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The class of demi Dunford-Pettis completely continuous operators

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ABSTRACT: In this paper, we introduce and study a new concept of operators that we call demi Dunford-Pettis completely continuous operators, use it to generalize known classes of operators which defined by Dunford-Pettis completely continuous operators. In addition, we establish some interesting properties of this class of operators.

Key Words: Demi-Dunford-Pettis completely continuous operator, Dunford-Pettis sets, the relatively compact Dunford-Pettis property.

Contents

1. Introduction

Petryshyn who first introduced the concept of demicompact operators in [10] as a general idea to generalize nonlinear compact operators, especially for solving fixed point theory problems. Later, the idea of demicompact operators has been extensively studied in the context of linear operators by various researchers. Jeribi [8,9] contributed significantly towards developing associated Fredholm and spectral theories.

Then, in [7], Krichen and O'Regan extended this work by creating the concept of weakly demicompact operators. This concept generalizes weakly compact operators since Petryshyn's earlier generalization of compact operators.
Benkhaled et al. in [5] expanded on this idea by introducing the demi Dunford-Pettis operators, which are a generalization of Dunford-Pettis operators.

Inspired by these generalizations, we introduce and study a new class of operators related to Dunford-Pettis completely continuous operators ($DPcc$ for short), which we call demi Dunford-Pettis completely continuous operators (demi-$DPcc$ for short) . Note that this new class contains Dunford-Pettis completely continuous operators as a special case (see Proposition 3.1).
The manuscript is organized as follows. After a preliminary, we will start by defining a new class of operators acting between Banach spaces called demi Dunford-Pettis completely continuous operators(see Definition 3.1). After we study some characterizations of the class of demi Dunford-Pettis completely continuous operators (see Theorem 3.1).We also provide interesting results about the sum of demi-$DPcc$ operators (see Theorem 3.2) and converse (see Theorem 3.3).Further, we establish more properties about the power of demi-$DPcc$ operators (see Theorem 3.4 and Theorem 3.5). Finally, we conclude our study by considering the domination property of this class of operators(see Theorem 3.6).

2. Preliminaries

A norm-bounded subset A of a Banach space X is said to be a Dunford-Pettis set if every weakly null sequence (f_n) of X' converges uniformly to zero on A, that is, $\lim_{n\to\infty} \sup_{x\in A} |f_n(x)| = 0$ [3]. It is well known that the class of Dunford-Pettis sets strictly contains the class of relatively compact sets, meaning

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that every relatively compact set is a Dunford-Pettis set. However, a Dunford-Pettis set is not necessarily relatively compact. For instance, the closed unit ball B_{c_0} is a Dunford-Pettis set in c_0 (since $(c_0)' = \ell^1$, which has the Schur property), but it is not relatively compact. Nevertheless, if X is a reflexive Banach space, the classes of Dunford-Pettis sets and relatively compact sets in X coincide.
Let us recall from [11] that, a sequence (x_n) in X is Dunford-Pettis if and only if $\lim_{n\to\infty} f_n(x_n) = 0$ for every weakly null sequence (f_n) in X'.
A Banach space X has:

- a relatively compact Dunford-Pettis property (DPrcP for short) if every Dunford-Pettis set in X is relatively compact [6]. For example, reflexive spaces, as well as Schur spaces, have the DPrcP. Equivalently, if and only if all weakly null DP sequences in X are norm null [11, Lemma 1.2].
- the Dunford-Pettis property (DP property for short) if and only if $\lim_{n\to\infty} f_n(x_n) = 0$ for every weakly null pair of sequences $((x_n), (f_n))$ in $X \times X'$. Equivalently, if every relatively weakly compact subset of X is DP.

Let us recall that an operator $T : \mathcal{D}(T) \subseteq X \to X$, where $\mathcal{D}(T)$ is a subspace of X is said to be:

- demicompact if, for every bounded sequence (x_n) in the domain $\mathcal{D}(T)$ such that $(x_n - T(x_n))$ converges to $x \in X$, there is a convergent subsequence of (x_n) [10].
- weakly demicompact if, for every bounded sequence (x_n) in $\mathcal{D}(T)$ such that $(x_n - T(x_n))$ weakly converges in X,has a weakly convergent subsequence of (x_n) [7].
- demi Dunford–Pettis (DDP for short), if for every sequence x_n in X such that $x_n \to 0$ in $\sigma(X, X')$ and $\|x_n - T(x_n)\| \to 0$ as $n \to \infty$, we have $\|x_n\| \to 0$ as $n \to \infty$ [5].

