_{n=0}^{\infty} (-1)^{n+k} q^{\binom{n+k}{2}} \mathbb{D}_{n+k}^{(\alpha-n-k)}(\boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, z) \frac{t^n}{(q;q)_n}.$$

Hence the proof is completed. $\square$

4. Rogers Formula for $\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$

In this section, we will describe the q-difference equations approach to Rogers' formula and its extension for the polynomials $\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$.

Theorem 4.1 (Roger's Formula for $\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$). *We have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}q^{\binom{n+m}{2}}\mathbb{D}_{n+m}^{(\alpha-n-m)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}=\frac{(x\ell;q)_\infty}{(t/\ell,y\ell;q)_\infty}$$
$$\times\sum_{k=0}^{\infty}\frac{(y\ell;q)_k}{(q,x\ell,q\ell/t;q)_k}q^k\sum_{\varrho=0}^{\infty}\begin{bmatrix}\alpha\\ \varrho\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_\varrho}{(\rho_1,\cdots,\rho_s;q)_\varrho}\left[(-1)^\varrho q^{\binom{\varrho}{2}}\right]^{1+s-r}(z\ell q^k)^\varrho, \tag{4.1}$$

where $\max\{|t/\ell|,|y\ell|\}<1$.

Proof: Let $g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$ denote RHS of (4.1). With the same technique as in Theorem 3.1, we can verify that $g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$ fulfills the q-difference equation (2.3). Therefore, we have

$$\begin{aligned}
&g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)\\
&=E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,0)\right\}\\
&=E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{\frac{(x\ell;q)_\infty}{(t/\ell,y\ell;q)_\infty}\sum_{k=0}^{\infty}\frac{(y\ell;q)_k}{(q,x\ell,q\ell/t;q)_k}q^k\right\}\\
&=E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{\frac{(x\ell;q)_\infty}{(y\ell;q)_\infty}\sum_{n=0}^{\infty}\frac{P_n(y,x)}{(x\ell;q)_n}\frac{t^n}{(q;q)_n}\right\}\quad\text{(by using (1.6))}\\
&=E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{\sum_{n=0}^{\infty}P_n(y,x)\frac{(x\ell q^n;q)_\infty}{(y\ell;q)_\infty}\frac{t^n}{(q;q)_n}\right\}\\
&=E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{\sum_{n=0}^{\infty}P_n(y,x)\frac{t^n}{(q;q)_n}\sum_{k=0}^{\infty}P_k(y,q^nx)\frac{\ell^k}{(q;q)_k}\right\}\\
&\qquad\text{(by using (1.7))}\\
&=E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}P_{n+k}(y,x)\frac{t^n}{(q;q)_n}\frac{\ell^k}{(q;q)_k}\right\}\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+k}q^{\binom{n+k}{2}}E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{(-1)^{n+k}q^{-\binom{n+k}{2}}P_{n+k}(y,x)\right\}\\
&\quad\times\frac{t^n}{(q;q)_n}\frac{\ell^k}{(q;q)_k}\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+k}q^{\binom{n+k}{2}}\mathbb{D}_{n+k}^{(\alpha-n-k)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)\frac{t^n}{(q;q)_n}\frac{\ell^k}{(q;q)_k}.\quad\text{(by using (2.10))}
\end{aligned}$$

□

Theorem 4.2 (Extended Rogers formula for $\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$). *we have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+m+k}q^{\binom{n+m+k}{2}}\mathbb{D}_{n+m+k}^{(\alpha-n-m-k)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}\frac{\tau^k}{(q;q)_k}$$
$$=\frac{(x\tau;q)_\infty}{(y\tau,t/\tau,\ell/\tau;q)_\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(y\tau,q)_{i+j}(-1)^{i+j}q^{-\binom{i+j}{2}}}{(x\tau,q^{-(i+j)}\ell/\tau;q)_{i+j}(q^{-i}t/\tau;q)_i}\frac{(t/\tau)^i}{(q;q)_i}\frac{(\ell/\tau)^j}{(q;q)_j}$$

$$\times \sum_{\varrho=0}^{\infty} \begin{bmatrix} \alpha \\ \varrho \end{bmatrix} \frac{(\sigma_1, \cdots, \sigma_r; q)_\varrho}{(\rho_1, \cdots, \rho_s; q)_\varrho} \left[(-1)^\varrho q^{\binom{\varrho}{2}}\right]^{1+s-r} (z\tau q^{i+j})^\varrho, \tag{4.2}$$

where $\max\{|y\tau|, |t/\tau|, |\ell/\tau|\} < 1$.

