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A new fixed point theorem in T_α-metric spaces with application

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ABSTRACT: In this article, we introduce the concept of T_α-metric space as a generalization that encompasses several existing generalized metric spaces, including b-metric, multiplicative metric and extended b-metric spaces. We establish necessary and sufficient conditions for a function defined on such spaces to satisfy Banach's contraction principle, thus proving the existence and uniqueness of a fixed point. Our approach unifies various extensions of Banach's fixed point theorem present in recent literature. We illustrate our theoretical results with examples and provide an application to solving nonlinear Fredholm integral equation.

Key Words: Metric spaces, T_α-metric, fixed point.

Contents

1. Introduction and Preliminary

Since the publication of the famous Banach fixed point theorem in 1922 [1], the study of fixed point theory in metric spaces has found numerous applications in mathematics, particularly in solving differential and functional equations. Over the years, several authors have introduced new classes of generalized metric spaces. Notably, Bakhtin (1989) [2] and Czerwik [3] introduced the concept of b-metric spaces as a rigorous generalization of metric spaces, and established an analogue of Banach's contraction principle within this framework.

In 2008, Bashirov et al. [4] introduced the concept of a multiplicative metric space, and in 2017, Kamran et al. [5] proposed the notion of an extended b-metric space by further relaxing the triangle inequality.

In this article, we begin by recalling the necessary definitions and properties related to b-metric spaces, multiplicative metric spaces, and extended b-metric spaces. We then introduce the notion of a T_α-metric space and provide the foundational tools required to establish a version of Banach's contraction principle in this new setting. Our results generalize and extend several recent fixed point theorems found in the literature. To illustrate the applicability of our approach, we present an application to the solution of nonlinear Fredholm integral equation.

Let us give some definitions known in the literature.

Definition 1.1 *[2] Let X be a non-empty set and $s \geq 1$ a given real number. A function $d : X \times X \longrightarrow \mathbb{R}^+$ is called a b-metric on X if the following conditions are satisfied:*
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.
Note that every metric space is a b-metric space with $s = 1$.

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Definition 1.2 *[4] Let X be a non-empty set. A function $d : X \times X \longrightarrow [1, \infty)$ is called a multiplicative metric on X if the following conditions are satisfied:*
(i) $d(x, y) = 1$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z)d(z, y)$ for all $x, y, z \in X$.

Definition 1.3 *[5] Let X be a non-empty set and $\theta : X \times X \to [1, \infty)$. A function $d_\theta : X \times X \to \mathbb{R}^+$ is called an extended b-metric on X if the following conditions are satisfied:*
(i) $d_\theta(x, y) = 0$ if and only if $x = y$,
(ii) $d_\theta(x, y) = d_\theta(y, x)$ for all $x, y \in X$,
(iii) $d_\theta(x, y) \leq \theta(x, y) [d_\theta(x, z) + d_\theta(z, y)]$ for all $x, y, z \in X$.
Note that every b-metric space is an extended b-metric space with $\theta(x, y) = s \geq 1$.

Definition 1.4 *Let X be a non-empty set. A symmetric on X is a non-negative real-valued function d on $X \times X$ such that:*
(i) $d(x, y) = 0$ iff $x = y$,
(ii) $d(x, y) = d(y, x) \quad \forall x, y \in X$.

2. Main Results

In this section, we begin by introducing and constructing the T_α-metric space, providing its fundamental properties and the necessary definitions. After that, we will proceed to prove a fixed point theorem that generalizes the classical Banach fixed point theorem in various spaces.

Definition 2.1 *Let $\alpha \in \mathbb{R}_+$ and Φ_α be the set of all functions $\phi : \mathbb{R}_+ \times [\alpha, +\infty) \longrightarrow [\alpha, +\infty)$ satisfying the following conditions:*
(ϕ_1) $\phi(., u)$ and $\phi(\lambda, .)$ are increasing,
(ϕ_2) $\lim_{\lambda \to 0} \phi(\lambda, u) = \lim_{u \to \alpha} \phi(\lambda, u) = \alpha$,
(ϕ_3) $\phi(\lambda_1, \phi(\lambda_2, u)) \leq \phi(\lambda_1 \lambda_2, u)$,
(ϕ_4) $\forall \lambda \in [0, 1)$, $\forall u > \alpha$, $\phi(\lambda, u) < u$.