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm, is also a Banach lattice.

We will use the term operator $T : E \longrightarrow F$ between two Banach lattices to mean a bounded linear mapping. If T is an operator from a Banach lattice E into another Banach lattice F then, its dual operator T' is defined from F' into E' by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$. We refer the reader to [2] for unexplained terminology on Banach lattice theory. Some useful and additional properties of Dunford-Pettis sets and Banach spaces with the DPrcP in [4,6].

3. Main results

We start with the following definition.

Definition 3.1 *Let X be a Banach space. An operator $T : X \to X$ is said to be demi Dunford-Pettis completely continuous (demi-DPcc for short), if for every Dunford-Pettis sequence (x_n) in X such that $x_n \to 0$ in $\sigma(X, X')$ and $\|x_n - T(x_n)\| \to 0$ as $n \to \infty$, we have $\|x_n\| \to 0$ as $n \to \infty$.*

Example 3.1 *For all $\alpha \neq 1$, αId_X is demi-DPcc .*

Example 3.2 *The identity operator $Id_{\ell^\infty} : \ell^\infty \to \ell^\infty$ is not demi DPcc. In fact, for the standard basis (e_n) in ℓ^∞, obviously, (e_n) is a Dunford-Pettis weakly null sequence in ℓ^∞ and $\|e_n - Id_{\ell^\infty}(e_n)\|_\infty = 0$ but $\|e_n\|_\infty = 1$.*

In [11] Y. Wen and J. Chen introduced a weak version of Dunford-Pettis operators, called Dunford-Pettis completely continuous operators (DPcc for short). An operator $T : X \to Y$ is $DPcc$, if T carries weakly null sequences which are DP in X to norm null ones.

Our following result proves that the class of demi-$DPcc$ operators contains that of $DPcc$ operators.

Proposition 3.1 *Let X be a Banach space. Every operator $T : X \to X$ that admits a $DPcc$ power is demi-$DPcc$.*

Proof: Let (x_n) be a Dunford-Pettis sequence of X such that $x_n \to 0$ in $\sigma(X, X')$ and $\|x_n - T(x_n)\| \to 0$ as $n \to \infty$. We have to show that $\|x_n\| \to 0$ as $n \to \infty$.
Since T is power $DPcc$, then there exists $m \in \mathbf{N}^\star$ such that T^m is $DPcc$. So $\|T^m(x_n)\| \to 0$ as $n \to \infty$.
Note that for all $n \in \mathbb{N}$, we have

$$\|x_n\| \leq \|x_n - T^m(x_n)\| + \|T^m(x_n)\|$$
$$= \|(I + T + ... + T^{m-1})(x_n - T(x_n))\| + \|T^m(x_n)\|$$
$$\leq \|I + T + ... + T^{m-1}\|\|x_n - T(x_n)\| + \|T^m(x_n)\|$$

So that $\|x_n\| \to 0$ as $n \to \infty$. □

An operator T from a Banach space X into another Y is said to be Dunford–Pettis if $\|T(x_n)\| \to 0$ as $n \to \infty$, for every weakly null sequence (x_n) in X.
It is obvious that every Dunford–Pettis operator is Dunford-Pettis completely continuous operator.
As a result, the class of Dunford–Pettis operators is contained in the class of demi-$DPcc$ operators.

The converse of Proposition 3.1 does not hold in general. To illustrate this, we present the following remark.

Remark 3.1 *We note that There exists a demi-DPcc operator of which all its powers are not DPcc. Indeed, let consider the identity operator of the Banach space ℓ^∞. It is clear that the operator $-Id_{\ell^\infty}$ is demi-DPcc (see Example 3.1). On the other side, $-Id_{\ell^\infty}$ is not DPcc.*
In fact, for the standard basis (e_n) in ℓ^∞, obviously, (e_n) is a Dunford-Pettis weakly null sequence in ℓ^∞. But $\| - Id_{\ell^\infty}(e_n)\|_\infty = \|e_n\|_\infty = 1$.