Proof: Let $g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, z)$ indicate the RHS of (4.2). Using the same approach as in Theorem 3.1, we can verify that $g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, z)$ satisfies (2.3). Hence, we have

$$\begin{aligned}
&g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, z) \\
&= E(q^{-\alpha}, \sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s; q, q^\alpha z\theta_{xy}) \left\{g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, 0)\right\} \\
&= E(q^{-\alpha}, \sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s; q, q^\alpha z\theta_{xy}) \Bigg\{ \frac{(x\tau; q)_\infty}{(y\tau, t/\tau, \ell/\tau; q)_\infty} \\
&\quad \times \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(y\tau, q)_{i+j}(-1)^{i+j} q^{-\binom{i+j}{2}}}{(x\tau, q^{-(i+j)}\ell/\tau; q)_{i+j} (q^{-i}t/\tau; q)_i} \frac{(t/\tau)^i}{(q;q)_i} \frac{(\ell/\tau)^j}{(q;q)_j} \Bigg\} \\
&= E(q^{-\alpha}, \sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s; q, q^\alpha z\theta_{xy}) \Bigg\{ \frac{(x\tau; q)_\infty}{(y\tau; q)_\infty} \\
&\quad \times \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(y\tau, q)_{i+j}(-1)^{i+j} q^{-\binom{i+j}{2}}}{(x\tau; q)_{i+j}} \frac{(t/\tau)^i}{(q;q)_i} \frac{(\ell/\tau)^j}{(q;q)_j} \frac{1}{(q^{-(i+j)}\ell/\tau; q)_\infty (q^{-i}t/\tau; q)_\infty} \Bigg\} \\
&= E(q^{-\alpha}, \sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s; q, q^\alpha z\theta_{xy}) \Bigg\{ \frac{(x\tau; q)_\infty}{(y\tau; q)_\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(y\tau, q)_{i+j}(-1)^{i+j} q^{-\binom{i+j}{2}}}{(x\tau; q)_{i+j}} \\
&\quad \times \frac{(t/\tau)^i}{(q;q)_i} \frac{(\ell/\tau)^j}{(q;q)_j} \sum_{m=0}^{\infty} \frac{(q^{-(i+j)}\ell/\tau)^m}{(q;q)_m} \sum_{n=0}^{\infty} \frac{(q^{-i}t/\tau)^n}{(q;q)_n} \Bigg\} \quad \text{(by using (1.2))} \\
&= E(q^{-\alpha}, \sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s; q, q^\alpha z\theta_{xy}) \Bigg\{ \frac{(x\tau; q)_\infty}{(y\tau; q)_\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(y\tau, q)_{i+j}(-1)^{i+j} q^{-\binom{i+j}{2}}}{(x\tau; q)_{i+j}} \\
&\quad \times \frac{(t/\tau)^{n+i}}{(q;q)_i} \frac{(\ell/\tau)^{m+j}}{(q;q)_j} \frac{q^{-m(i+j)}}{(q;q)_m} \frac{q^{-ni}}{(q;q)_n} \Bigg\} \\
&= E(q^{-\alpha}, \sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s; q, q^\alpha z\theta_{xy}) \Bigg\{ \frac{(x\tau; q)_\infty}{(y\tau; q)_\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{n=i}^{\infty} \sum_{m=j}^{\infty} \frac{(y\tau, q)_{i+j}(-1)^{i+j} q^{-\binom{i+j}{2}}}{(x\tau; q)_{i+j}} \\
&\quad \times \frac{(t/\tau)^{n}}{(q;q)_i} \frac{(\ell/\tau)^{m}}{(q;q)_j} \frac{q^{-(m-j)(i+j)}}{(q;q)_{m-j}} \frac{q^{-(n-i)i}}{(q;q)_{n-i}} \Bigg\} \\
&= E(q^{-\alpha}, \sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s; q, q^\alpha z\theta_{xy}) \Bigg\{ \frac{(x\tau; q)_\infty}{(y\tau; q)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(t/\tau)^n}{(q;q)_n} \frac{(\ell/\tau)^m}{(q;q)_m} \\
&\quad \times \sum_{i=0}^{n} \sum_{j=0}^{m} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} \frac{(y\tau, q)_{i+j}(-1)^{i+j} q^{-\binom{i+j}{2}} q^{-(m-j)(i+j)} q^{-(n-i)i}}{(x\tau; q)_{i+j}} \Bigg\} \\
&= E(q^{-\alpha}, \sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s; q, q^\alpha z\theta_{xy}) \Bigg\{ \frac{(x\tau; q)_\infty}{(y\tau; q)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(t/\tau)^n}{(q;q)_n} \frac{(\ell/\tau)^m}{(q;q)_m} \\
&\quad \times \sum_{i=0}^{n} \sum_{k-i=0}^{m} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} m \\ k-i \end{bmatrix} \frac{(y\tau, q)_{k}(-1)^{k} q^{\binom{k}{2}+(n-i)(k-i)-(n+m)k+k}}{(x\tau; q)_{k}} \Bigg\} \\
&= E(q^{-\alpha}, \sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s; q, q^\alpha z\theta_{xy}) \Bigg\{ \frac{(x\tau; q)_\infty}{(y\tau; q)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(t/\tau)^n}{(q;q)_n} \frac{(\ell/\tau)^m}{(q;q)_m}
\end{aligned}$$

