Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44)** : 1–10.
ISSN-0037-8712
doi:10.5269/bspm.77418

On Statistical Convergence of Topological $\mathfrak{q}$-Henstock-Kurzweil Integral

Gülizar Gülenay Zengin[1] *, Ekrem Savas[2], and Hemanta Kalita[3]

ABSTRACT: In this paper, we present the concept of an $\mathfrak{q}$-Henstock-Kurzweil-type integrable function (hereafter referred to as a topological $\mathfrak{q}$-Henstock-Kurzweil integrable function) within a topological vector space linked to a Radon measure μ. Fundamental findings concerning the topological $\mathfrak{q}$-Henstock-Kurzweil integrable function are examined. Furthermore, the connection between the topological $\mathfrak{q}$-Henstock-Kurzweil integral and the topological Henstock-Kurzweil integral is analyzed. Finally, we extend the concept of statistical convergence to topological $\mathfrak{q}$-Henstock-Kurzweil integrable functions within a μ-subcell of a topological vector space.

Keywords: $\mathfrak{q}$-Henstock-Kurzweil integral, statistical convergence, topological vector spaces, Radon measure.

Contents

1. Introduction

Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a bounded function. In the year 1854, Riemann first introduced the concept of Riemann integral of f through the concept of tagged partitions in the interval $[a, b]$ and the limit of Riemann sums as the norm of the tagged partitions goes to zero. During 1950, Henstock and Kurzweil developed an integral, known as Henstock-Kurzweil integral which uses the concept of δ-fine tagged partition of the interval $[a, b]$ where $\delta : [a, b] \to \mathbb{R}_+$. Henstock-Kurzweil integral is more general than Lebesgue integral as well as similar to that of Riemann integral (see [8, 10,18]). Moreover, the Henstock-Kurzweil integral is non-absolute, in the sense that, some functions are Henstock-Kurzweil integrable but not absolutely Henstock-Kurzweil integrable. G. Carrao [1] investigated Henstock-Kurzweil type integral on a complete measure metric space endowed with a Radon measure μ and with a family $\mathcal{F}$ of "intervals" satisfying the Vitali covering theorem called μ-cell (see [1, Definition 2.14]). Recently, H. Kalita et al. expanded the idea of μ-cell for Henstock-Kurzweil integrals in various settings in [12,13].

In the context of linear topological vector space, Ch. Klein et al. [15] presented the Riemann integral with respect to any non-atomic measure of functions. R. Paluga et al. [16] presented several properties of Henstock-Kurzweil integrals on a topological space. Recently, H. Kalita et al. have introduced ap-Henstock-Kurzweil integrals on topological vector space in [11].

On the other hand, the concept of statistical convergence for sequences of real or complex numbers has been investigated by Fast in [3]. It uses the notion of natural density of subsets of natural numbers. If $K \subseteq \mathbb{N}$, then the natural density of K is denoted by $d(K)$ and is defined by $d(K) = \lim\limits_{n\to\infty} \frac{1}{n} |\{k \le n : k \in K\}|$. Since then, it has been the subject of investigation in other articles, including [4,5,6], and [7]. Maio et al. [9] examined statistical uniform convergence in topological spaces, and illustrate the applications of this convergence to function spaces, hyperspaces, and selection principles. Also, the notion of μ-statistically convergent function sequences was presented in [2]. Recently, statistical Riemann and

---

* Corresponding author.
2020 *Mathematics Subject Classification:* 40A35, 28C05, 37B02, 46A35.
Submitted June 20, 2025. Published March 22, 2026

Lebesgue integrable sequences of functions have been studied by Srivastava et al. in [17] and investigated Korovkin type approximation theorems for such class of functions.

The idea of [9] motivated us to introduce topological $\mathfrak{q}$-Henstock-Kurzweil integrals on a μ-cell of a topological vector space. The fascinating ideas of [17] impels us to extent the convergence of topological $\mathfrak{q}$-Henstock-Kurzweil integrable functions to statistical convergence in our setting.

The paper is organized as follows: in Section 2, some basic notions and terminologies has been introduced which will be needed throughout the paper. In Section 3, we introduce the notion of topological $\mathfrak{q}$-Henstock-Kurzweil integral (denoted as $\mathfrak{q}$-THK integral) of a μ-cell-valued functions along with some properties. In Section 4, we extend the theory of convergence of $\mathfrak{q}$-THK integral to statistically convergence of $\mathfrak{q}$-THK integrable functions. Moreover, we introduce statistically equi-integrability for $\mathfrak{q}$-THK integrable functions to prove sequence of equi-integrable $\mathfrak{q}$-THK integrable functions are statistically Cauchy.

2. Preliminaries

The $\mathfrak{q}$-differential of an arbitrary f function is defined by $d_{\mathfrak{q}}f(x) = f(\mathfrak{q}x) - f(x)$. In particular let $d_{\mathfrak{q}}x = (\mathfrak{q}-1)x$. Then the $\mathfrak{q}$-derivative of f defined by

$$D_{\mathfrak{q}}f(x) = \frac{d_{\mathfrak{q}}f(x)}{d_{\mathfrak{q}}x} = \frac{f(\mathfrak{q}x) - f(x)}{(\mathfrak{q}-1)x}$$

where $x \neq 0$ and $0 < \mathfrak{q} < 1$. Note that if f is differentiable function, then

$$\lim_{\mathfrak{q}\to 1} D_{\mathfrak{q}}f(x) = \lim_{\mathfrak{q}\to 1} \frac{f(\mathfrak{q}x) - f(x)}{(\mathfrak{q}-1)x} = \frac{xf'(x)}{x} = f'(x) = \frac{df(x)}{dx}.$$

Let δ be a positive function on the closed interval $[a,b]$. We say $P = \Big\{([x_{i-1},x_i],t_i) \ : \ 1 \leq i \leq n\Big\}$ is δ-fine tagged partition of $[a,b]$ if $\Big\{[x_{i-1},x_i] \ : \ 1 \leq i \leq n\Big\}$ is a partition of $[a,b]$, $t_i \in [a,b]$ and $[x_{i-1},x_i] \subseteq (t_i - \delta(t_i), t_i + \delta(t_i))$ for every i, $1 \leq i \leq n$. It is clear that if $t_i \in [a_i,b_i]$ then for $0 < \mathfrak{q} < 1$, $\mathfrak{q}_j t_i \in [a_i,b_i]$ $j = 1,2,\cdots$. The expression $S_{\mathfrak{q}}(f,P) = \sum\limits_{i=1}^{n} f(\mathfrak{q}_i t_i)$; $j = 1,2,\cdots$ will be called $\mathfrak{q}$-Riemann sum of f on $\mathcal{I} = [a,b]$ if $S_{\mathfrak{q}}(f,P)$ is finite. We are now ready to define $\mathfrak{q}$-Henstock-Kurzweil integral as follows.