The next result gives a necessary and sufficient condition under which each operator is demi-$DPcc$.

Theorem 3.1 *Let X be a Banach space, then the following assertions are equivalent:*

1. *Every operator $T : X \to X$ is $DPcc$;*
2. *Every operator $T : X \to X$ is demi-$DPcc$;*
3. *The identity operator of X is demi-$DPcc$;*
4. *X has the DPrcP.*

Proof: (1) $\Longrightarrow$ (2) Follows from Proposition 3.1.
(2) $\Longrightarrow$ (3) Obvious.
(3) $\Longrightarrow$ (4) We have to show that X has the DPrcP. It suffices to prove that $\|x_n\| \to 0$ as $n \to \infty$ for every Dunford-Pettis weakly null sequence (x_n) in X.
Let (x_n) be a Dunford-Pettis weakly null sequence. It is clear that $\|x_n - Id_X(x_n)\| = 0$. Since, the identity operator $Id_X : X \to X$ is demi-$DPcc$, then $\|x_n\| \to 0$ as $n \to \infty$.
(4) $\Longrightarrow$ (1) Follows from Theorem 1.3 [11]. □

Remark 3.2 *In general, the sum of two demi-DPcc operators is not necessarily demi-DPcc.*
In fact, let $X = \ell^\infty$ and $Id_X : X \to X$ be the identity operator. Clearly, $T = 2Id_X$ and $S = -Id_X$ are demi-DPcc operators (see Example 3.1). However, the operator $T + S = Id_X$ is not demi-DPcc (see Example 3.2).

The following result asserts that a $DPcc$ perturbation of a demi-$DPcc$ operator is demi-$DPcc$.

Theorem 3.2 *Let X a Banach space. For every demi-DPcc operator T from X into X and for every $DPcc$ operator S from X into X, the operator $T+S$ is demi-DPcc.*

Proof: Let (x_n) be a Dunford-Pettis sequence of X such that $x_n \to 0$ in $\sigma(X, X')$ and $\|x_n - (T+S)(x_n)\| \to 0$ as $n \to \infty$. We have to show that $\|x_n\| \to 0$ as $n \to \infty$.
As S is $DPcc$, we obtain $\|S(x_n)\| \to 0$ as $n \to \infty$ and by the following inequality

$$\|x_n - T(x_n)\| = \|x_n - T(x_n) - S(x_n) + S(x_n)\| \leq \|x_n - (S+T)(x_n)\| + \|S(x_n)\|,$$

we conclude that $\|x_n - T(x_n)\| \to 0$ as $n \to \infty$. Thus, since T is demi-$DPcc$ operator then $\|x_n\| \to 0$ as $n \to \infty$. Therefore, $T+S$ is demi-$DPcc$. □

Now, we study the reciprocal of the Theorem 3.2.

Theorem 3.3 *Let T and S be operators on a Banach space X. If $T+S$ is demi-DPcc and S is DPcc, then T is demi-DPcc.*

Proof: Let (x_n) be a Dunford-Pettis sequence of X such that $x_n \to 0$ in $\sigma(X, X')$ and $\|x_n - T(x_n)\| \to 0$ as $n \to \infty$. We have to show that $\|x_n\| \to 0$ as $n \to \infty$.
Since S is $DPcc$, we obtain $\|S(x_n)\| \to 0$ as $n \to \infty$ and by the following inequality:

$$\|x_n - (T+S)(x_n)\| = \|x_n - T(x_n) - S(x_n)\| \leq \|x_n - T(x_n)\| + \|S(x_n)\|,$$

we obtain $\|x_n - (T+S)(x_n)\| \to 0$ as $n \to \infty$. Thus, since $T+S$ is demi-$DPcc$ operator then $\|x_n\| \to 0$ as $n \to \infty$. Therefore, T is demi-$DPcc$. □

As consequences of Theorem 3.2 and Theorem 3.3, we have the following characterizations.