$$\times \sum_{k=0}^{n+m}\sum_{i=0}^{k}\begin{bmatrix} n\\ i\end{bmatrix}\begin{bmatrix} m\\ k-i\end{bmatrix} q^{(n-i)(k-i)}\frac{(y\tau,q)_k(-1)^k q^{\binom{k}{2}-(n+m)k+k}}{(x\tau;q)_k}\Big\}$$

$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\Big\{\frac{(x\tau;q)_\infty}{(y\tau;q)_\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(t/\tau)^n}{(q;q)_n}\frac{(\ell/\tau)^m}{(q;q)_m}$$

$$\times \sum_{k=0}^{n+m}\begin{bmatrix} n+m\\ k\end{bmatrix}\frac{(y\tau,q)_k(-1)^k q^{\binom{k}{2}-(n+m)k+k}}{(x\tau;q)_k}\Big\} \quad \text{(by using (1.5))}$$

$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\Big\{\frac{(x\tau;q)_\infty}{(y\tau;q)_\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(t/\tau)^n}{(q;q)_n}\frac{(\ell/\tau)^m}{(q;q)_m}$$

$$\times \sum_{k=0}^{n+m}\frac{(q^{-(n+m)},y\tau,q)_k q^k}{(x\tau;q)_k}\Big\} \quad \text{(by using (1.3))}$$

$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\Big\{\frac{(x\tau;q)_\infty}{(y\tau;q)_\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(t/\tau)^n}{(q;q)_n}\frac{(\ell/\tau)^m}{(q;q)_m}$$

$$\times \frac{(x/y,q)_{n+m}(y\tau)^{n+m}}{(x\tau;q)_{n+m}}\Big\} \quad \text{(by using (1.4))}$$

$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\Big\{\frac{(x\tau;q)_\infty}{(y\tau;q)_\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}\frac{P_{n+m}(y,x)}{(x\tau;q)_{n+m}}\Big\}$$

$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\Big\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{P_{n+m}(y,x)}{(x\tau;q)_{n+m}}\frac{(x\tau;q)_\infty}{(y\tau;q)_\infty}\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}\Big\}$$

$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\Big\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}P_{n+m}(y,x)\frac{(q^{n+m}x\tau;q)_\infty}{(y\tau;q)_\infty}\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}\Big\}$$

$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\Big\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}P_{n+m}(y,x)\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}$$

$$\times \sum_{k=0}^{\infty}P_k(y,q^{n+m}x)\frac{\tau^k}{(q;q)_k}\Big\} \quad \text{(by using (1.7))}$$

$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\Big\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}P_{n+m+k}(y,x)\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}\frac{\tau^k}{(q;q)_k}\Big\}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+m+k}q^{\binom{n+m+k}{2}}$$

$$\times E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\Big\{(-1)^{n+m+k}q^{-\binom{n+m+k}{2}}P_{n+m+k}(y,x)\Big\}$$

$$\times \frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}\frac{\tau^k}{(q;q)_k}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+m+k}q^{\binom{-n+m+k}{2}}\mathbb{D}_{n+m+k}^{(\alpha-n-m-k)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}\frac{\tau^k}{(q;q)_k}.$$

(by using (2.10))

□

5. Mehler's Formula for $\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$

In this section, we demonstrate Mehler's formula and its extension for the polynomials $\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$ utilizing q-difference equation method.