Definition 2.1 *[19, Definition 2.1] Let $f : [a,b] \to \mathbb{R}$ be a given function. The function f is called a $\mathfrak{q}$-Henstock-Kurzweil integrable on $\mathcal{I} = [a,b]$ if for each $\epsilon > 0$, we can find a δ-fine tagged partition P, and a real number $A_{\mathfrak{q}}$ such that $|S_{\mathfrak{q}}(f,P) - A_{\mathfrak{q}}| < \epsilon$ whenever $0 < \mathfrak{q} < 1$.*

The real value $A_{\mathfrak{q}}$ is called $\mathfrak{q}$-Henstock-Kurzweil integral of f and $A_{\mathfrak{q}} = \int_a^b f d_{\mathfrak{q}}x$.

Theorem 2.2 *[19] Let f_1 and f_2 are $\mathfrak{q}$-Henstock-Kurzweil integrable functions on $\mathcal{I} = [a,b]$ then,*

1. *kf_1 is $\mathfrak{q}$-Henstock-Kurzweil integrable function on $[a,b]$ for every $k \in \mathbb{R}$ with $\int_a^b kf_1 = k\int_a^b fd_{\mathfrak{q}}x$.*
2. *$f_1 + f_2$ is $\mathfrak{q}$-Henstock-Kurzweil integrable functions on $[a,b]$ with $\int_a^b (f_1+f_2)d_{\mathfrak{q}}x = \int_a^b f_1 d_{\mathfrak{q}}x + \int_a^b f_2 d_{\mathfrak{q}}x$.*
3. *If $f(r) \geq 0$ for every r in the interval $[a,b]$ and $r = \mathfrak{q}x$, then it follows that $\int_a^b fd_{\mathfrak{q}}r \geq 0$ for any $0 < \mathfrak{q} < 1$.*
4. *If $f_1 \leqslant f_2$ almost everywhere on $\mathcal{I} = [a,b]$, then $\int_a^b f_1 d_{\mathfrak{q}}x \leqslant \int_a^b f_2 d_{\mathfrak{q}}x$.*
5. *If f and $|f|$ are $\mathfrak{q}$-Henstock-Kurzweil integrable functions on $[a,b]$, then $\left|\int_a^b fd_{\mathfrak{q}}x\right| \leqslant \int_a^b \Big|f\Big| d_{\mathfrak{q}}x$.*

Let us define $\mathfrak{q}$-Henstock-Kurzweil equi-integrable functions as follows.

Definition 2.3 *A sequence $(f_n)_{n=1}^{\infty}$ of $\mathfrak{q}$-Henstock-Kurzweil integrable functions on $[a, b]$ is said to be $\mathfrak{q}$-Henstock-Kurzweil equi-integrable on $[a, b]$ if for each $\epsilon > 0$ there exists a gauge δ independent of n, on $[a, b]$ such that*

$$\sup_{n\in\mathbb{N}} \left| S_{\mathfrak{q}}(f_n, P) - \int_a^b f_n \right| < \epsilon$$

for each δ-fine partition P of $[a, b]$.

Let $\mathcal{X}$ be a compact Hausdorff topological vector space over real numbers. We denote support of μ by $supp(\mu) = \big\{x \in \mathcal{X} : \ \mu(\mathcal{U}) > 0$ for every θ nbd $\mathcal{U}$ of $x\big\}$. Throughout our work $\mu(\mathcal{U}) > 0$ is understood. If $A \subset \mathcal{X}$, then $\overline{A}$ denotes the closure of A in $\mathcal{X}$. ∂A denotes boundary of A where $\partial A = \overline{A} \cap \overline{\mathcal{X} \setminus A}$. Let $Bo(\mathcal{X})$ denotes the Borel sigma algebra of $\mathcal{X}$ containing all compact subsets of $\mathcal{X}$. Recalling an element $A \in Bo(\mathcal{X})$ is called μ-continuity set if $\mu(\partial A) = 0$.

Lemma 2.4 *[11] Let $\mathcal{X}$ be a topological vector space. Then there is a local base $\mathcal{B}$ of θ (the zero vector), satisfying the following:*

1. *If $\mathcal{U}, \mathcal{V} \in \mathcal{B}$, then there is a $\mathcal{W} \in \mathcal{B}$ with $\mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V}$.*
2. *If $\mathcal{U} \in \mathcal{B}$ and $x \in \mathcal{U}$, there is a $\mathcal{V} \in \mathcal{B}$ such that $x + \mathcal{V} \subseteq \mathcal{U}$.*
3. *If $\mathcal{U} \in \mathcal{B}$, there is a $\mathcal{V} \in \mathcal{B}$ such that $\mathcal{V} + \mathcal{V} \subseteq \mathcal{U}$.*
4. *If $\mathcal{U} \in \mathcal{B}$ and $x \in \mathcal{X}$, then there is $k \in \mathbb{R}$ such that $x \in k\mathcal{U}$.*
5. *If $\mathcal{U} \in \mathcal{B}$ and $0 < |k| \leq 1$, then $k\mathcal{U} \subseteq \mathcal{U}$ and $k\mathcal{U} \in \mathcal{B}$.*
6. *$T\big\{\mathcal{U} : \mathcal{U} \in \mathcal{B}\big\} = \{\theta\}$.*

Conversely, given a collection $\mathcal{B}$ of subsets containing θ and satisfying the above conditions, there is a topology for $\mathcal{X}$ making $\mathcal{X}$ a topological vector space and having B as a local base at θ.

Let $\Big(\mathcal{X}, \mathcal{T}\Big)$ be a topological space. A family $\mathcal{F} = \{\mathcal{Q}_i : i \in \mathbb{N}\}$ of subsets of $\mathcal{X}$ is a filter in $\mathcal{X}$ if the following are satisfies:

1. For every $i \in \mathbb{N}$, $\mathcal{Q}_i \neq \emptyset$.
2. For $\mathcal{Q}_i, \mathcal{Q}_j \in \mathcal{F}$ then $\mathcal{Q}_i \cap \mathcal{Q}_j \in \mathcal{F}$.
3. If $\mathcal{Q} \in \mathcal{F}$, and $\mathcal{Q} \subseteq \mathcal{B}$ then $\mathcal{B} \in \mathcal{F}$.

The filter $\mathcal{F}$ converges to $x \in \mathcal{X}$ if for every θ-nbd $\mathcal{U}$ (θ is zero vector of $\mathcal{X}$) there exists $\mathcal{Q} \in \mathcal{F}$ such that $\mathcal{Q} - x \subseteq \mathcal{U}$. We say $\mathcal{F}$ is Cauchy if for every θ-nbd $\mathcal{U}$ there exists $\mathcal{Q} \in \mathcal{F}$ such that $\mathcal{Q} \subseteq \mathcal{U}$. Let $\mathcal{X}$ be a topological vector space. We say that $\mathcal{X}$ is complete if every Cauchy filter in $\mathcal{X}$ converges. We say that $\mathcal{X}$ is locally convex if there is a local base at θ whose members are convex.