Definition 2.2 *Let X be a non-empty set and $\theta : X \times X \times X \longrightarrow \mathbb{R}_+$.*
Let $\alpha \in \mathbb{R}_+$ and $\phi \in \Phi_\alpha$. We say that $T : X \times X \times X \times [\alpha, +\infty) \times [\alpha, +\infty) \longrightarrow [\alpha, +\infty)$, satisfy the condition $P_\alpha(\phi, \theta)$ if for all sequences $(x_n), (y_n), (z_n) \subset X$ and all $(t_n), (t'_n) \subset [\alpha, +\infty)$, we have

$$\liminf_{n \to +\infty} t_n = \alpha \ \Rightarrow \ \big(\liminf_{n \to +\infty} T(x_n, y_n, z_n, t'_n, t_n) \leq \liminf_{n \to +\infty} \phi(\theta(x_n, y_n, z_n), t'_n)\big). \tag{2.1}$$

Example 2.1 *Let $X = \mathbb{R}$, $\alpha = 0$, $T(x, y, z, t_1, t_2) = t_1 + t_2$, $\phi(\lambda, u) = \lambda u$ and $\theta(x, y, z) = 1$.*

Example 2.2 *Let $X = \mathbb{R}$, $\alpha = 1$, $T(x, y, z, t_1, t_2) = t_1 t_2$, $\phi(\lambda, u) = u^\lambda$ and $\theta(x, y, z) = 1$.*

Definition 2.3 *Let X be a non-empty set and T satisfying $P_\alpha(\phi, \theta)$ as in Definition 2.2. A function $d : X \times X \longrightarrow [\alpha, +\infty)$ is called a T_α-metric, if the following conditions are satisfied for all $x, y, z \in X$,*
(i) $d(x, y) = \alpha \Leftrightarrow x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leq T(x, y, z, d(x, z), d(z, y))$.
The pair (X, d) is a T_α-metric space.

Example 2.3 *Let $X = [1, 2]$ and*

$$\begin{aligned} d : \quad X \times X &\longrightarrow \mathbb{R}_+ \\ (x, y) &\longrightarrow |xy|(x - y)^2. \end{aligned}$$

Then for all $x, y, z \in X$, we have:

$$
\begin{aligned}
(i)\, d(x,y) &= 0 \Leftrightarrow x = y,\\
(ii)\, d(x,y) &= d(y,x),\\
(iii)\, d(x,y) &= |xy|(x-y)^2\\
&\le 2|xy|[(x-z)^2+(z-y)^2]\\
&= 2|xy|(x-z)^2+2|xy|(z-y)^2\\
&\le 2\left|\frac{y}{z}\right||xz|(x-z)^2+2\left|\frac{x}{z}\right||zy|(z-y)^2\\
&\le 2\left|\frac{y}{z}\right|d(x,z)+2\left|\frac{x}{z}\right|d(z,y).
\end{aligned}
$$

Therefore (X,d) is a T_0-metric space, with

$$T\,(x,y,z,t_1,t_2) = 2\left|\frac{y}{z}\right|t_1 + 2\left|\frac{x}{z}\right|t_2,\ \theta(x,y,z) = 2\sup\left(\left|\frac{y}{z}\right|,\left|\frac{x}{z}\right|\right)\ \text{and}\ \phi(\lambda,u)=\lambda u.$$

Example 2.4 *Let (X,d) be a metric space and $\alpha > 0$.*
Consider the following function

$$
\begin{aligned}
d' :\quad X\times X &\longrightarrow [\alpha,+\infty)\\
(x,y) &\longrightarrow \alpha\exp(d(x,y));
\end{aligned}
$$

then (X,d') is a T_α-metric space, with

$$T(x,y,z,t_1,t_2) = \frac{t_1t_2}{\alpha},\ \theta(x,y,z)=1\ \text{and}\ \phi(\lambda,u)=u^\lambda.$$

Example 2.5 *The multiplicative metric space is a T_1-metric space, with*

$$T\,(x,y,z,t_1,t_2) = t_1t_2,\ \theta(x,y,z)=1\ \text{and}\ \phi(\lambda,u)=u^\lambda.$$

Example 2.6 *The symmetric space is a T_0-metric space, with*

$$T\,(x,y,z,t_1,t_2) = \theta(x,y,z)(t_1+t_2),\ \phi(\lambda,u)=\lambda u$$

and

$$\theta(x,y,z) = \begin{cases} \dfrac{d(x,y)}{d(x,z)+d(z,y)} & \text{if } x\ne y,\\ 0 & \text{otherwise.}\end{cases}.$$

Remark 2.1 Metric $\to$ b-metric $\to$ extended b-metric $\to$ Symmetric $\leftrightarrows$ T_0-metric. Also multiplicative metric $\to$ T_1-metric.