Theorem 5.1 (Mehler's formula for $\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$). *We have*

$$\sum_{n=0}^{\infty} q^{n^2-n}\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)\mathbb{D}_n^{(\beta-n)}(\boldsymbol{c},\boldsymbol{d},u,v,w)\frac{t^n}{(q;q)_n}$$
$$= \frac{(xvt;q)_\infty}{(yvt;q)_\infty}\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}\frac{(c_1,\ldots,c_r;q)_k}{(d_1,\ldots,d_s;q)_k}(w/v)^k\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(u/v)^n}{(q;q)_n}$$
$$\times\sum_{\jmath=0}^{n+k}\frac{(q^{-(n+k)},yvt;q)_\jmath}{(q,xvt;q)_\jmath}\,q^\jmath\sum_{\varrho=0}^{\infty}\begin{bmatrix}\alpha\\\varrho\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_\varrho}{(\rho_1,\cdots,\rho_s;q)_\varrho}\left[(-1)^\varrho q^{\binom{\varrho}{2}}\right]^{1+s-r}(zvtq^\jmath)^\varrho, \tag{5.1}$$

where $|yvt|<1$.

Proof: Let $g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$ denote RHS of (5.1). Using the same procedure as in Theorem 3.1, we can establish that $g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$ satisfies the q-difference equation (2.3). Hence we have

$$g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$$
$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,0)\right\}$$
$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{\frac{(xvt;q)_\infty}{(yvt;q)_\infty}\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}\frac{(c_1,\ldots,c_r;q)_k}{(d_1,\ldots,d_s;q)_k}(w/v)^k\right.$$
$$\left.\times\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(u/v)^n}{(q;q)_n}\sum_{\jmath=0}^{n+k}\frac{(q^{-(n+k)},yvt;q)_\jmath}{(q,xvt;q)_\jmath}\,q^\jmath\right\}$$
$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{\frac{(xvt;q)_\infty}{(yvt;q)_\infty}\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}\frac{(c_1,\ldots,c_r;q)_k}{(d_1,\ldots,d_s;q)_k}(w/v)^k\right.$$
$$\left.\times\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(u/v)^n}{(q;q)_n}\frac{P_{n+k}(y,x)}{(xvt;q)_{n+k}}(vt)^{n+k}\right\}\quad\text{(by using (1.4))}$$
$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}\frac{(c_1,\ldots,c_r;q)_k}{(d_1,\ldots,d_s;q)_k}(wt)^k P_k(y,x)\right.$$
$$\left.\times\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(ut)^n}{(q;q)_n}\frac{P_n(y,q^k x)}{(xvtq^k;q)_n}\frac{(xvtq^k;q)_\infty}{(yvt;q)_\infty}\right\}$$
$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}\frac{(c_1,\ldots,c_r;q)_k}{(d_1,\ldots,d_s;q)_k}P_k(y,x)(wt)^k\right.$$
$$\left.\times\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\frac{(xvtq^k;q)_\infty}{(yvt;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(ut)^n}{(q;q)_n}\frac{P_n(y,q^k x)}{(xvtq^k;q)_n}\right\}$$
$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}\frac{(c_1,\ldots,c_r;q)_k}{(d_1,\ldots,d_s;q)_k}P_k(y,x)(wt)^k\right.$$
$$\left.\times\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\frac{(xvtq^k;q)_\infty}{(yvt;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(q^k y/x;q)_n(yut)^n}{(q,xvtq^k;q)_n}\right\}$$
$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}\frac{(c_1,\ldots,c_r;q)_k}{(d_1,\ldots,d_s;q)_k}(wt)^k P_k(y,x)(wt)^k\right.$$
$$\left.\times\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\sum_{n=0}^{\infty}P_n(v,u)P_n(y,q^k x)\frac{t^n}{(q;q)_n}\right\}\quad\text{(by using (1.8))}$$