Definition 2.5 1. *Let $\mathcal{Q}, \mathcal{R} \in \mathcal{F}$. We say $\mathcal{Q}, \mathcal{R}$ are non overlapping if $\mathcal{Q} \cap \mathcal{R} = \emptyset$.*

2. *Let $\mathcal{G}$ be a subfamily of $\mathcal{F}$. We say that $\mathcal{G}$ is a fine cover of $E \subset \mathcal{X}$ if $\mu(\mathcal{Q}) \to 0$ whenever $x \in \mathcal{Q}$ for all $x \in E$.*

Definition 2.6 *We call $\mathcal{F}$ is a family of μ-cells if it satisfies the following conditions:*

1. *Given $\mathcal{Q} \in \mathcal{F}$ and a constant $a > 0$ there exists a division $\Big\{\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_m\Big\}$ and there exists $\mathcal{U}$, such that $\inf \Big[\mu\Big(\mathcal{U}(\mathcal{Q}_i)\Big)\Big] < a$ for $i = 1, 2, ..., m$;*

2. *Given* $A,\ \mathcal{Q} \in \mathcal{F}$ *and* $A \subset \mathcal{Q}$, *there exists a division* $\left\{\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_m\right\}$ *of* $\mathcal{Q}$ *and there exists* $\mathcal{U}$, *such that* $\mathcal{U}(A) = \mathcal{U}(\mathcal{Q}_1)$;
3. $\mu(\partial\mathcal{Q}) = 0$ *for each* $\mathcal{Q} \in \mathcal{F}$ *where* $\partial\mathcal{Q}$ *is the boundary of* $\mathcal{Q}$.

We call $\mathcal{F}$ is μ-filter if for each subset E of $\mathcal{X}$ and for each subfamily $\mathcal{G}$ of $\mathcal{F}$ that is a fine cover of E, there exists a countable system $\left\{\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_j, ..\right\}$ of pairwise non-overlapping cells of $\mathcal{G}$ such $\mu(E) \setminus \mu(\cup\mathcal{Q}_j) \geq 0$.

Definition 2.7 *Let* $\mathcal{Q} \in \mathcal{F}$, $E \subset \mathcal{Q}$ *and* δ *be a gauge on* $\mathcal{Q}$. *A collection* $P = \left\{(x_i, \mathcal{Q}_i)\right\}_{i=1}^{m}$ *of finite ordered pairs of points and cells is said to be*

1. *partition of* $\mathcal{Q}$ *if* $\left\{\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_m\right\} \subseteq Bo(\mathcal{X})$ *is a division of* $\mathcal{Q}$ *and* $x_i \in \mathcal{Q}_i$ *for* $i = 1, 2, ..., m$;
2. *a partial partition of* $\mathcal{Q}$ *if* $\left\{\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_m\right\} \subseteq Bo(\mathcal{X})$ *is a subsystem of a division of* $\mathcal{Q}$ *and* $x_i \in \mathcal{Q}_i$ *for* $i = 1, 2, ..., m$;
3. δ*-fine if* $\mu\left(\mathcal{Q}_i\right) < \delta(x_i)$ *for* $i = 1, 2, ..., m$;
4. E*-anchored if the points* $x_1, x_2, ..., x_m$ *belongs to* E.

3. $\mathfrak{q}$-Henstock-Kurzweil integrals on topological vector spaces

In this section, we introduce topological $\mathfrak{q}$-Henstock-Kurzweil integral on a μ-cell $\mathcal{Q}$ of a topological vector space $\mathcal{X}$. The following Cousin's type lemma addresses the existence of δ-fine partitions of a given cell $\mathcal{Q}$.

Lemma 3.1 *If* δ *is a gauge on* $\mathcal{Q}$, *then there exists a* δ*-fine partition of* $\mathcal{Q}$.

Proof: The proof follows similar technique of [1, Lemma 2.2.1], so omitted.

Let $P = \left\{(t_i, \mathcal{Q}_i)\right\}_{i=1}^{m}$ be a partition of $\mathcal{Q} \in \mathcal{F}$ and $f : \mathcal{Q} \to \mathcal{X}$ be a given function. We consider $\mathfrak{q} \in (0, 1)$. We define Riemann sum in associate with $\mathfrak{q}$ as $S_{\mathfrak{q}}(f, P) = \sum_{i=1}^{m} f(\mathfrak{q}_i t_i)\mu\left(\mathcal{Q}_i\right)$. We are ready to define topological $\mathfrak{q}$-Henstock-Kurzweil integral as follows:

Definition 3.2 *A function* $f : \mathcal{Q} \to \mathbb{R}$ *is said to be topological* $\mathfrak{q}$*-Henstock-Kurzweil integrable on* $\mathcal{Q}$ *with respect to* μ *if there exists a real number* A *such that for each* $\mathcal{U}$, *there exists a gauge* δ *on* $\mathcal{Q}$, *with* $|S_{\mathfrak{q}}(f, P) - A| < \mu(\mathcal{U})$ *whenever* $P = \left\{(t_i, \mathcal{Q}_i)\right\}_{i=1}^{m}$ *is* δ*-fine partition on* $\mathcal{Q}$ *and* $\mu(\mathcal{U}) > 0$.

The number A is said to be the topological $\mathfrak{q}$-Henstock-Kurzweil integral of f on $\mathcal{Q}$ with respect to μ (in short, $\mathfrak{q}$-THK integral) on $\mathcal{Q}$ with respect to μ and we write $A = \int_{\mathcal{Q}} f d\mu$. Now-onwards, we denote $\mathfrak{q}$-THK for topological $\mathfrak{q}$-Henstock-Kurzweil integral of f on $\mathcal{Q}$ with respect to μ. It is not hard to find the uniqueness of the integral value of a given $\mathfrak{q}$-THK integrable function $f : \mathcal{Q} \to \mathbb{R}$. The collection of all $\mathfrak{q}$-THK integrable functions on $\mathcal{Q}$ with respect to μ shall be denoted by $\mathfrak{q}$-THK$(\mathcal{Q})$. The Alexiewicz type semi-norm on $\mathfrak{q}$-THK$(\mathcal{Q})$ can be defined by

$$||f|| = \sup_{\mathcal{Q} \in \mathcal{F}} \left| \int_{\mathcal{Q}} f d_{\mathfrak{q}}\mu(t) \right|$$

where the integral is in the sense of $\mathfrak{q}$-THK. It is easy to see $(\mathfrak{q}\text{-}THK(\mathcal{Q}), ||.||)$ is a linear space.

Example 3.3 *The function* $f(x)=c$ *with* $c\in\mathbb{R}$*, for all* $x\in\mathcal{Q}$ *is* $\mathfrak{q}$*-THK integrable on* $\mathcal{Q}$ *with* $\int_{\mathcal{Q}} f d\mu = c\mu(\mathcal{Q})$*.*

Few simple properties of $\mathfrak{q}$-THK integrals are as follows.