Corollary 3.1 *Let X a Banach space, and $T, S : X \to X$ be two operators such that S is DPcc, then the following assertions are equivalent:*

1. *T is demi-DPcc;*
2. *$T+S$ is demi-DPcc;*

In the following result, we will be focusing on power demi-$DPcc$ operators.

Theorem 3.4 *Let X a Banach space and $T : X \to X$ be an operator. If T^2 is demi-DPcc, then T is demi-DPcc.*

Proof: Let (x_n) be a Dunford-Pettis weakly null sequence of X and $\|x_n - T(x_n)\| \to 0$ as $n \to \infty$.
We have to show that $\|x_n\| \to 0$ as $n \to \infty$.
First, we claim that $(T(x_n))$ is Dunford-Pettis sequence.
Given that (x_n) is weakly null, it follows easily that $(T(x_n))$ is also weakly null in X. We now claim that $(T(x_n))$ is a Dunford-Pettis sequence of X.
To see this, we proceed as follows.
Let (f_n) to be a weakly null sequence in X'.
Then $\lim_{n\to\infty} f_n(T(x_n)) = \lim_{n\to\infty} T'(f_n(x_n))$. Now, since (x_n) is a Dunford-Pettis sequence of X then, $\lim_{n\to\infty} f_n(x_n) = 0$ and hence $\lim_{n\to\infty} f_n(T(x_n)) = 0$. Therefore $(T(x_n))$ is a Dunford-Pettis weakly null sequence in X. Thus, $\|T(x_n) - T^2(x_n)\| \to 0$ as $n \to \infty$.
From the following inequality:

$$\|x_n - T^2(x_n)\| \leq \|x_n - T(x_n)\| + \|T(x_n) - T^2(x_n)\|,$$

we obtain $\|x_n - T^2(x_n)\| \to 0$ as $n \to \infty$ and T^2 is demi-$DPcc$ operator implies that $\|x_n\| \to 0$ as $n \to \infty$. Therefore, T is demi-$DPcc$.

□

We obtain also the following result:

Theorem 3.5 *Let X a Banach space and $T : X \to X$ be an operator. If T and $-T$ are both demi-DPcc, then T^2 is demi-DPcc.*

Proof: Let (x_n) be a Dunford-Pettis weakly null sequence of X and $\|x_n - T^2(x_n)\| \to 0$ as $n \to \infty$. We set $z_n = x_n - T(x_n)$, clearly $z_n \to 0$ in $\sigma(X, X')$.
Now, from $z_n + T(z_n) = z_n - (-T(z_n)) = x_n - T^2(x_n)$, we obtain that

$$\|z_n + T(z_n)\| = \|z_n - (-T(z_n))\| = \|x_n - T^2(x_n)\| \to 0$$

On the other hand, since the operator $-T$ is demi-$DPcc$ then, $\|z_n\| = \|x_n - T(x_n)\| \to 0$ as $n \to \infty$. Now, as T is demi-$DPcc$ then $\|x_n\| \to 0$ as $n \to \infty$. This implies that T^2 is demi-$DPcc$.

□

We should note that the domination problem does not hold in the class of demi-DPcc operators, as demonstrated by the following example.

Remark 3.3 *Let $Id_{\ell^\infty} : \ell^\infty \to \ell^\infty$ be the identity operator of ℓ^∞. We can see that $0 \leq Id_{\ell^\infty} \leq 2Id_{\ell^\infty}$. It is clear that $2Id_{\ell^\infty}$ is demi-DPcc (see Example 3.1). But, Id_{ℓ^∞} is not demi-DPcc (see Example 3.2).*

In [11], the authors proved that the class of positive $DPcc$ operators satisfies the domination property (see [11, Theorem 3.1]). Such a property can be stated as well in the setting for central demi-$DPcc$ operators.

Recall from [1, Definition 3.28] that an operator $T : E \to E$ is called central if it is dominated by a multiple of the identity operator that is, T is a central operator if and only if there exists some scalar $\lambda > 0$ such that $|Tx| \leq \lambda |x|$ holds for all $x \in E$.

In the following result, we prove that the domination problem for central demi-$DPcc$ operators is valid.