$$
\begin{aligned}
&= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\Bigg\{\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}\frac{(c_1,...,c_r;q)_k}{(d_1,...,d_s;q)_k}(wt)^k\\
&\quad\times\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\sum_{n=0}^{\infty}P_n(u,v)P_{n+k}(x,y)\frac{t^n}{(q;q)_n}\Bigg\}\\
&= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\Bigg\{\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}\frac{(c_1,...,c_r;q)_k}{(d_1,...,d_s;q)_k}(wt)^k\\
&\quad\times\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\sum_{n=k}^{\infty}P_{n-k}(v,u)P_n(y,x)\frac{t^{n-k}}{(q;q)_{n-k}}\Bigg\}\\
&= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\Bigg\{\sum_{n=0}^{\infty}P_n(y,x)\sum_{k=0}^{n}\begin{bmatrix}\beta\\k\end{bmatrix}\begin{bmatrix}n\\k\end{bmatrix}\frac{(c_1,...,c_r;q)_k}{(d_1,...,d_s;q)_k}\\
&\quad\times(q,q)_kP_{n-k}(v,u)w^k\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\frac{t^n}{(q;q)_n}\Bigg\}\\
&= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\left\{\sum_{n=0}^{\infty}(-1)^nq^{\binom{n}{2}}P_n(y,x)\right\}\\
&\quad\times\mathbb{D}_n^{(\beta-n)}(\mathbf{c},\mathbf{d},u,v,w)\frac{t^n}{(q;q)_n}\quad\text{(by using (2.2))}\\
&=\sum_{n=0}^{\infty}q^{n^2-n}E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\left\{(-1)^nq^{-\binom{n}{2}}P_n(y,x)\right\}\\
&\quad\times\mathbb{D}_n^{(\beta-n)}(\mathbf{c},\mathbf{d},u,v,w)\frac{t^n}{(q;q)_n}\\
&=\sum_{n=0}^{\infty}q^{n^2-n}\,\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)\mathbb{D}_n^{(\beta-n)}(\mathbf{c},\mathbf{d},u,v,w)\frac{t^n}{(q;q)_n}.\quad\text{(by using (2.10))}
\end{aligned}
$$

□

Theorem 5.2 (Extended Mehler's formula for $\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$). *We have*

$$
\begin{aligned}
&\sum_{n=0}^{\infty}q^{n^2-n}\mathbb{D}_{n+m}^{(\alpha-n-m)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)\mathbb{D}_n^{(\beta-n)}(\boldsymbol{c},\boldsymbol{d},u,v,w)\frac{t^n}{(q;q)_n}\\
&\quad=\frac{(q^{-m}xvt;q)_\infty}{(q^{-m}yvt;q)_\infty}\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}\frac{(c_1,...,c_r;q)_k}{(d_1,...,d_s;q)_k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}(w/v)^k(-1)^mq^{\binom{m+1}{2}}(vt)^{-m}\\
&\qquad\times\sum_{n=0}^{\infty}\frac{(-1)^nq^{\binom{n}{2}}(u/v)^n}{(q;q)_n}\sum_{j=0}^{n+m+k}\frac{(q^{-(n+m+k)},q^{-m}yvt,q)_jq^j}{(q,q^{-m}xvt;q)_j}\\
&\qquad\times\sum_{\varrho=0}^{\infty}\begin{bmatrix}\alpha\\\varrho\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_\varrho}{(\rho_1,\cdots,\rho_s;q)_\varrho}\left[(-1)^\varrho q^{\binom{\varrho}{2}}\right]^{1+s-r}(zvtq^{j-m})^\varrho,\quad|yvt|<1.
\end{aligned}\tag{5.2}
$$

Proof: Let $g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$ denote RHS of (5.2). By the same technique used in Theorem 3.1, we may check that $g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$ satisfies (2.3), so we have

$$
\begin{aligned}
&g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,0)\\
&=\frac{(q^{-m}xvt;q)_\infty}{(q^{-m}yvt;q)_\infty}\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}\frac{(c_1,...,c_r;q)_k}{(d_1,...,d_s;q)_k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}(w/v)^k(-1)^mq^{\binom{m+1}{2}}(vt)^{-m}
\end{aligned}
$$

$$\times \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (u/v)^n}{(q;q)_n} \sum_{j=0}^{n+m+k} \frac{(q^{-(n+m+k)}, q^{-m}yvt, q)_j q^j}{(q, q^{-m}xvt; q)_j}$$

$$= \frac{(q^{-m}xvt;q)_\infty}{(q^{-m}yvt;q)_\infty} \sum_{k=0}^{\infty} \begin{bmatrix}\beta\\k\end{bmatrix} \frac{(c_1,...,c_r;q)_k}{(d_1,...,d_s;q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} (w/v)^k (-1)^m q^{\binom{m+1}{2}} (vt)^{-m}$$

$$\times \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (u/v)^n}{(q;q)_n} \frac{(x/y;q)_{n+m+k}}{(q^{-m}xvt;q)_{n+m+k}} (q^{-m}vt)^{n+m+k}$$

$$= \frac{(q^{-m}xvt;q)_\infty}{(q^{-m}yvt;q)_\infty} \sum_{k=0}^{\infty} \begin{bmatrix}\beta\\k\end{bmatrix} \frac{(c_1,...,c_r;q)_k}{(d_1,...,d_s;q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} (wt)^k (-1)^m q^{\binom{m+1}{2}}$$