Theorem 3.4 *Let* $f,g\in \mathfrak{q}\text{-}THK(\mathcal{Q})$*, then* $f+g\in THK(\mathcal{Q})$*, and* $\int_{\mathcal{Q}}(f+g)d\mu=\int_{\mathcal{Q}} f d\mu+\int_{\mathcal{Q}} g d\mu$*.*

Theorem 3.5 *If* $f\in \mathfrak{q}\text{-}THK(\mathcal{Q})$ *and* $k\in\mathbb{R}$*, then* $kf\in \mathfrak{q}\text{-}THK(\mathcal{Q})$ *and* $\int_{\mathcal{Q}} kf d\mu = k\int_{\mathcal{Q}} f d\mu$*.*

Theorem 3.6 *If* $f\in \mathfrak{q}\text{-}THK(\mathcal{Q})$ *and* $f(x)\geq 0$ *for each* $x\in\mathcal{Q}$*, then* $\int_{\mathcal{Q}} f d\mu\geq 0$*.*

Corollary 3.7 *Let* $f,g\in \mathfrak{q}\text{-}THK(\mathcal{Q})$*. If* $f\geq g$ *for each* $x\in\mathcal{Q}$*, then* $\int_{\mathcal{Q}} f d\mu\geq\int_{\mathcal{Q}} g d\mu$*.*

Proposition 3.8 *If* $f:\mathcal{Q}\to\mathbb{R}$ *be* $\mathfrak{q}$*-THK integrable on* $\mathcal{Q}$ *and* $|f(x)|<M$ *with* $M\in\mathbb{R}$ *for all* $x\in\mathcal{Q}$*, then* $\left|\int_{\mathcal{Q}} f\right|\leq M\big(\mu(\mathcal{Q})\big)$*.*

Next, we prove Cauchy criterion for $\mathfrak{q}$-THK integrable functions on $\mathcal{Q}$. The proof is similar techniques of [14, Theorem 3.4], so we have ommited.

Theorem 3.9 (The Cauchy Criterion) A function $f:\mathcal{Q}\to\mathbb{R}$ is $\mathfrak{q}$-THK integrable on $\mathcal{Q}$ if and only if for each $\mathcal{U}$, there exists a gauge δ on $\mathcal{Q}$ such that $|S_{\mathfrak{q}}(f,P_1)-S_{\mathfrak{q}}(f,P_2)|<\mu(\mathcal{U})$ whenever P_1 and P_2 are δ-fine partitions of $\mathcal{Q}$.

In the following theorem, we shall prove that $\mathfrak{q}$-THK integrability of f on a set $\mathcal{Q}$ implies its $\mathfrak{q}$-THK integrability on each subcells of $\mathcal{Q}$.

Theorem 3.10 *If* $f\in \mathfrak{q}\text{-}THK(\mathcal{Q})$*, and if* $\mathcal{A}$ *is a subcell of* $\mathcal{Q}$*, then* $f\in \mathfrak{q}\text{-}THK(\mathcal{Q})$ *and* $\int_{\mathcal{A}} f d_{\mathfrak{q}}\mu=\int_{\mathcal{Q}} f\chi_{\mathcal{A}} d_{\mathfrak{q}}\mu$*.*

Proof: Let $\mathcal{U}$ be given. By Theorem 3.9, there exists a gauge δ on $\mathcal{Q}$ so that $|S_{\mathfrak{q}}(f,P_1)-S_{\mathfrak{q}}(f,P_2)|<\mu(\mathcal{U})$ for each pair of δ-fine partitions P_1 and P_2 of $\mathfrak{Q}$. Given that there exists a division $P=\{\mathfrak{Q}_1,\mathfrak{Q}_2,...,\mathfrak{Q}_m\}$ of $\mathfrak{Q}$ and $\mathcal{A}\subset\mathfrak{Q}$, such that $\mathcal{A}=\mathfrak{Q}_1$. For each $k\in\{2,3,...,m\}$ we fix a δ-fine partition P_k of $\mathfrak{Q}_k$. If R_1 and R_2 are δ-fine partitions of $\mathcal{A}$, then $R_1\cup\bigcup\limits_{k=2}^{m}P_k$ and $R_2\cup\bigcup\limits_{k=2}^{m}P_k$ are δ-fine partitions of $\mathfrak{Q}$. Thus,

$$\begin{aligned}|S_{\mathfrak{q}}(f,R_1)-S_{\mathfrak{q}}(f,R_2)|&=|S_{\mathfrak{q}}(f,R_1)+\sum_{k=2}^{m}S_{\mathfrak{q}}(f,P_k)-S_{\mathfrak{q}}(f,R_2)-\sum_{k=2}^{m}S_{\mathfrak{q}}(f,P_k)|\\&\leq\Big|S_{\mathfrak{q}}\Big(f,R_1\cup\bigcup_{k=2}^{m}P_k\Big)-S_{\mathfrak{q}}\Big(f,R_2\cup\bigcup_{k=2}^{m}P_k\Big)\Big|<\mu(\mathcal{U}).\end{aligned}$$

Thus by Theorem 3.9, $f\in\mathfrak{q}\text{-}THK(\mathcal{A})$.

Proposition 3.11 *Let* $f:\mathfrak{Q}\to\mathbb{R}$ *be a* $\mathfrak{q}$*-THK integrable function on* $\mathfrak{Q}$*. If* $\Big\{\mathfrak{Q}_1,\mathfrak{Q}_2,...,\mathfrak{Q}_m\Big\}$ *is a division of* $\mathfrak{Q}$*, then* $f\in\mathfrak{q}\text{-}THK(\mathfrak{Q}_1)\cap...\cap\mathfrak{q}\text{-}THK(\mathfrak{Q}_m)$ *and* $\int_{\mathfrak{Q}} f d_{\mathfrak{q}}\mu=\sum\limits_{i=1}^{m}\int_{\mathfrak{Q}_i} f d_{\mathfrak{q}}\mu$*.*

Proof: Given $\mathcal{U}$, there exists a gauge δ on $\mathcal{Q}$ such that $|S_{\mathfrak{q}}(f,P)-\int_{\mathcal{Q}} f d_{\mathfrak{q}}\mu|<\mu(\mathcal{U})$, for each δ-fine partition P of $\mathcal{Q}$. By Theorem 3.10, $f\in\mathfrak{q}\text{-}THK(\mathfrak{Q}_i)$ for $i=1,2,...,m$ such that $\delta_i(\mathfrak{q}_i x_i)<\delta(x)$ for each $x\in\mathcal{Q}_i$, $0<\mathfrak{q}<1$ and such that $|S_{\mathfrak{q}}(f,P_i)-\int_{\mathfrak{Q}_i} f d_{\mathfrak{q}}\mu|<\frac{\mu(\mathcal{U})}{m}$, for each δ_i-fine partitions P_i of $\mathfrak{Q}$. Therefore $P=P_1\cup...\cup P_m$ is a δ-fine partition of $\mathfrak{Q}$. Consequently,

$$|S_{\mathfrak{q}}(f,P)-\sum_{i=1}^{m}\int_{\mathfrak{Q}_i} f d\mu|\leq|S_{\mathfrak{q}}(f,P_1)-\int_{\mathfrak{Q}_1} f d_{\mathfrak{q}}\mu|+...+|S_{\mathfrak{q}}(f,P_m)-\int_{\mathfrak{Q}_m} f d_{\mathfrak{q}}\mu|<\mu(\mathcal{U}).$$

Thus $\int_{\mathfrak{Q}} f d_{\mathfrak{q}}\mu=\sum\limits_{i=1}^{m}\int_{\mathfrak{Q}_i} f d_{\mathfrak{q}}\mu$.