Theorem 3.6 *Let E be a Banach lattice and $S, T : E \to E$ be two positive operators.*

i. If $0 \leq S \leq T \leq Id_E$ and T is demi-DPcc, then S is likewise demi-DPcc.

ii. If $Id_E \leq R \leq S$ such that S is disjointness preserving and R is demi-DPcc, then S is demi-DPcc.

Proof:

i. Let (x_n) be a Dunford-Pettis sequence of E such that $x_n \to 0$ for the weak topology $\sigma(E, E')$ and $\|x_n - S(x_n)\| \to 0$ as $n \to \infty$. We have to show that $\|x_n\| \to 0$ as $n \to \infty$.
In view of [[1], Theorem 3.30], we obtain

$$|x_n - S(x_n)| = |(Id_E - S)(x_n)| = |(Id_E - S)(|x_n|)| = (Id_E - S)(|x_n|) \quad \text{for all } n.$$

Now, by using the inequality

$$|(Id_E - T)(x_n)| = (Id_E - T)(|x_n|) \leq (Id_E - S)(|x_n|),$$

We see that $\|(Id_E - T)(x_n)\| \leq \|(Id_E - S)(x_n)\|$ for all n, from which we get $\|x_n - T(x_n)\| \to 0$ as $n \to \infty$. The fact that T is demi-$DPcc$, we infer that $\|x_n\| \to 0$ as $n \to \infty$ and so S is demi-$DPcc$.

ii. Let (x_n) be a Dunford-Pettis sequence of E such that $x_n \to 0$ for the weak topology $\sigma(E, E')$ and $\|x_n - S(x_n)\| \to 0$ as $n \to \infty$. We have to show that $\|x_n\| \to 0$ as $n \to \infty$.
By [[2], Theorem 2.40], we obtain

$$|S(x_n) - x_n| = |(S - Id_E)(x_n)| = |(S - Id_E)|(|x_n|) = (S - Id_E)(|x_n|) \quad \text{for all } n.$$

Using the inequality

$$|(Id_E - R)(x_n)| \leq (R - Id_E)(|x_n|) \leq (S - Id_E)(|x_n|),$$

We infer that $\|(Id_E - R)(x_n)\| \leq \|(S - Id_E)(x_n)\|$ for all n,
from which we get $\|(Id_E - R)(x_n)\| \to 0$ as $n \to \infty$.
The fact that R is demi-$DPcc$, we infer that $\|x_n\| \to 0$ as $n \to \infty$ and so S is demi-$DPcc$.

□

Remark 3.4 *Let E a Banach lattice, and $T, S, R : E \to E$ be operators such that $R \leq S \leq T \leq R + Id_E$. If R and T are demi-DPcc operators, we cannot deduce that the operator S is demi DPcc.*
In fact, this problem is equivalent to $0 \leq S - R \leq T - R \leq Id_E$, where operator $T - R$ is demi-DPcc.
However, if R and T are both demi-DPcc operators, then $T - R$ is not necessarily a demi-DPcc operator.
In fact, the identity operator $Id_{\ell^\infty} : \ell^\infty \to \ell^\infty$.
Let $T = 3Id_{\ell^\infty}$ and $S = 2Id_{\ell^\infty}$ are demi-DPcc operators (see Example 3.1).
However, $T - R = 3Id_{\ell^\infty} - 2Id_{\ell^\infty} = Id_{\ell^\infty}$ is not demi-DPcc(see Example 3.2).

Corollary 3.2 *Let E a Banach lattice, and $T, S, R : E \to E$ be operators such that $R \leq S \leq T \leq R + Id_E$. If R is DPcc and T is demi-DPcc, then S is demi-DPcc.*

Proof: Let $R \leq S \leq T \leq R + Id_E$ is equivalent to $0 \leq S - R \leq T - R \leq Id_E$.
Since R is $DPcc$ and T is demi-$DPcc$ then $T - R$ is demi-$DPcc$.
As a result of Theorem 3.6, it follows that $S - R$ is demi-$DPcc$.
Since R is $DPcc$ and from equality $S = R + (S - R)$, it follows from Theorem 3.2 that S is demi-$DPcc$.
□

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