$$\times \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (ut)^n}{(q;q)_n} \frac{P_{n+m+k}(y,x)}{(q^{-m}xvt;q)_{n+m+k}} q^{-m(n+m+k)}$$

$$= \frac{(q^{-m}xvt;q)_\infty}{(q^{-m}yvt;q)_\infty} \sum_{k=0}^{\infty} \begin{bmatrix}\beta\\k\end{bmatrix} \frac{(c_1,...,c_r;q)_k}{(d_1,...,d_s;q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} (wt)^k (-1)^m q^{\binom{m+1}{2}}$$

$$\times q^{-m(m+k)} \frac{P_{m+k}(y,x)}{(q^{-m}xvt;q)_{m+k}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (q^{-m}ut)^n}{(q;q)_n} \frac{P_n(y, q^{m+k}x)}{(q^{-m}xvtq^{m+k};q)_n}$$

$$= \frac{(q^{-m}xvt;q)_\infty}{(q^{-m}yvt;q)_\infty} \sum_{k=0}^{\infty} \begin{bmatrix}\beta\\k\end{bmatrix} \frac{(c_1,...,c_r;q)_k}{(d_1,...,d_s;q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} (wt)^k (-1)^m q^{\binom{m+1}{2}}$$

$$\times q^{-m(m+k)} \frac{P_{m+k}(y,x)}{(q^{-m}xvt;q)_{m+k}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (q^{m+k}x/y;q)_n (yutq^{-m})^n}{(q, q^{-m}xvtq^{m+k};q)_n}$$

$$= \sum_{k=0}^{\infty} \begin{bmatrix}\beta\\k\end{bmatrix} \frac{(c_1,...,c_r;q)_k}{(d_1,...,d_s;q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} (wt)^k (-1)^m q^{\binom{m+1}{2}}$$

$$\times q^{-m(m+k)} P_{m+k}(y,x) \sum_{n=0}^{\infty} P_n(v,u) P_n(y, q^{m+k}x) \frac{(q^{-m}t)^n}{(q;q)_n} \quad \text{(by using (1.8))}$$

$$= \sum_{k=0}^{\infty} \begin{bmatrix}\beta\\k\end{bmatrix} \frac{(c_1,...,c_r;q)_k}{(d_1,...,d_s;q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} (wt)^k (-1)^m q^{\binom{m+1}{2}}$$

$$\times q^{-m(m+k)} \sum_{n=0}^{\infty} P_n(v,u) P_{n+m+k}(y,x) \frac{(q^{-m}t)^n}{(q;q)_n}$$

$$= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \begin{bmatrix}\beta\\k\end{bmatrix} \frac{(c_1,...,c_r;q)_k}{(d_1,...,d_s;q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} (wt)^k (-1)^m q^{-\binom{m}{2}-mk}$$

$$\times P_{n-k}(v,u) P_{n+m}(y,x) \frac{(q^{-m}t)^{n-k}}{(q;q)_{n-k}}$$

$$= \sum_{n=0}^{\infty} (-1)^m q^{-\binom{m}{2}-nm} P_{n+m}(y,x) \sum_{k=0}^{n} \begin{bmatrix}n\\k\end{bmatrix} \begin{bmatrix}\beta\\k\end{bmatrix} \frac{(c_1,...,c_r;q)_k}{(d_1,...,d_s;q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}$$

$$\times w^k (q;q)_k P_{n-k}(v,u) \frac{t^n}{(q;q)_n}$$

$$= \sum_{n=0}^{\infty} (-1)^{n+m} q^{\binom{n}{2}-\binom{m}{2}-nm} P_{n+m}(y,x) \mathbb{D}_n^{(\beta-n)}(\mathbf{c}, \mathbf{d}, u, v, w) \frac{t^n}{(q;q)_n}.$$

By using (2.14), we obtain

$$g(\alpha, \boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, z) = \sum_{n=0}^{\infty} q^{n^2-n} \mathbb{D}_{n+m}^{(\alpha-n-m)}(\boldsymbol{\sigma}, \boldsymbol{\rho}, x, y, z) \mathbb{D}_n^{(\beta-n)}(\mathbf{c}, \mathbf{d}, u, v, w) \frac{t^n}{(q;q)_n}.$$

□

6. A Transformational Identity Involving Generating Functions for $\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$

In this section, we use the q-difference equation approach to derive the transformational identity for the polynomials $\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$.