Definition 2.4 *Let (X,d) be a T_α-metric space.*
1) *A sequence $(x_n)\subset X$ is said to converge to x, and we write $x_n\to x$, if and only if $d(x_n,x)\to\alpha$ as $n\to+\infty$.*
2) *A sequence $(x_n)\subset X$ is said to be Cauchy when $d(x_n,x_m)\to\alpha$ as $n,m\to+\infty$.*
3) *(X,d) is said to be complete if every Cauchy sequence is convergent.*
4) *A subset $B\subset X$ is said to be closed if for every sequence $(x_n)\subset B$ converging to $x\in X$, we have $x\in B$.*
5) *A subset $B\subset X$ is said to be compact if every sequence in B has a convergent subsequence.*
6) *A subset $B\subset X$ is said to be bounded if $\sup\{d(x,y)\,:\,x,y\in B\}<+\infty$.*

Theorem 2.1 *Let (X,d) be a complete T_α-metric space, whith T satisfying $P_\alpha(\phi,\theta)$ as in Definition 2.2 and such that*

$$\sup\{\theta(x,y,z)\ ;\ (x,y,z)\in X\times X\times X\}\leq \eta$$

for some constant $\eta\geq 1$.

Let $f:X\to X$ be a function and suppose the existence of a $\lambda\in[0,\frac{1}{\eta})$ such that

$$d(f(x),f(y))\leq \phi(\lambda,d(x,y)),$$

for all $x,y\in X$. Then f admits a unique fixed point x in X.

Proof.

To prove the existence of a fixed point, let $(x_n)_{n\in\mathbb{N}}\subset X$ such that $x_0\in X$ and $x_{n+1}=f(x_n)$, then for $n\geq 1$, we have:

$$d(x_{n+1},x_n)=d(f(x_n),f(x_{n-1}))\leq \phi(\lambda,d(x_n,x_{n-1}))\leq \phi(\lambda^n,d(x_1,x_0)).$$

Therefore,

$$\lim_{n\to+\infty} d(x_{n+1},x_n)=\alpha.$$

Now, we must show that (x_n) is Cauchy. Let us proceed by contradiction. There exists $\varepsilon>\alpha$, for which we can find two subsequences $\{x_{m(i)}\},\{x_{n(i)}\}$ such that $n(i)$ is the smallest index for which

$$n(i)>m(i)\geq i \text{ and } d(x_{n(i)},x_{m(i)})>\varepsilon. \tag{2.2}$$

This means that

$$d(x_{n(i)-1},x_{m(i)})\leq\varepsilon.$$

We have therefore

$$\varepsilon<d(x_{n(i)},x_{m(i)})\leq T(x_{n(i)},x_{m(i)},x_{m(i)+1},d(x_{n(i)},x_{m(i)+1}),d(x_{m(i)+1},x_{m(i)})).$$

Thus, as $i\to+\infty$ and (2.1) we obtain

$$\begin{aligned}
\varepsilon &\leq \liminf T(x_{n(i)},x_{m(i)},x_{m(i)+1},d(x_{n(i)},x_{m(i)+1}),d(x_{m(i)+1},x_{m(i)}))\\
&\leq \liminf \phi(\theta(x_{n(i)},x_{m(i)},x_{m(i)+1}),d(x_{n(i)},x_{m(i)+1}))\\
&\leq \liminf \phi(\eta,d(x_{n(i)},x_{m(i)+1}))\\
&\leq \liminf \phi(\eta,\phi(\lambda,d(x_{n(i)-1},x_{m(i)})))\\
&\leq \liminf \phi(\eta\lambda,d(x_{n(i)-1},x_{m(i)}))\\
&\leq \phi(\eta\lambda,\varepsilon)\\
&< \varepsilon.
\end{aligned}$$