We define indefinite $\mathfrak{q}$-THK integral of a given $\mathfrak{q}$-THK integrable function f as follows:

Definition 3.12 *Let $\mathcal{M}$ be the collection of all subcells of $\mathfrak{Q}$ and $\mathcal{A}$ be any subcell of $\mathfrak{Q}$. A function $F : \mathcal{M} \to \mathbb{R}$ defined by $F(\mathcal{A}) = \int_{\mathcal{A}} f d_{\mathfrak{q}}\mu$ is called an indefinite $\mathfrak{q}$-THK integral of f.*

It is easy to see by Proposition 3.11, each additive $\mathfrak{q}$-THK integrable function is an indefinite $\mathfrak{q}$-THK integrable on $\mathfrak{Q}$.

Lemma 3.13 *(Saks-Henstock Lemma) A function $f : \mathfrak{Q} \to \mathbb{R}$ is $\mathfrak{q}$-THK integrable on $\mathfrak{Q}$ if and only if there exists an additive cell function π defined on the family of subcells of $\mathfrak{Q}$ such that for each $\mathcal{U}$, there exists a gauge δ on $\mathfrak{Q}$ with $\sum\limits_{(t_i,\mathfrak{Q}_i)\in P} |\pi(\mathfrak{Q}_i) - f(\mathfrak{q}_i t_i)\mu(\mathfrak{Q}_i)| < \mu(\mathcal{U})$, for each δ-fine partial partition P of $\mathfrak{Q}$.*

Next, we see topological Henstock-Kurzweil integrable function on each cell $\mathfrak{Q}$ is $\mathfrak{q}$-THK integrable and the two integrals coincide. Let us denote $(THK)\int_{\mathfrak{Q}} f d\mu$ be topological Henstock-Kurzweil integrable functions with respect to μ.

Theorem 3.14 *Let $f : \mathfrak{Q} \to \mathbb{R}$ be a function. If f is topological Henstock-Kurzweil integrable on $\mathfrak{Q}$, with respect to μ, then f is $\mathfrak{q}$-THK integrable on $\mathfrak{Q}$ and $(THK)\int_{\mathfrak{Q}} f d_{\mathfrak{q}}\mu = \int_{\mathfrak{Q}} f d_{\mathfrak{q}}\mu$.*

Proof: Let $\mathcal{U}$ be any θ-nbd. Let $f : \mathfrak{Q} \to \mathbb{R}$ be a topological Henstock-Kurzweil intagrable on $\mathfrak{Q}$ and $(THK)\int_{\mathfrak{Q}} f d_{\mathfrak{q}}\mu = A$. By the definition of topological Henstock-Kurzweil integral, there exists a δ-fine tagged partition $P = \{(t_i, \mathfrak{Q}_i) : 1 \leqslant i \leqslant n\}$ of $\mathfrak{Q}$ so that $|S(f, P) - A| < \mu(\mathcal{U})$. Let $0 < \mathfrak{q} < 1$. Since P is δ-fine tagged partition, $t_i \in \mathfrak{Q}_i$, and $\mu(\mathfrak{Q}_i) \leqslant \delta(t_i)$ for all $i = 1, 2, ..., n$. It is clear that $\mathfrak{q}_i t_i \in \mathfrak{Q}_i$ whenever $0 < \mathfrak{q} < 1$. Let us construct $\gamma(\mathfrak{q}_i t_i) = f(t_i)$. Clearly $\mu(\mathfrak{Q}_i) \leqslant \gamma(\mathfrak{q}_i t_i)$ for all $1 \leqslant i \leqslant n$ and

$$\begin{aligned} S(f, P) &= \sum_{i=1}^{n} f(\mathfrak{q}_i t_i)\mu(\mathcal{U}) \\ &\leq \sum_{i=1}^{n} f(t_i)\mu(\mathfrak{Q}_i). \end{aligned}$$

So, for each θ-nbd $\mathcal{U}$ we can construct any δ-fine tagged partition Q on $\mathfrak{Q}$ such that $\left|S(f, Q) - \int_{\mathfrak{Q}} f d_{\mathfrak{q}}\mu(x)\right| < \mu(\mathcal{U})$ whenever $0 < \mathfrak{q} < 1$. Hence, we can conclude that f is $\mathfrak{q}$-topological Henstock-Kurzweil integrable function.

The following counter example shows that the converse of Theorem 3.14 is always not true.

Example 3.15 *Let $\mathcal{C}$ be Cantor topological subspace of $[0, 1]$, where $\mathcal{C} = \cap\mathcal{C}_n$, and $\mathcal{C}_n$ is open intervals from $\mathcal{C}_{n-1}$ where $\mathcal{C}_0 = [0, 1]$. It is easy to see the Cantor set $\mathcal{C}$ with the interval components of the space $\mathcal{C}_n$ form the basis for the topology on $\mathcal{C}$. Let $f : \mathcal{C} \to \mathbb{R}$ be the function by*

$$f(x) = \begin{cases} \frac{(-1)^n 3^n}{x}, & if \ x \in \left[\frac{2}{3^n}, \frac{1}{3^{n-1}}\right] \cap \mathcal{C}, \ n = 1, 2, 3... \\ 0, & if x = 0. \end{cases}.$$

In the given range, for topological Henstock-Kurzweil integrability, the function must not blow up to infinity on any topological subspaces of $[0, 1]$. Since there are infinite discontinuity near the θ-nbd of $x = 0$, and the divergence at θ-nbd of $x = 0$ makes the function non-integrable in the topological Henstock-Kurzweil sense. We can find that the function is $\mathfrak{q}$-topological Henstock-Kurzweil integrable on $[0, 1]$. In order to find $\mathfrak{q}$-topological Henstock-Kurzweil integrability of f, Let $\mathcal{U}$ be any θ-nbd and let $N \in \mathbb{N}$ be such that $\mu(\mathcal{U})N \geq 2$ and $\left|\sum\limits_{j=n+1}^{\infty} \frac{(-1)^{j+1}}{j}\right| < \frac{\mu(\mathcal{U})}{2}$ for each $n \geq N$. Let us construct a gauge δ on $\mathcal{C}$ such that

$$\delta(x) = \begin{cases} K, \ x \in (x - \delta(x), x + \delta(x)) \cap C \\ \\ \frac{1}{3^{n-1}}, \ x = 0. \end{cases}$$