Theorem 6.1 *Let the coefficients $A(k)$ and $B(k)$ satisfy the following relation:*

$$\sum_{k=0}^{\infty} A(k)P_k(y,x) = \sum_{k=0}^{\infty} B(k)\frac{(xutq^{1-k};q)_\infty}{(yutq^{1-k};q)_\infty}. \tag{6.1}$$

Then

$$\begin{aligned}&\sum_{k=0}^{\infty}(-1)^k q^{\binom{k}{2}} A(k)\mathbb{D}_k^{(\alpha-k)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)\\ &\quad= \sum_{k=0}^{\infty} B(k)\frac{(xutq^{1-k};q)_\infty}{(yutq^{1-k};q)_\infty}\sum_{n=0}^{\infty}\begin{bmatrix}\alpha\\ n\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_n}{(\rho_1,\cdots,\rho_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}(zutq^{1-k})^n,\end{aligned} \tag{6.2}$$

where each of the series in (6.1) and (6.2) are absolutely convergent.

Proof: Let $g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$ the right-hand side of (6.2). We can check that $g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$ satisfies equation (2.3).

$$\begin{aligned}&g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)\\ &\quad= \sum_{k=0}^{\infty} B(k)\frac{(xutq^{1-k};q)_\infty}{(yutq^{1-k};q)_\infty}\sum_{n=0}^{\infty}\begin{bmatrix}\alpha\\ n\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_n}{(\rho_1,\cdots,\rho_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}(zutq^{1-k})^n.\end{aligned}$$

Then

$$\begin{aligned}&(q^{-1}x-y)\sum_{j=0}^{s}(-q)^{-j}B_j\Big(g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,zq^j)-g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,zq^{j+1})\Big)\\
&= (q^{-1}x-y)\sum_{j=0}^{s}(-q)^{-j}B_j\Bigg(\sum_{k=0}^{\infty} B(k)\frac{(xutq^{1-k};q)_\infty}{(yutq^{1-k};q)_\infty}\sum_{n=0}^{\infty}\begin{bmatrix}\alpha\\ n\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_n}{(\rho_1,\cdots,\rho_s;q)_n}\\
&\quad\times\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}(zutq^{1+j-k})^n\\
&\quad-\sum_{k=0}^{\infty} B(k)\frac{(xutq^{1-k};q)_\infty}{(yutq^{1-k};q)_\infty}\sum_{n=0}^{\infty}\begin{bmatrix}\alpha\\ n\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_n}{(\rho_1,\cdots,\rho_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}(zutq^{2+j-k})^n\Bigg)\\
&= (q^{-1}x-y)\sum_{j=0}^{s}(-q)^{-j}B_j\Bigg(\sum_{k=0}^{\infty} B(k)\frac{(xutq^{1-k};q)_\infty}{(yutq^{1-k};q)_\infty}\sum_{n=0}^{\infty}\begin{bmatrix}\alpha\\ n\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_n}{(\rho_1,\cdots,\rho_s;q)_n}\\
&\quad\times\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}(zutq^{1+j-k})^n(1-q^n)\\
&= (q^{-1}x-y)\Bigg(\sum_{k=0}^{\infty} B(k)\frac{(xutq^{1-k};q)_\infty}{(yutq^{1-k};q)_\infty}\sum_{n=0}^{\infty}\begin{bmatrix}\alpha\\ n\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_n}{(\rho_1,\cdots,\rho_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\\
&\quad\times(zutq^{1-k})^n(1-q^n)\sum_{j=0}^{s}(-1)^j B_j q^{j(n-1)}\\
&= (q^{-1}x-y)\Bigg(\sum_{k=0}^{\infty} B(k)\frac{(xutq^{1-k};q)_\infty}{(yutq^{1-k};q)_\infty}\sum_{n=0}^{\infty}\begin{bmatrix}\alpha\\ n\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_n}{(\rho_1,\cdots,\rho_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\end{aligned}$$