This is a contradiction. Therefore, (x_n) is Cauchy, and consequently, there exists $x\in X$ such that

$$\lim_{n\to+\infty} d(x_n,x)=\alpha.$$

Let us show that x is a fixed point of f. We have:

$$d(f(x),x)\leq T(f(x),x,x_{n+1},d(f(x),x_{n+1}),d(x_{n+1},x)).$$

Therefore,

$$\begin{aligned}
d(f(x),x) &\leq \liminf_{n\to+\infty} T(f(x),x,x_{n+1},d(f(x),f(x_n)),d(x_{n+1},x))\\
&\leq \liminf_{n\to+\infty} \phi(\theta(f(x),x,x_{n+1}),d(f(x),f(x_n)))\\
&\leq \liminf_{n\to+\infty} \phi(\eta,d(f(x),f(x_n)))\\
&\leq \liminf_{n\to+\infty} \phi(\eta\lambda,d(x,x_n))\\
&= \alpha.
\end{aligned}$$

Thus $x = f(x)$.

Now, suppose there exists another fixed point $y \neq x$ of f, then

$$\begin{aligned} d(x,y) &= d(f(x),f(y)) \\ &\leq \phi(\lambda, d(x,y)) \\ &< d(x,y), \end{aligned}$$

which is a contradiction.

Example 2.7 *Let $X = [1,2]$, $d : X \times X \longrightarrow \mathbb{R}_+$ such that $d(x,y) = |xy|(x-y)^2$, $\forall x,y \in X$. According to example 2.3, (X,d) is a T_0-metric space, with*

$$T(x,y,z,t_1,t_2) = 2\left|\frac{y}{z}\right|t_1 + 2\left|\frac{x}{z}\right|t_2, \quad \phi(\lambda,u) = \lambda u$$

and

$$\theta(x,y,z) = \sup\left(\left|\frac{y}{z}\right|, \left|\frac{x}{z}\right|\right) \leq 2 = \eta.$$

It is clear that (X,d) is a complete T_0-metric space. Consider the following function

$$f : X \longrightarrow X, \quad x \longrightarrow \sqrt{x}.$$

Then, for all $x,y \in X$, we have:

$$\begin{aligned} d(f(x),f(y)) &= d(\sqrt{x},\sqrt{y}) \\ &= |\sqrt{x}\sqrt{y}|(\sqrt{x}-\sqrt{y})^2 \\ &= \frac{1}{|\sqrt{x}\sqrt{y}|(\sqrt{x}+\sqrt{y})^2} \times |xy|(x-y)^2 \\ &\leq \frac{1}{4} \times |xy|(x-y)^2 \\ &= \frac{1}{4}d(x,y). \end{aligned}$$

Therefore,

$$d(f(x),f(y)) \leq \phi(\lambda, d(x,y)) \quad \text{with} \quad \lambda = \frac{1}{4} \in \left[0, \frac{1}{\eta}\right).$$

According to Theorem 1, f admits a unique fixed point $x = 1$ in X.

3. Application

In this section, we apply our main fixed point theorem to establish the existence and uniqueness of solutions to a nonlinear Fredholm integral equation.

Consider the following equation:

$$x(t) = g(t) + \int_0^1 K(t,s)f(s,x(s))ds, \quad t \in [0,1],$$

where:

- $g : [0,1] \to \mathbb{R}$ is a given function;
- $K : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous function;
- $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a function satisfying the following generalized Lipschitz condition:

$$|f(s,u) - f(s,v)| \leq M(s)|u-v|, \quad \forall s \in [0,1], \ \forall u,v \in \mathbb{R},$$

where $M(s) > 0$ is a continuous function on $[0,1]$.

Let us consider the space $X = C([0,1],\mathbb{R})$ of continuous functions on $[0,1]$ equipped with:

$$d(x,y) = \sup_{t\in[0,1]} \left(\frac{|x(t)y(t)|+1}{|x(t)y(t)|+2} |x(t)-y(t)| \right).$$

Proposition 3.1 *The space (X,d) is a complete T_0-metric space with the control function*

$$T(x,y,z,t_1,t_2) = 2\frac{\|x\|_\infty \|y\|_\infty + 1}{\|x\|_\infty \|y\|_\infty + 2} [t_1 + t_2],$$

and the associated control function

$$\theta(x,y,z) = 2\frac{\|x\|_\infty \|y\|_\infty + 1}{\|x\|_\infty \|y\|_\infty + 2} \le \eta = 2,$$

where $\|x\|_\infty = \sup_{t\in[0,1]} |x(t)|$.