Next consider a δ-fine partition $P = \Big\{(\mathfrak{q}_1 t_1, \mathfrak{Q}_1), (\mathfrak{q}_2 t_2, \mathfrak{Q}_2), ..., (\mathfrak{q}_m t_m, \mathfrak{Q}_m)\Big\}$ *of* $\mathcal{C}$ *such that* $\mathfrak{Q}_1 = [0, c] \cap \mathcal{C}$ *where* $[0, c] \subset [0, 1]$*. Then* $\mathfrak{q}_1 t_1 = 0$ *and* $c < \frac{1}{3^{n-1}}$*. Next, if we consider* $n \in \mathbb{N}$ *such that* $\frac{1}{3^n} < c < \frac{1}{3^{n-1}}$ *then* $n \geq N$*. In this situation*

$$\bigcup_{i=2}^{m} \mathfrak{Q}_j = \begin{cases} \Big([c, \frac{1}{3^{n-1}}] \cup [\frac{2}{3^{n-1}}, 1]\Big) \cap \mathcal{C} \ \textit{if}\ c > \frac{2}{3^n} \\ \\ ([\frac{2}{3^n}, 1] \cap \mathcal{C}) \ \textit{if}\ c < \frac{2}{3^n}. \end{cases}$$

Then

$$\Big|S(f, P) - log2\Big| = \begin{cases} \mu(\mathcal{U}) \ \textit{if}\ c \geq \frac{2}{3^n} \\ \\ \frac{\mu(\mathcal{U})}{2} \ \textit{if}\ c < \frac{2}{3^n} \end{cases}$$

Hence f *is* $\mathfrak{q}$*-THK on* $\mathcal{C}$*.*

4. Statistical convergence

In this section, we present the concept of statistical convergence of topological $\mathfrak{q}$-Henstock-Kurzweil integrals within a topological vector space. We demonstrate that any convergence associated with a topological $\mathfrak{q}$-Henstock-Kurzweil integrable function is also statistically convergent. Additionally, we present the concept of statistically equi-integrability for topological $\mathfrak{q}$- Henstock-Kurzweil integrable functions to demonstrate statistical Cauchy on $\mathfrak{q}$-$THK(\mathfrak{Q})$.

Let $A \subset \mathbb{N}$ and $n \in \mathbb{N}$. Let $A(n) = \Big\{k \in A : \ k \leq n\Big\}$. Then natural density $d(A(n))$ is given as $d(A(n)) = \lim\limits_{n\to\infty} \frac{|A(n)|}{n} = \alpha$ where α is a finite real number and $|A(n)|$ denotes the cardinality of the enclosed set.

Definition 4.1 *Let* $\mathfrak{Q} \in Bo(\mathfrak{X})$*. A sequence of* $\mathfrak{q}$*-THK integrable functions* $f_n : \mathfrak{Q} \to \mathbb{R}$ *is said to be stastistically convergence to* f *if for any* θ*-nbd* $\mathcal{U}$ *and a* δ*-fine partition* P*,*

$$d\Bigg(\Big\{n \in \mathbb{N} : \ |S(f_n, P) - \int_{\mathfrak{Q}} f| \geq \mu(\mathfrak{Q})\Big\}\Bigg) = 0.$$

We denote *st-* $\lim\limits_{n\to\infty} f_n = f$. It is not hard to see if *st-limit* is unique. The following result are true.

Theorem 4.2 *If a sequence of* $\mathfrak{q}$*-THK integrable function* (f_n) *of stastistically convergence, then its* *st-*$\lim\limits_{n}$ *is unique.*

Proof: Suppose st-$\lim\limits_{n} f_n = f$ and st-$\lim\limits_{n} f_n = g$. Given $\epsilon > 0$, define

$$K_1(\mu(\mathcal{U})) = \Big\{n \in \mathbb{N} : \ \Big|S_{\mathfrak{q}}(f_n, P) - \int_{\mathfrak{Q}} f\Big| \geq \mu(\mathcal{U})\Big\}$$

and

$$K_2(\mu(\mathcal{U})) = \{n \in \mathbb{N} : \ \Big|S_{\mathfrak{q}}(f_n, P) - \int_{\mathfrak{Q}} g\Big| \geq \mu(\mathcal{U})\}.$$

Clearly, $d\big(K_1(\mu(\mathcal{U}))\big) = 0$ and $d\big(K_2(\mu(\mathcal{U}))\big) = 0$. Let $K\big(\mu(\mathcal{U})\big) = K_1\big(\mu(\mathcal{U})\big) \bigcup K_2\big(\mu(\mathcal{U})\big)$. Then $d\Bigg(K(\mu(\mathcal{U})\Bigg) = 0$, gives $\mathbb{N} \setminus d\big(K(\mu(\mathcal{U}))\big) = 1$. Next, if $k \in \mathbb{N} \setminus K(\mu(\mathcal{U}))$, we have

$$\Big|\int_{\mathfrak{Q}} f - \int_{\mathfrak{Q}} g\Big| \leq |\int_{\mathfrak{Q}} f - S_{\mathfrak{q}}(f_k, P)| + |S_{\mathfrak{q}}(f_k, P) - \int_{\mathfrak{Q}} g|$$
$$< \frac{\mu(\mathcal{U})}{2} + \frac{\mu(\mathcal{U})}{2} = \mu(\mathcal{U}).$$

Since $\mu(\mathcal{U})$ is arbitrary, we can find $\left|\int_{\mathfrak{Q}}(f-g)\right| = 0$. Hence $\int_{\mathfrak{Q}}(f-g) = 0$ gives $f = g$.

Definition 4.3 *We say that* (f_n) *of* $\mathfrak{q}$-*THK integrable functions is statistically Cauchy if for any* θ-*nbd* $\mathcal{U}$, *and a* δ-*fine partition* P *there exists* $M \in \mathbb{N}$ *such that*

$$d\Big(\{n \in \mathbb{N} : \ |S_{\mathfrak{q}}(f_n, P) - S_{\mathfrak{q}}(f_M, P)| \geq \mu(\mathcal{U})\}\Big) = 0.$$

Next, we state several fundamental properties below.

Theorem 4.4 *Let* (f_n) *and* $g_n)$ *are in* $\mathfrak{q}\text{-}THK([a,b], \mathcal{X})$ *with* $f = st\text{-}\lim_n f_n$ *and* $g = st\text{-}\lim_n g_n$. *Then following holds:*

1. $st\text{-}\lim_n (f_n + g_n) = f + g$.
2. $st\text{-}\lim_n (\alpha f_n) = \alpha(st\text{-}\lim_n f_n) = \alpha f$.
3. $st\text{-}\lim_n f_n g_n = fg$.