$$\times (zutq^{1-k})^n(1-q^n)(1-\rho_1q^{n-1})\cdots(1-\rho_sq^{n-1})$$
$$= (q^{-1}x-y)\left(\sum_{k=0}^{\infty}B(k)\frac{(xutq^{1-k};q)_\infty}{(yutq^{1-k};q)_\infty}\sum_{n=1}^{\infty}\frac{(q^{-\alpha},\sigma_1,\cdots,\sigma_r;q)_{n-1}}{(q,\rho_1,\cdots,\rho_s;q)_{n-1}}\left[(-1)^nq^{\binom{n}{2}}\right]^{s-r}\right.$$
$$\times q^{\alpha n}(1-q^{-\alpha+n-1})(zutq^{1-k})^n(1-\sigma_1q^{n-1})\cdots(1-\sigma_rq^{n-1})$$
$$= (q^{-1}x-y)\sum_{k=0}^{\infty}B(k)\frac{(xutq^{1-k};q)_\infty}{(yutq^{1-k};q)_\infty}\sum_{n=0}^{\infty}\frac{(q^{-\alpha},\sigma_1,\cdots,\sigma_r;q)_n}{(q,\rho_1,\cdots,\rho_s;q)_n}\left[(-1)^nq^{\binom{n}{2}}\right]^{s-r}$$
$$\times q^{\alpha(n+1)}(1-q^{-\alpha+n})(zutq^{1-k})^{n+1}(-1)^{s-r}q^{n(s-r)}(1-\sigma_1q^n)\cdots(1-\sigma_rq^n)$$
$$= (q^{-1}x-y)(-1)^{s-r}z\sum_{k=0}^{\infty}B(k)(utq^{1-k})\frac{(xutq^{1-k};q)_\infty}{(yutq^{1-k};q)_\infty}\sum_{n=0}^{\infty}\frac{(q^{-\alpha},\sigma_1,\cdots,\sigma_r)_n}{(q,\rho_1,\cdots,\rho_s;q)_n}\left[(-1)^nq^{\binom{n}{2}}\right]^{s-r}$$
$$\times q^{\alpha n}(q^\alpha-q^n)(zutq^{1-k})^nq^{n(s-r)}\sum_{j=0}^{r}(-1)^jA_jq^{nj}$$
$$= (q^{-1}x-y)(-1)^{s-r}z\sum_{j=0}^{r}(-1)^jA_j\sum_{k=0}^{\infty}B(k)(utq^{1-k})\frac{(xutq^{1-k};q)_\infty}{(yutq^{1-k};q)_\infty}\sum_{n=0}^{\infty}\begin{bmatrix}\alpha\\ n\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_n}{(\rho_1,\cdots,\rho_s;q)_n}$$
$$\times\left[(-1)^nq^{\binom{n}{2}}\right]^{1+s-r}(q^\alpha-q^n)(zutq^{1-k})^nq^{n(j+s-r)}$$
$$= (-1)^{1+s-r}z\sum_{j=0}^{r}(-1)^jA_j\sum_{k=0}^{\infty}B(k)\left(\frac{(q^{-1}xutq^{1-k};q)_\infty}{(yutq^{1-k};q)_\infty}-\frac{(xutq^{1-k};q)_\infty}{(qyutq^{1-k};q)_\infty}\right)$$
$$\times\sum_{n=0}^{\infty}\begin{bmatrix}\alpha\\ n\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_n}{(\rho_1,\cdots,\rho_s;q)_n}\left[(-1)^nq^{\binom{n}{2}}\right]^{1+s-r}(q^\alpha-q^n)(zutq^{1-k})^nq^{n(j+s-r)}$$
$$= (-1)^{1+s-r}z\sum_{j=0}^{r}(-1)^jA_j\left[q^\alpha\left(g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,zq^{j+s-r})-g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,zq^{j+s-r})\right)\right.$$
$$\left.-\left(g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,zq^{j+1+s-r})-g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,zq^{j+1+s-r})\right)\right].$$

Thus, $g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$ satisfies equation (2.3). By equation (2.5), we have

$$g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$$
$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{g(\alpha,\boldsymbol{\sigma},\boldsymbol{\rho},x,y,0)\right\}$$
$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{\sum_{k=0}^{\infty}B(k)\frac{(xutq^{1-k};q)_\infty}{(yutq^{1-k};q)_\infty}\right\}$$
$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{\sum_{k=0}^{\infty}A(k)P_k(y,x)\right\}\quad\text{(by using (6.1))}$$
$$= \sum_{k=0}^{\infty}(-1)^kq^{\binom{k}{2}}A(k)E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{(-1)^kq^{-\binom{k}{2}}P_k(y,x)\right\}$$
$$= \sum_{k=0}^{\infty}(-1)^kq^{\binom{k}{2}}A(k)\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z),\quad\text{(by using (2.10))}$$

which is the left-hand side of (6.2). Hence the proof is completed. □

7. Conclusions

Using the method of q-difference equations, we derived several q-identities, such as generating function and its extension, Rogers formula and its extension, Mehler's formula and its extension, and a transformational identity involving generating functions for $\mathbb{D}_n^{(\alpha-n)}(\boldsymbol{\sigma},\boldsymbol{\rho},x,y,z)$.

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