Proof.

The verification of the properties of a T_0-metric space is straightforward.

To prove completeness, let $x, y \in X$. It is easy to verify that the function d satisfies the following inequalities:

$$\frac{1}{2}\|x-y\|_\infty \le d(x,y) \le \|x-y\|_\infty.$$

Since $(X, \|.\|_\infty)$ is a Banach space, we conclude that (X,d) is a complete metric space.

Now, define the operator $F : X \to X$ by

$$F(x)(t) = g(t) + \int_0^1 K(t,s) f(s,x(s))\, ds.$$

Theorem 3.1 *Suppose the following conditions are satisfied:*

- *K is a continuous,*
- *f satisfies the generalized Lipschitz condition mentioned above,*
- $$\max_{t,s\in[0,1]} |K(t,s)| \le \frac{1}{5\int_0^1 M(s)\, ds}.$$

Then the integral equation admits a unique solution in $X = C([0,1],\mathbb{R})$.

Proof.

Let us show that the operator F satisfies the conditions of Theorem 2.1.

For all $x, y \in X$ and all $t \in [0,1]$, we have:

$$\begin{aligned}
|F(x)(t) - F(y)(t)| &= \left| \int_0^1 K(t,s)[f(s,x(s)) - f(s,y(s))]\, ds \right| \\
&\le \int_0^1 |K(t,s)| \cdot |f(s,x(s)) - f(s,y(s))|\, ds \\
&\le \int_0^1 |K(t,s)| \cdot M(s)|x(s)-y(s)|\, ds \\
&\le \max_{t,s\in[0,1]} |K(t,s)| \int_0^1 M(s)|x(s)-y(s)|\, ds \\
&\le \max_{t,s\in[0,1]} |K(t,s)| \int_0^1 M(s) \cdot \frac{|x(s)y(s)|+2}{|x(s)y(s)|+1} \left(\frac{|x(s)y(s)|+1}{|x(s)y(s)|+2} |x(s)-y(s)| \right) ds \\
&\le \max_{t,s\in[0,1]} |K(t,s)| \int_0^1 M(s) \cdot \frac{|x(s)y(s)|+2}{|x(s)y(s)|+1} \cdot \sup_{s\in[0,1]} \left(\frac{|x(s)y(s)|+1}{|x(s)y(s)|+2} |x(s)-y(s)| \right) ds \\
&\le d(x,y) \cdot \max_{t,s\in[0,1]} |K(t,s)| \cdot 2 \int_0^1 M(s)\, ds.
\end{aligned}$$

Let us denote

$$\lambda = \max_{t,s\in[0,1]} |K(t,s)| \cdot 2 \int_0^1 M(s)\,ds.$$

Then,

$$|F(x)(t) - F(y)(t)| \leq \lambda d(x,y).$$

Therefore,

$$\begin{aligned} d(Fx,Fy) &= \sup_{t\in[0,1]} \left(\frac{|Fx(t)Fy(t)|+1}{|Fx(t)Fy(t)|+2} |F(x)(t) - F(y)(t)| \right) \\ &\leq \sup_{t\in[0,1]} |F(x)(t) - F(y)(t)| \\ &\leq \phi(\lambda, d(x,y)), \end{aligned}$$

where $\phi(\lambda, u) = \lambda u$.

From our hypothesis,

$$\max_{t,s\in[0,1]} |K(t,s)| \leq \frac{1}{5\int_0^1 M(s)\,ds},$$

which implies

$$\lambda \leq \frac{2}{5} < \frac{1}{2} = \frac{1}{\eta}.$$

By applying Theorem 2.1, we conclude that the operator F has a unique fixed point in X, which corresponds to the unique solution of the integral equation.

4. Conclusion and Perspectives

In this work, we introduced the concept of T_α-metric space, as a generalization that encompasses several existing generalized metric spaces. This structure unifies and extends various versions of Banach's fixed point theorem found in recent literature. We established a general fixed point theorem within this framework and provided illustrative examples and an application.

Future research directions include:

1. The study of the topological properties of T_α-metric spaces.
2. The exploration of different classes of contractions in T_α-metric spaces.

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