Proof: For (1): Let $f_n : \mathfrak{Q} \to \mathbb{R}$ and $g_n : \mathfrak{Q} \to \mathbb{R}$ are sequences of $\mathfrak{q}$-THK integrable functions. Let $\text{st-}\lim_n f_n = f$ and $\text{st-}\lim_n g_n = g$. Then for any θ-nbd $\mathcal{U}$, and δ-fine partition P, we have

$$d\bigg(\bigg\{k \in \mathbb{N} : \ |S_{\mathfrak{q}}(f_k, P) - \int_{\mathfrak{Q}} f| < \mu(\mathcal{U})\bigg\}\bigg) = 1$$

and

$$d\bigg(\bigg\{k \in \mathbb{N} : \ |S_{\mathfrak{q}}(g_k, P) - \int_{\mathfrak{Q}} g| < \mu(\mathcal{U})\bigg\}\bigg) = 1.$$

It is easy to see,

$$d\bigg(\{k \in \mathbb{N} : \ |S_{\mathfrak{q}}(f_k, P) - \int_{\mathfrak{Q}} f| \geq \mu(\mathcal{U})\} \cup \{k \in \mathbb{N} : \ |S_{\mathfrak{q}}(g_k, P) - \int_{\mathfrak{Q}} g| \geq \mu(\mathcal{U})\}\bigg) = 1.$$

Let $\mathcal{P} = \Big\{k \in \mathbb{N} : \ |S_{\mathfrak{q}}(f_k, P) - \int_{\mathfrak{Q}} f| < \mu(\mathcal{U})\Big\} \cup \Big\{k \in \mathbb{N} : \ |S_{\mathfrak{q}}(g_k, P) - \int_{\mathfrak{Q}} g| < \mu(\mathcal{U})\Big\}$. Then for any $k \in \mathcal{P}$, we have $|S_{\mathfrak{q}}(f_n, P) + S_{\mathfrak{q}}(g_n, P) - \int_{\mathfrak{Q}}(f+g)| < 2\mu(\mathcal{U})$. Thus

$$d(\mathcal{P}) = 1 \leq d\bigg(\{k \in \mathbb{N} : \ |S_{\mathfrak{q}}(f_n, P) + S_{\mathfrak{q}}(g_n, P) - \int_{\mathfrak{Q}}(f+g)| < 2\mu(\mathcal{U})\bigg) \leq 1.$$

Since $\mu(\mathcal{U})$ is arbitrary, we get $\text{st-}\lim_n (f_n + g_n) = f + g$.
Proof of (2) and (3) is straightforward, so ommitted.

Next, we show that every convergent sequence of $\mathfrak{q}$-THK integrable functions is statistically convergent.

Theorem 4.5 *If a sequence* (f_n) *of* $\mathfrak{q}$-*THK integrable functions in* $\mathfrak{q}\text{-}THK(\mathfrak{Q}, \mathcal{X})$ *converges to* $f \in \mathcal{X}$, *then* (f_n) *is statistically convergent to* f.

Proof: Let $\mathcal{U}$ be a θ-nbd. Since (f_n) is a sequence of $\mathfrak{q}$-THK integrable functions on $\mathfrak{q}\text{-}THK(\mathfrak{Q}, \mathcal{X})$ converges to f, so, there exists $\mathcal{N} \subset \mathbb{N}$ with $\delta(\mathcal{N}) = 1$ and $n_0 = n_0(\mathcal{U})$ such that $n \geq n_o$ and $n \in \mathcal{N}$ implies $|S_{\mathfrak{q}}(f_n, P) - \int_{\mathfrak{Q}} f| < \mu(\mathcal{U})$ whenever P is a free tagged partition of $\mathfrak{Q}$. Clearly $|f_n| < \mu(\mathcal{U})$. Again,

$$\bigg\{n \in \mathbb{N} : \ |S_{\mathfrak{q}}(f_n, P) - \int_{\mathfrak{Q}} f| \geq \mu(\mathcal{U})\bigg\} \subset \bigg\{1, 2, ..., n_o\bigg\} \cup (\mathbb{N} \setminus \mathcal{N}).$$

Since $\delta\bigg\{1, 2, ..., n_0\bigg\} \cup (\mathbb{N} \setminus \mathcal{N}) = 0$, it follows $f = s\text{-}\lim_n f_n$.

The following example shows that converse of Theorem 4.5 does not hold.

Example 4.6 *Consider a sequence (f_n) of $\mathfrak{q}$-THK integrable functions whose terms are*

$$f_n = \begin{cases} n & \text{if } n = i^2,\ i = 1, 2, ..., \\ \frac{1}{n} & \text{otherwise.} \end{cases}$$

It is easy to see the sequence (f_n) is divergence. Let $K = \{i^2 :\ i = 1, 2, ...\}$, then $\delta(K) = 0$, it follows $0 = s\text{-}\lim_n f_n$.

We introduce statistical equi-integrability for $\mathfrak{q}$-THK integrable function as follows:

Definition 4.7 *A sequence of statistical $\mathfrak{q}$-THK integrable functions $f_n : \mathfrak{Q} \to \mathbb{R}$ is said to be statistically equi-integrable if for any θ-nbd $\mathcal{U}$ and a δ-fine partition P, we have $d\Big(\Big\{n \in \mathbb{N} :\ \Big|S_{\mathfrak{q}}(f_n, P) - \int_{\mathfrak{Q}} f_n\Big| \geq \mu(\mathcal{U})\Big\}\Big) = 0$.*

Proposition 4.8 *Let (f_n) be statistically convergent sequence of $\mathfrak{q}$-THK integrable functions on $\mathfrak{Q}$. If (f_n) is statistically equi-integrable then (f_n) is statistically Cauchy on $\mathfrak{q}\text{-}THK(\mathfrak{Q})$.*

Proof: Let st-$\lim_n f_n = f$. Consider $\mathcal{U}_n$ be a sequence of nested base of θ-nbd.

Let $W^{(j)} = \Big\{w \in \mathbb{N} :\ w \leq n, |S_{\mathfrak{q}}(f_n, P) - \int_{\mathfrak{Q}} f| \geq \mu(\mathcal{U}_j)\Big\}$ for any positive integer j. Clearly for each $W^{(j+1)} \subset W^{(j)} < \lim_{n\to\infty} \frac{1}{n}|W^{(j)}| = 1$. Let us choose $m \in \mathbb{N}$ such that $n > m$. Then $\frac{1}{n}|W^{(j)}| > 0$. This shows that $W^{(1)} \neq \emptyset$. In general we can find natural numbers $m(p+1) > m(p)$ such that we can a positive number $r > m(p+1)$ implies $W^{(p+1)} \neq \emptyset$. Further By Lemma 2.4, we have for every θ-nbd $\mathcal{U}$, there is a symmetric θ-nbd $\mathcal{V}$ such that $\mathcal{V} + \mathcal{V} \subseteq \mathcal{U}$. Let us consider (f_n) be a statistically equi-integrable $\mathfrak{q}$-THK integrable functions on $\mathfrak{q}\text{-}THK(\mathfrak{Q})$. Then by definition of equi-integrable $\mathfrak{q}$-THK integrable function, for any θ-nbd $\mathcal{U}$ and a δ-fine partition P such that $d\Big(\Big\{n \in \mathbb{N} :\ \Big|S_{\mathfrak{q}}(f_n, P) - \int_{\mathfrak{Q}} f_n\Big| \geq \mu(\mathcal{U})\Big\}\Big) = 0$. So,

$$d\Big(\Big\{n \in \mathbb{N} :\ |S_{\mathfrak{q}}(f_n, P) - \int_{\mathfrak{Q}} f_n| \geq \mu(\mathcal{V})\Big\}\Big) = 0 \text{ and} \tag{4.1}$$

$$d\Big(\Big\{n \in \mathbb{N} :\ |S_{\mathfrak{q}}(f_m, P) - \int_{\mathfrak{Q}} f_m| \geq \mu(\mathcal{V})\Big\}\Big) = 0. \tag{4.2}$$

Thus we have,

$$\begin{aligned} d\Big(\Big\{k \leq n :\ |S_{\mathfrak{q}}(f_n, P) - S_{\mathfrak{q}}(f_m, P)| \geq \mu(\mathcal{U})\Big\}\Big) &\leq d\Big(\Big\{n \in \mathbb{N} :\ |S_{\mathfrak{q}}(f_n, P) - \int_{\mathfrak{Q}} f_n| \geq \mu(\mathcal{V})\Big\}\Big) \\ &+ d\Big(\Big\{n \in \mathbb{N} :\ |S_{\mathfrak{q}}(f_m, P) - \int_{\mathfrak{Q}} f_m| \geq \mu(\mathcal{V})\Big\}\Big) \\ &\to 0 \quad \text{using (4.1), (11).} \end{aligned}$$

Hence (f_n) is statistically Cauchy on $\mathfrak{q}\text{-}THK(\mathfrak{Q})$.

Remark 4.9 *If we consider $\mathfrak{q} = 1$ then all results of statistical topological $\mathfrak{q}$-Henstock-Kurzweil integral on a μ-cell $\mathfrak{Q}$ of a topological vector space are statistical topological Henstock-Kurzweil integral on a μ-cell $\mathfrak{Q}$ of a topological vector space of [13].*

Conclusion

In this study, the topological $\mathfrak{q}$-Henstock-Kurzweil integral on a μ-cell $\mathfrak{Q}$ of a topological vector space $\mathcal{X}$ has been examined. Various properties are addressed in this context. In conclusion, we broaden the conventional convergence of topological $\mathfrak{q}$-Henstock-Kurzweil integrable functions to encompass statistical convergence, and we have established a connection between statistically equi-integrable topological $\mathfrak{q}$-Henstock-Kurzweil integrals and statistical Cauchy convergence.

Conflicts of Interest

The authors declare no conflict of interest.
Data availability: Our manuscript does not have associated data.

Acknowledgments

The authors thank to the Journal editors and reviewers for their constructive suggestions for better presentation of the article. Also first author like to thanks Tobitak for supporting this research.

References

1. G. Corrao, *An Henstock-Kurzweil type integral on a measure metric space*. Doctoral Thesis, Universita Degli Studi Di Palermo, Palermo, (2013).
2. O. Duman, C. Orhan, μ-statistically convergent function sequences. *Czec. Math. J.* **54**(2), 413–422, (2004).
3. H. Fast, Sur la convergence statistique. *Colloq. Math.* **2**(3), 241–244, (1951).
4. J. A. Fridy, On statistical convergence. *Analysis.* **5**(4), 301–313, (1985).
5. J. A. Fridy, Statistical limit points. *Proc. Amer. Math. Soc.* **118**(4), 1187–1192, (1993).
6. J. A. Fridy, H. I. Miller, A matrix characterization of statistical convergence. *Analysis.* **11**(1), 59–66, (1991).
7. J. A. Fridy, C. Orhan, Lacunary statistical convergence. *Pacific J. Math.* **160**(1), 43–51, (1993).
8. R. G. Gardon, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*. American Mathematical Society, (1994).
9. D. M. Giuseppe, L. D. R. Kocinac, Statistical convergence in topology. *Topol Appl.* **156**(1), 28–45, (2008).
10. R. Henstock, *The general theory of integration*. Oxford University Press, Oxford, UK, (1991).
11. H. Kalita, B. Hazarika, A convergence theorem for ap-Henstock-Kurzweil integral and its relation to topology. *Filomat.* **36**(20), 1–10, (2022).
12. H. Kalita, B. Hazarika, T. P. Becerra, On AP-Henstock-Kurzweil Integrals and Non-Atomic Radon Measure. *Mathematics.* **11**(6), 1–16, (2023).
13. H. Kalita, R. P. Agarwal, B. Hazarika, Convergence of ap-Henstock-Kurzweil integral on locally compact spaces. *Czec. Math. J.* 75, 103–121 (2025), https://doi.org/10.21136/CMJ.2023.0450-22.
14. H. Kalita, S. Som, B. Hazarika, On statistical convergence of topological Henstock-Kurzweil integral, Carpathian J. Math. 41(2), 393–408, (2025).
15. Ch. Klein, S. Rolewicz, On Riemann integration of functions with values in topological linear spaces. *Studia Mathematica.* **80**(4), 109–118, (1984).
16. R. Paluga, S. Canoy Jr, The Henstock Integral in topological vector spaces. *Mat. Matematika.* **24**(3), 34–47, (2001).
17. H. M. Srivastava, B. B. Jena, S. K. Paikray, Statistical Riemann and Lebesgue Integrable Sequence of Functions with Korovkin Type Approximation Theorems. *Axioms.* **10**(10), 1–16, (2021).
18. L. T. Yeong, *Henstock-Kurzweil integration on Euclidean spaces*. Series in Real Analysis, no. 12, World Scientific, (2011).
19. G. G. Zengin, E. Savas, H. Kalita, H. Bharali, An introduction to q-Henstock-Kurzweil integral and applications, Thermal Science no. (2025), 971.

[1,2] *Department of Mathematics, Usak University, Usak, Turkey*
[3] *Mathematics Division, VIT Bhopal University, Bhopal-Indore Highway, Kothrikalan, Sehore, Madhya Pradesh 466114, India*

E-mail address: `glnyzngn@gmail.com; ekremsavas@yahoo.com; hemanta30kalita@gmail.com`